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ON TOTALIZATION OF THE H_1-INTEGRAL

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Abstract. Based on the total H_1-integrability concept, which is established in this paper, we shall try to show that at any point of a compact interval $(a, b]$ in $\mathbb{R}$, at which a point function F has no a discontinuity, F is the total H_1-indefinite integral of a function dF_{ex} being the limit of $\Delta F_{ex}(I)$, where $I \subseteq [a, b]$, on $[a, b]$, without additional hypotheses on F. A residue function of F is introduced. The paper ends with a few of examples that illustrate the theory.

1. INTRODUCTION

Let F: $[a, b] \mapsto \mathbb{R}$ be a differentiable function and let f be its derivative. The problem of recovering F from f is called problem of primitives. The generalized *Riemann* integral, or the *Kurzweil-Henstock* integral [1, 3], solves this problem in formulating the fundamental theorem of calculus in $\mathbb{R}$ whenever f exists on $[a, b]$. Notion of the so called H_1-integral, as the *Moore-Smith* limit of *Riemann* sums, was introduced by *Garces*, *Lee* and *Zhao*, [2]. This integral has the property that f is *Kurzweil-Henstock* integrable on $[a, b]$ if and only if there is an H_1-integrable function g such that $f(x) = g(x)$ almost everywhere on $[a, b]$. In contrast to the one-dimensional case, the *Kurzweil-Henstock* integral in $\mathbb{R}^n$ does not integrate all derivatives (see *Pfeffer* [9]). In order to remove this flow, *Mawhin* [7] added a condition restricting the class of admissible partitions of an n-dimensional interval. This led to another *Riemann* type integral named regular partition integral, that would integrate all derivatives in $\mathbb{R}^n$. *Macdonald* [6] used the regular partition integral to overcome the deficiency in *Hestenes'* proof of the fundamental theorem of calculus. Namely, *Hestenes* gave a heuristic demonstration of the fundamental theorem, [4]. The demonstration is wonderfully brief and offers great intuitive

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insight, but it is not a rigorous proof. In particular, *Hestenes* proves his result using the usual definition of a *Riemann* integral, as well as the integral definition of f. However, in the fundamental theorem proof the derivative f is approximated by the interval function without any conditions on f. Accordingly, in what follows, we will try to give a rigorous proof of the fundamental theorem in $\mathbb{R}$, for a large-scale class of F, and in the spirit of *Hestenes'* appealing proof. To do this, we must firstly define a linear differential form dF_{ex} of an extension F_{ex} of F. After that it remains to define a new integral named total H_1-integral that would integrate this differential form without additional hypotheses on F.

2. Preliminaries

The extended set $\mathbb{R} \cup \{-\infty, +\infty\}$ of real numbers $\mathbb{R}$ is denoted by $\overline{\mathbb{R}}$. By $\mathbb{N}$ we denote the set of natural numbers. The *Lebesgue* measure in $\mathbb{R}$ is denoted by μ, however, for $E \subset \mathbb{R}$ we write $|E|$ instead of $\mu(E)$.

Given a compact interval $[a, b]$ in $\mathbb{R}$, let the collection $\mathcal{I}([a, b])$ be a family of all compact subintervals I of $[a, b]$. Any real valued function defined on $\mathcal{I}([a, b])$ is an interval function. For $f : [a, b] \mapsto \mathbb{R}$ the associated interval function of f is an interval function $f : \mathcal{I}([a, b]) \mapsto \mathbb{R}$, again denoted by f, [10]. A partition $P[a, b]$ of $[a, b]$ is a finite set (collection) of interval-point pairs $([a_i, b_i], x_i)$, $i = 1, ..., \nu$, such that the subintervals $[a_i, b_i]$ are non-overlapping ($(a_i, b_i) \cap (a_j, b_j) = \emptyset$ for $i \neq j$, where (a_i, b_i) is the interior of $[a_i, b_i]$), $\cup_{i \leq \nu} [a_i, b_i] = [a, b]$ and $x_i \in (a_i, b_i)$ if x_i is an interior point of $[a, b]$. The points $\{x_i\}_{i \leq \nu}$ are the tags of $P[a, b]$. If E is a subset of $[a, b]$, then the restriction of $P[a, b]$ to E is a finite collection of $([a_i, b_i], x_i) \in P[a, b]$ such that each $x_i \in E$. In symbols, $P[a, b]|_E = \{([a_i, b_i], x_i) \in P[a, b] \mid x_i \in E\}$. It is evident that a given partition of $[a, b]$ can be tagged in infinitely many ways by choosing different points as tags. Given $\delta : [a, b] \mapsto \mathbb{R}_+$, named a gauge, a partition $P[a, b]$ is called *δ-fine* if $[a_i, b_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i))$. By *Cousin's* lemma the set of *δ-fine* partitions of $[a, b]$ is nonempty, [1]. Let $\mathcal{P}[a, b]$ be the family of all partitions $P[a, b]$ of $[a, b]$. Then, by $\mathcal{P}_\delta[a, b]$ we denote the family of all *δ-fine* partitions of $[a, b]$ for some given $\delta : [a, b] \mapsto \mathbb{R}_+$.

For the infinite set of partitions $\{P_n[a, b] \mid P_n[a, b] = \{([a_{i_n}, b_{i_n}], x_{i_n})\}, n \in \mathbb{N}\}$, denoted by $\langle P_n[a, b] \rangle$, we write $\langle P_n[a, b] \rangle \in (\mathcal{P}[a, b], \prec)$, if $P_n[a, b] \prec P_{n+1}[a, b]$ for each $n \in \mathbb{N}$. The statement $P_n[a, b] \prec P_{n+1}[a, b]$ means that for each interval-point pair $([a_{i_{n+1}}, b_{i_{n+1}}], x_{i_{n+1}}) \in P_{n+1}[a, b]$ there exists a corresponding interval-point pair $([a_{i_n}, b_{i_n}], x_{i_n}) \in P_n[a, b]$ such that $[a_{i_{n+1}}, b_{i_{n+1}}] \subset [a_{i_n}, b_{i_n}]$, and

$$\{x_{i_n} \mid ([a_{i_n}, b_{i_n}], x_{i_n}) \in P_n[a, b]\} \subset \{x_{i_{n+1}} \mid ([a_{i_{n+1}}, b_{i_{n+1}}], x_{i_{n+1}}) \in P_{n+1}[a, b]\}.$$

Then, $(\mathcal{P}[a, b], \prec)$ is the family of directed sets, [2]. Clearly, for any $x \in [a, b]$ there exists a directed set $\langle P_n[a, b] \rangle \in (\mathcal{P}[a, b], \prec)$ so that x is a tag for it.

If $f : [a, b] \mapsto \mathbb{R}$ is a point function and $\phi : \mathcal{I}[a, b] \mapsto \mathbb{R}$ is an interval function, which assign to each interval-point pair $([a_{i_n}, b_{i_n}], x_{i_n})$ of each partition $P_n[a, b]$ in the partition set $\langle P_n[a, b]\rangle \in (\mathcal{P}[a, b], \prec)$ the real numbers $f(x_{i_n})$ and $\phi([a_{i_n}, b_{i_n}])$, respectively, we shall call both $\phi : \langle P_n[a, b]\rangle \mapsto \mathbb{R}$ and $f\phi : \langle P_n[a, b]\rangle \mapsto \mathbb{R}$ a net of real numbers, [8]. The statement $f\phi : \langle P_n[a, b]\rangle \mapsto \mathbb{R}$ means that each interval-point pair $([a_{i_n}, b_{i_n}], x_{i_n})$ of each partition $P_n[a, b]$ in the partition set $\langle P_n[a, b]\rangle \in (\mathcal{P}[a, b], \prec)$ is mapped by the so called interval-point function $f\phi : [a, b] \times \mathcal{I}[a, b] \mapsto \mathbb{R}$ being the product of $f : [a, b] \mapsto \mathbb{R}$ and $\phi : \mathcal{I}[a, b] \mapsto \mathbb{R}$ to the real number $f(x_{i_n})\,\phi([a_{i_n}, b_{i_n}])$.

For a real-valued function F, the derivative f could be defined as the limit of an interval function associated to F. There are a number of different ways to do that. However, before that we have to define the limit of some interval function on $[a, b]$. Accordingly, the definition given below comes from *Definition* of the *Moore-Smith* limit in [8].

Definition 1. Let $\phi : \mathcal{I}[a, b] \mapsto \mathbb{R}$ and $E \subseteq [a, b]$. Then, a function $f : [a, b] \mapsto \mathbb{R}$ is the limit of ϕ on $[a, b]\backslash E$ if there exists a gauge δ on $[a, b]$ such that for each $\langle P_n[a, b]\rangle \in (\mathcal{P}_\delta[a, b], \prec)$ and for every $\varepsilon > 0$ there exists a partition $P_{n_\varepsilon}[a, b] \in \langle P_n[a, b]\rangle$ such that

$$|\phi([a_{i_n}, b_{i_n}]) - f(x_{i_n})| < \varepsilon, \tag{2.1}$$

whenever $([a_{i_n}, b_{i_n}], x_{i_n}) \in P_n[a, b]\,\big|_{[a,b]\backslash E}$ and $P_{n_\varepsilon}[a, b] \prec P_n[a, b]$.

In what follows we will use the following notations $\Delta F(I) = F(v) - F(u)$, where u and v are the endpoints of $I \in \mathcal{I}([a, b])$, $\sum_i \Delta F([a_i, b_i]) = \Delta F(P[a, b]\,|_E)$ and $\sum_i f(x_i)\,|[a_i, b_i]| = \delta F(P[a, b]\,|_E)$ whenever $([a_i, b_i], x_i) \in P[a, b]\,|_E$.

Definition 2. Let $F : [a, b] \mapsto \mathbb{R}$ and let $f : [a, b] \mapsto \mathbb{R}$. Then, F is said to be differentiable to f on $[a, b]$, if f is the limit of ϕ on $[a, b]$, and ϕ is defined by

$$\phi(I) = \frac{\Delta F(I)}{\Delta x(I)}, \tag{2.2}$$

where $\Delta x(I) = |I|$ and $I \in \mathcal{I}([a, b])$.

If $f : [a, b] \mapsto \mathbb{R}$ is the limit of a convergent interval function ϕ on $[a, b]$, then f is a *Baire* class one function (see *Theorem* 5.22 in [3]). On the other hand, if ϕ converges to the limit function f almost everywhere on $[a, b]$, that means for every $x \in [a, b]$ except for a set $E \subset [a, b]$ of *Lebesgue* measure zero, then at the points belonging to E the limit f of ϕ can take values $\pm\infty$ or not be defined at all. Hence, the domain of f may not be all of $[a, b]$. If the set E is a countable set, then ϕ is said to converge to f nearly everywhere on $[a, b]$. Unless otherwise stated in what follows, we assume that F is defined and differentiable on $[a, b] \setminus E$, as well as that the endpoints of $[a, b]$ do not belong to E.

When working with functions, which have a finite number of discontinuities on $[a, b]$, it does not really matter how these functions will be defined on the set of discontinuities. The validity of this statement will be clarified as the theory unfolds. As this situation will arise frequently, we adopt the convention that, unless mentioned otherwise, such functions are equal to 0 at all points at which they can take values $\pm\infty$ or not be defined at all. Accordingly, we may define point functions $F_{ex} : [a, b] \mapsto \mathbb{R}$ and $f_{ex} : [a, b] \mapsto \mathbb{R}$ by extending F and its derivative f from $[a, b] \setminus E$ to E by $F_{ex}(x) = 0$ and $f_{ex}(x) = 0$ for $x \in E$, so that

$$F_{ex}(x) = \begin{cases} F(x), \text{ if } x \in [a, b] \setminus E \\ 0, \text{ if } x \in E \end{cases} \quad \text{and} \tag{2.3}$$

$$f_{ex}(x) = \begin{cases} f(x), \text{ if } x \in [a, b] \setminus E \\ 0, \text{ if } x \in E \end{cases} .$$

3. Main Results

Let $F : [a, b] \mapsto \mathbb{R}$. It is an old result that F is continuous on $[a, b]$ if and only if $\triangle F(I)$, where $I \in \mathcal{I}[a, b]$, converges to 0 at all points of $[a, b]$, [3]. Accordingly, we are now in a position to define the notion of the linear differential form on $[a, b]$.

Definition 3. For $F : [a, b] \mapsto \mathbb{R}$ let ϕ be defined by (2.2). Then, the limit dF of the interval function

$$\triangle F(I) = \phi(I)\triangle x(I), \tag{3.1}$$

where $I \in \mathcal{I}[a, b]$, on $[a, b]$, is a linear differential form on $[a, b]$.

Clearly, if F is continuous on $[a, b]$ then dF vanishes identically on $[a, b]$. If in addition F is differentiable to $f : [a, b] \mapsto \mathbb{R}$ on $[a, b]$, then we can introduce into the analysis an interval-point function $\delta F : [a, b] \times \mathcal{I}([a, b]) \mapsto \mathbb{R}$, being the product of the point function f and the interval function $\triangle x : \mathcal{I}[a, b] \mapsto \mathbb{R}$, as follows

$$\delta F(I, x) = f(x)\triangle x(I). \tag{3.2}$$

As we can see, there is a difference between the interval-point function $\delta F(I, x)$ and the interval function $\triangle F(I)$. However, by *Definition 1*, for the case f is the limit of ϕ on $[a, b]$, there exists a gauge δ on $[a, b]$ such that for every $\varepsilon > 0$ there exists a *δ-fine* partition $P_{n_\varepsilon}[a, b]$ such that

$$|\delta F([a_{i_n}, b_{i_n}], x_{i_n}) - \triangle F([a_{i_n}, b_{i_n}])| < \varepsilon\triangle x(I), \tag{3.3}$$

whenever $([a_{i_n}, b_{i_n}], x_{i_n}) \in P_n\,[a,b]$, $\langle P_n\,[a,b]\rangle \in (\mathcal{P}_\delta\,[a,b]\,, \prec)$ and $P_{n_\varepsilon}\,[a,b] \prec P_n\,[a,b]$. So, in this emphasized case, the point function fdx, as the limit of δF, is identically equal to dF, as the limit of $\triangle F$, on $[a,b]$.

The following definition of the H_1-integral comes from [2].

Definition 4. A point function $f : [a,b] \mapsto \mathbb{R}$ is H_1-integrable to a real point $\mathcal{F}$ on $[a,b]$ if there exists a gauge δ on $[a,b]$ such that for every $\varepsilon > 0$ there exists a δ-*fine* partition $P_{n_\varepsilon}\,[a,b]$ such that

$$|\delta F(P_n\,[a,b]) - \mathcal{F}| < \varepsilon, \tag{3.4}$$

whenever $P_n\,[a,b] \in \langle P_n\,[a,b]\rangle$, $\langle P_n\,[a,b]\rangle \in (\mathcal{P}_\delta\,[a,b]\,, \prec)$ and $P_{n_\varepsilon}\,[a,b] \prec P_n\,[a,b]$. In symbols, $\mathcal{F} := \mathrm{H}_1 - \int_a^b f dx$.

Based on this definition we obtain, in aforementioned case, that

$$\mathrm{H}_1 - \int_a^b f dx = \mathrm{H}_1 - \int_a^b dF = \triangle F\,([a,b])\,. \tag{3.5}$$

It remains to consider the case when F has a certain number of discontinuities within $[a,b]$, at which its derivative f, as the limit of (2.2), can take values $\pm\infty$ or not be defined at all, just as the primitive F. Now, in spite of the fact that the limit dx of $\triangle x(I)$ vanishes identically on $[a,b]$, the limit dF_{ex} of $\triangle F_{ex}(I) = \phi_{ex}(I)\triangle x(I)$ could be a null function on $[a,b]$ (*A function* $dF_{ex} : [a,b] \mapsto \mathbb{R}$ *is said to be a null function on* $[a,b]$ *if the set* $\{x \in [a,b] \mid dF_{ex}\,(x) \neq 0\}$ *is a set of Lebesgue measure zero*, see *2.4 Definition* in [1]). Clearly, $\{x \in [a,b] \mid dF_{ex}\,(x) \neq 0\} \subseteq E$, where E is a set at whose points F and its derivative f can take values $\pm\infty$ or not be defined at all. So, it would be reasonable, in this case, to make use of $\triangle F_{ex}$ instead of δF_{ex} to define an integral of dF_{ex}. This is obviously our way of attempting to totalize the H_1-integral. The definition of the total H_1-integral which follows is more general one since it includes one more interval function.

Definition 5. Let $\gamma : \mathcal{I}\,[a,b] \mapsto \mathbb{R}$ be an arbitrary interval function and for $F : [a,b] \mapsto \mathbb{R}$ let $\Delta F : \mathcal{I}\,[a,b] \mapsto \mathbb{R}$ be an interval function defined by (3.4) that converge to $g\,(x)$ and $dF\,(x)$, respectively, almost everywhere on $[a,b]$. A point function $g\,(x)$ is totally H_1-integrable, with respect to the differential form $dF\,(x)$, to a real point $\mathcal{F}$ on $[a,b]$ if there exists a gauge δ on $[a,b]$ such that for every $\varepsilon > 0$ there exists a δ-*fine* partition $P_{n_\varepsilon}\,[a,b]$ such that

$$\left|\sum_{i_n} \gamma\,([a_{i_n}, b_{i_n}])\,\triangle F\,([a_{i_n}, b_{i_n}]) - \mathcal{F}\right| < \varepsilon, \tag{3.6}$$

whenever $([a_{i_n}, b_{i_n}], x_{i_n}) \in P_n\,[a,b]$, $P_n\,[a,b] \in \langle P_n\,[a,b]\rangle$, $\langle P_n\,[a,b]\rangle \in (\mathcal{P}_\delta\,[a,b]\,, \prec)$ and $P_{n_\varepsilon}\,[a,b] \prec P_n\,[a,b]$. In symbols, $\mathcal{F} := \mathrm{H}_1 - vt\int_a^b gdF$.

The crucial advantage of the integration process established by *Definition 5*, in comparison with any other integration process defined up to now, including all the generalized *Riemann* approach to integration, lies in the fact that it is not necessary that g and dF, as the limits of γ and $\triangle F$, respectively, to be defined at all points of $[a,b]$. This fact, upon which our theory is based in what follows, gives us the possibility to include the calculus of residues in the process of integration of real valued functions.

Before that, we are prepared to prove the extended version of the fundamental theorem of calculus. As we will see, the proof becomes trivial if the definition of the total H_1-integral is applied. In fact, in this way, we will attempt to put into a rigorous form *Hestenes'* proof based on the integral definition of the derivative and on the *Riemann* integral. A major motivation for the formulation of integration in this manuscript has been to achieve as simple and general a statement of the fundamental theorem as possible, just as was *Hestenes'* motivation too.

Theorem 1. *For a some compact interval* $[a,b] \in \mathbb{R}$ *let* $E \subset [a,b]$ *be a set of Lebesgue measure zero at whose points a point function* F *defined on* $[a,b] \backslash E$ *can take values* $\pm\infty$ *or not be defined at all. For every* $\hat{x} \in (a,b]$ *that is not a point of discontinuity of* F *the linear differential form* dF_{ex} *is totally* H_1-*integrable to* $F(\hat{x}) - F(a)$ *on* $[a,\hat{x}]$.

Proof. As F_{ex} is defined on $[a,b]$ and hence $\Delta F_{ex}(P_n([a,\hat{x}])) = \Delta F([a,\hat{x}])$, where $\hat{x} \in (a,b]$ is a point at which F has no discontinuity, for each $P_n([a,\hat{x}]) \in \mathcal{P}[a,\hat{x}]$, it follows from *Definition 5* that

$$\mathrm{H}_1 - vt \int_a^{\hat{x}} dF_{ex} = \Delta F([a,\hat{x}]) \tag{3.7}$$

for every $\hat{x} \in (a,b]$ at which F has no discontinuity. ■

If $dF_{ex} = f_{ex}dx$ on $[a,\hat{x}]$, it means that dF_{ex}, as the limit of $\phi_{ex}(I)\Delta x(I) = \Delta F_{ex}(I)$ on $[a,\hat{x}]$, where $I \in \mathcal{I}([a,b])$, vanishes identically on $[a,\hat{x}] \cap E$, then it follows from (3.7) that $\mathrm{H}_1 - vt\int_a^{\hat{x}} f_{ex}(x)\,dx = \Delta F([a,\hat{x}])$. In opposite, there are two cases, one of which is the extreme case when dF_{ex} takes values $\pm\infty$ at some points of $[a,\hat{x}] \cap E$. In the second one, dF_{ex} is a null function on $[a,\hat{x}] \cap E$. However, in both cases, by (3.7), $\mathrm{H}_1 - vt\int_a^{\hat{x}} dF_{ex}(x) = \Delta F([a,\hat{x}])$. All this refer us to the following definitions.

Definition 6. For some compact interval $[a,b] \in \mathbb{R}$ let $F : [a,b] \mapsto \mathbb{R}$ and $E \subset [a,b]$. The linear differential form dF, as the limit of $\Delta F(I)$, where $I \in \mathcal{I}([a,b])$, on $[a,b]$, is said to be basically summable (BS_δ) to a real number $\Re$ on E if there exists a gauge δ on $[a,b]$ such that for every $\varepsilon > 0$ there exists a δ*-fine* partition $P_{n_\varepsilon}[a,b]$ such that $|\Delta F(P_n[a,b]|_E) - \Re| < \varepsilon$, whenever $P_n[a,b] \in \langle P_n[a,b] \rangle$,

$\langle P_n [a, b]\rangle \in (\mathcal{P}_\delta [a, b], \prec)$ and $P_{n_\varepsilon} [a, b] \prec P_n [a, b]$. If in addition E can be written as a countable union of sets on each of which the linear differential form dF is $\mathrm{BS}_{\delta_\varepsilon}$, then dF is said to be BSG_δ on E. In symbols, $\Re := \sum_{x\in E} dF(x)$.

Definition 7. For $F : [a, b] \mapsto \mathbb{R}$ the linear differential form dF is the residue function $\mathcal{R}$ of F on $[a, b]$.

Comparing the two previous definitions with *Definition 5* we may conclude that for any compact interval $I \in \mathcal{I}([a, b])$, whose endpoints are points of continuity of F, the sum of residues of F on I is the total H_1-integral of dF_{ex} on the same interval I, that is,

$$\mathrm{H}_1 - vt \int_I dF_{ex} = \sum_{x\in I} \mathcal{R}(x) . \tag{3.8}$$

On the other hand, if there exists a sum of residues $\Re$ of a primitive F on any set of points $E \subset [a, b]$ of *Lebesgue* measure zero (according to *Definition 6*), at which F and its derivative f can take values $\pm\infty$ or not be defined at all, then $\Re$ is a sum of discrete values that, under certain conditions, can belong to either an at most countable or an uncountable set of real numbers and

$$\Re_I = \sum_{x\in I\cap E} \mathcal{R}(x) . \tag{3.9}$$

By combining the two previous results with the result (3.7) in the proof of *Theorem 1* we obtain that in the aforementioned case

$$\Delta F(I) = \mathrm{H}_1 - vt \int_I dF_{ex} = \sum_{x\in I\setminus E} \mathcal{R}(x) + \sum_{x\in I\cap E} \mathcal{R}(x) . \tag{3.10}$$

This further implies that

$$\Delta F([a, b]) = \mathrm{H}_1 - vt \int_a^b dF_{ex} = \mathrm{H}_1 - \int_a^b f_{ex} dx + \sum_{x\in E} \mathcal{R}(x) , \tag{3.11}$$

since by (2.3) $\sum_{x\in E} f_{ex}(x)\, dx = 0$. In what follows we shall formulate the result (3.11) as a theorem and prove it explicitly.

Theorem 2. *For a some compact interval $[a, b] \in \mathbb{R}$ let $E \subset [a, b]$ be a set of Lebesgue measure zero at whose points a primitive F defined and differentiable on $[a, b] \setminus E$ and its derivative f can take values $\pm\infty$ or not be defined at all. If dF_{ex} is basically summable (BS_δ) on E to the sum $\Re$, then f_{ex} is H_1-integrable on $[a, b]$ and*

$$H_1 - \int_a^b f_{ex} dx + \Re = \Delta F([a, b]) = H_1 - vt \int_a^b dF_{ex}. \tag{3.12}$$

Proof. Let F_{ex} and f_{ex} be defined by (2.3). Since dF_{ex} is BS_δ on E to $\Re$ it follows from *Definition 6* that there exists a gauge δ on $[a,b]$ such that for every $\varepsilon > 0$ there exists a δ-*fine* partition $P_{n_\varepsilon}[a,b]$ such that $|\Delta F_{ex}(P_n[a,b]|_E) - \Re| < \varepsilon$, whenever $P_n[a,b] \in \langle P_n[a,b]\rangle$, $\langle P_n[a,b]\rangle \in (\mathcal{P}_\delta[a,b], \prec)$ and $P_{n_\varepsilon}[a,b] \prec P_n[a,b]$. In addition, $f_{ex}(x) \equiv 0$ on E and $\Delta F_{ex}(P_n[a,b]) = \Delta F([a,b])$, whenever $P_n[a,b] \in \mathcal{P}[a,b]$. Take the result (3.3) into consideration it is readily seen that

$$\begin{aligned} &|\delta F_{ex}(P_n[a,b]) - [\Delta F([a,b]) - \Re]| \\ &\le \left|\delta F(P_n[a,b]|_{[a,b]\setminus E} - \Delta F(P_n[a,b]|_{[a,b]\setminus E}\right| \\ &\quad + |\Delta F_{ex}(P_n[a,b]|_E) - \Re| < \varepsilon(|[a,b]| + 1). \end{aligned}$$

Therefore, f_{ex} is H_1-integrable on $[a,b]$ and $\mathrm{H}_1 - \int_a^b f_{ex}dx = \Delta F([a,b]) - \Re$, that is,

$$\mathrm{H}_1 - \int_a^b f_{ex}dx + \Re = \Delta F([a,b]) = \mathrm{H}_1{-}vt\int_a^b dF_{ex}. \qquad \blacksquare$$

For a primitive F let f be its derivative and let $E \subset [a,b]$ be a set at whose point f can take values $\pm\infty$ or not be defined at all.

Definition 8. The set $vp[a,b] \subseteq [a,b]$ of points, at which F is differentiable, is said to be a regular domain of F in $[a,b]$. Points belonging to $vp[a,b]$ of F are said to be regular points of F in $[a,b]$. The complement of $vp[a,b]$, with respect to $[a,b]$, is said to be a singular domain $vs[a,b]$ of F in $[a,b]$. Points belonging to $vs[a,b]$ of F are said to be singular points of F in $[a,b]$. If f is bounded on a singular point neighbourhood, then this singular point of F is said to be a seeming singular point of F. If f is a bounded function on $vs[a,b]$ of F, then $vs[a,b]$ of F is said to be a seeming singular domain of F.

Definition 9. For $F:[a,b]\mapsto\mathbb{R}$ let $\mathcal{R} : [a,b] \mapsto \mathbb{R}$ be a function of residues of F. The principal value of the H_1-integral of dF on $[a,b]$ is the sum $\sum_{x\in vp[a,b]} \mathcal{R}(x)$ and will be written as $\mathrm{H}_1{-}vp\int_a^b dF$. The singular value of the H_1-integral of dF on $[a,b]$ is the sum $\sum_{x\in vs[a,b]} \mathcal{R}(x)$ and will be written as $\mathrm{H}_1{-}vs\int_a^b dF$.

By *Definition 9*, the result (3.12) of *Theorem 2* can be rewritten as

$$\begin{aligned} \Delta F([a,b]) &= \mathrm{H}_1{-}vt\int_a^b f_{ex}dx = \mathrm{H}_1{-}vp\int_a^b f_{ex}dx + \sum_{x\in vs[a,b]} \mathcal{R}(x) \\ &= \mathrm{H}_1{-}vp\int_a^b f_{ex}dx + \mathrm{H}_1{-}vs\int_a^b f_{ex}dx. \end{aligned} \tag{3.13}$$

In case f vanishes identically on $vp[a,b]$, it follows from (3.13) that

$$\Delta F([a,b]) = \sum_{x\in vs[a,b]} \mathcal{R}(x). \tag{3.14}$$

The obtained result provides an extension of *Cauchy's* result from the calculus of residues in $\mathbb{R}$ (compare with results in [5]).

4. Examples

(1) Let $[a, b] \subset \mathbb{R}$ be an arbitrary compact interval, such that $0 \in (a, b)$, and let F be *Heaviside's* unit function defined on $[a, b]$ as follows

$$F(x) := \begin{cases} 0, \text{ if } a \leq x \leq 0 \\ 1, \text{ if } 0 < x \leq b \end{cases}.$$

The following result $\mathrm{H}_1\text{--}vt \int_a^b dF = 1$ (dF is the limit of $\triangle F(I) = \phi(I)\Delta x(I)$, where $I \in \mathcal{I}([a, b])$, on $[a, b]$, and ϕ is defined by (2.2)) is an immediate consequence of *Theorem* 1. So, in spite of the fact that ϕ converges to $\delta(x) : [a, b] \mapsto \overline{\mathbb{R}}$, which is defined on $[a, b]$ as follows

$$\delta(x) = \begin{cases} +\infty, \text{ if } x = 0 \\ 0, \text{ otherwise} \end{cases},$$

nearly everywhere on $[a, b]$, more precisely on $[a, b] / \{0\}$, the limit of $\phi(I)\Delta x(I)$, where $I \in \mathcal{I}([a, b])$, at the point of discontinuity of F, goes to 1. Hence, $\Re = 1$ and $\mathrm{H}_1\text{--}vt \int_a^b dF = \mathrm{H}_1\text{--} \int_a^b \delta_{ex} dx + \Re = 1$, since $\mathrm{H}_1\text{--} \int_a^b \delta_{ex} dx = 0$. ■

(2) Let $[a, b]$ be as in the previous example. Then, by *Definition 2* the point function $F(x) = \ln |x|$ is differentiable to $1/x$ at all but the set $\{0\}$ of (a, b). Recall that our convection is that F_{ex} is defined on $[a, b]$ and $F_{ex}(0) = 0$. In this case, the limit of $\triangle F_{ex}([I]) = \phi_{ex}(I)\Delta x(I)$, where $I \in \mathcal{I}([a, b])$, at the point of discontinuity of F, is not defined. However, the sum $\mathrm{H}_1\text{--}vp \int_a^b f_{ex} dx + \Re$ now reduces to the so called indeterminate expression $\infty - \infty$ that, according to *Theorem* 1, takes the value $\ln |b/a|$, so that $\mathrm{H}_1\text{--}vt \int_a^b dF_{ex} = \mathrm{H}_1\text{--}vp \int_a^b f_{ex} dx + \Re = \Delta F([a, b]) = \ln |b/a|$. ■

(3) Let $[a, b]$ be as before. Then, the point function $F(x) = 1/x$ is differentiable to $-\left(1/x^2\right)$ at all but the exceptional set $\{0\}$ (the set where the limit of ϕ defined by (2.2) does not exist in this case) of $[a, b]$. Note again that by our convection F_{ex} is defined on $[a, b]$ and $F_{ex}(0) = 0$. Hence, $\mathrm{H}_1\text{--}vt \int_a^b dF_{ex} = \mathrm{H}_1\text{--}vp \int_a^b f_{ex} dx + \Re$, as the so called indeterminate expression of type $\infty - \infty$, is equal to $(a - b) / (ab)$. ■

References

1. R. G. Bartle, *A Modern Theory of Integration*, Graduate Studies in Math., Vol. 32, AMS, Providence, 2001.

2. I. J. L. Garces and P. Y. Lee, Convergence theorems for the H_1-integral, *Taiwanese J. Math.*, **4(3)** (2000), 439-445.

3. R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron and Henstock*, Graduate Studies in Math., Vol. 4, AMS, Providence, 1994.

4. D. Hestenes, Multivector Calculus, *J. Math. Anal. Appl.*, **24** (1968), 313-325.

5. D. Hestenes, Multivector Functions, *J. Math. Anal. Appl.*, **24** (1968), 467-473.

6. A. Macdonald, Stokes' theorem, *Real Analysis Exchange*, **27** (2002), 739-747.

7. J. Mawhin, Generalized Riemann integrals and the divergence theorem for differentiable vector fields, E. B. Christofel, ed., Aachen/Monschau, 1979, *Birkhauser*, Basel, 1981, pp. 704-714.

8. E. J. McShane, Partial Orderings and Moore-Smith Limits, *Am. Math. Mon.*, **59** (1952), 1-11.

9. W. Pfeffer, The multidimensional fundamental theorem of calculus, *J. Austral. Math. Soc.*, Ser. A, **43** (1987), 143-170.

10. V. Sinha and I. K. Rana, On the continuity of associated interval functions, *Real Analysis Exchange* **29(2)** (2003/2004), 979-981.

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