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DISTRIBUTION OF NORMALIZED ZERO-SETS OF RANDOM ENTIRE FUNCTIONS WITH SMALL RANDOM PERTURBATION

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Abstract. This paper extends our recent result about the distribution of normalized zero-sets of random entire functions[Y–200811] to the context of random entire functions with small random perturbation.

1. INTRODUCTION

In this paper, we extend our recent result about the distribution of normalized zero-sets of random entire functions[Y–200811] to the context of random entire functions with small random perturbation. The small perturbation problem naturally arose from the proof of the fundamental theorem of algebra as explained below and is also known as the important *moving targets* problem in Nevanlinna theory. Recall that the proof of the fundamental theorem of algebra comes from the following observation: when we write $P(z) = a_n z^n + Q_{n-1}(z)$, where $n = \deg P$ and $\deg Q_{n-1} = n-1$. Then $|Q_{n-1}(z)| < |a_n z^n|$ on $|z| = r$ for r large enough, hence Rouché's theorem implies that the the zeros of P is the same as the zeros of $a_n z^n$. In other words, $P(z)$ can be obtained from $a_n z^n$ through a small perturbation by Q_{n-1}. Similarly, one can easily prove that, in the Nevanlinna theory, the growth (characteristic function) of f is the same as $f + g$, the function obtained by small perturbation by g. (Here, by small perturbations we mean $T_g(r) = o(T_f(r))$). Problems of these types are called *small perturbation problems* or called *problems of slowly moving targets*. In 1983, Steinmetz successfully extended Nevanlinna's SMT to slowly moving targets, and in 1990, Ru-Stoll extended H. Cartan's theorem to slowly moving hyperlanes.

The theory of zeros distributions of polynomials, especially the fundamental theorem of algebra, was extended by Nevanlinna to meromorphic functions, now known as Nevanlinna's theory. Analogous to the fundamental theorem of algebra,

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the First Main Theorem of Nevanlinna asserts that, $N_f(r,a)+O(1)_a \leq T_f(r)$ for all $a \in \mathbb{P}^1(\mathbb{C})$, and equality holds for almost all a. Here the constant $O(1)_a$ depends on a and $N_f(r,a)$ is the generalization of the (logarithmic average of the) number of zeros of a polynomials to holomorphic maps. The First Main Theorem extends to meromorphic maps into $\mathbb{P}^n(\mathbb{C})$ with respect to hyperplanes a in $\mathbb{P}^n(\mathbb{C})$. The First Main Theorem in this case implies (using the transitivity of the unitary group acting on $\mathbb{P}^n(\mathbb{C})$) the Crofton formula

$$\int_{a\in(\mathbb{P}^n(\mathbb{C}))^*} N_f(T,a)\,\omega^n = T_f(r)$$

where $(\mathbb{P}^n(\mathbb{C}))^*$ is the space of hyperplanes in $\mathbb{P}^n(\mathbb{C})$ and ω is the Fubini-Study metric. The classical theory of Nevanlinna Theory is based on the classical potential theory (such as electrical charges and also the classical theory of mechanics, see H. Weyl and J. Weyl).

In our paper [Y–200811], we studied the the distribution of normalized zero-sets of random entire functions. As noted in that paper, our result can be viewed as the extension of the First Main Theorem to random polynomials (or random entire functions). To state the result, we introduce the following set up: let $f_1(z),\cdots,f_\ell(z)$ be a finite number of fixed entire functions; Let

$$G_n(z) = \sum_{\nu=0}^{n}\sum_{j_1=1}^{\ell}\cdots\sum_{j_\nu=1}^{\ell} a_{j_1,\cdots,j_\nu} f_{j_1}(z)\cdots f_{j_\nu}(z)$$

be a random polynomial, where each coefficient $a_{j_1,\cdots,j_\nu}$ is an indeterminate which satisfies the Gaussian distribution

$$\frac{1}{\pi}\,e^{-|z|^2}$$

on $\mathbb{C}$; Note that, by the usual notational convention, for $\nu = 0$ the coefficient $a_{j_1,\cdots,j_\nu}$ is a single indeterminate with the same Gaussian distribution though we have no values for $j_1,\cdots,j_\nu$; We define the normalized counting divisor $\mathbf{Z}\,(r,G_n)$ of $G_n(z)$ on the punctured disk $0<|z|<r$ by

$$\mathbf{Z}\,(r,G_n) = \frac{1}{n}\sum_{\substack{G_n(z)=0,\\ |z|<r}} \delta_z,$$

where δ_z is the Dirac delta on $\mathbb{C}^\ell$ at the point z of $\mathbb{C}^\ell$. Let $\mathbf{E}(\mathbf{Z}(r,G_n))$ be the expectation of $\mathbf{Z}(r,G_n)$ respect to a probability measure (in the sequel this will be denoted by $d\mu_{\ell,n}$) on the Euclidean space $\mathbb{C}^{N_\ell}$ of random coefficients $\left(a_{j_1,\ldots,j_\nu}\right)_{1\leq j_1,\ldots,j_\nu\leq \ell;\,0\leq\nu\leq n}$; On the complex Euclidean space $\mathbb{C}^k$ we denote by χ_E the characteristic function on a Lebesgue measurable subset $E\subset\mathbb{C}^k$, and by $B^k\subset\mathbb{C}^k$ the unit ball. The main result of our paper [Y–200811] is as follows:

Theorem A. *Let C be the smooth (possibly non-closed) curve in $\mathbb{C}$ consisting of all points z such that $|f(z)| := \left(\sum_{j=1}^{\ell} |f_j(z)|^2\right)^{\frac{1}{2}} = 1$ and such that $f'(z) = (f_1'(z), ..., f_\ell'(z)) \neq 0$. Then the limit of $\mathbf{E}(\mathbf{Z}(r, G_n))$ as a $(1,1)$-current is equal to the sum of*

$$\chi_{f^{-1}(\mathbb{C}^\ell - B^\ell)} \left(\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|f(z)|\right)$$

and the measure on C defined by the 1-form

$$\frac{\sqrt{-1}}{2}\sum_{j=1}^{\ell}\left(f_j(z)d\overline{f_j(z)} - \overline{f_j(z)}df_j(z)\right).$$

This paper, we extend the Theorem A to the context of random entire functions with small perturbation. The Main Theorem is stated in section 3, where the precise meaning of "small perturbations" is explained.

2. Some Propositions

In this section, we recall the following key result known as the *Complex Version of Lemma on the Convergence of Integrals as Distributions*, which is obtained in [Y-200811].

Proposition 1. [Y-200811]. (Complex Version of Lemma on the Convergence of Integrals as Distributions).

$$\lim_{n\to\infty}\frac{1}{n}\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\sum_{k=0}^{n}|z|^{2k}\right)$$
$$= \chi_{\mathbb{C}^\ell - B^\ell}\left(\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log r\right) + \left[\delta_{S^{2\ell-1}}\right] \wedge \frac{\sqrt{-1}}{2}\sum_{j=1}^{\ell}\left(z_j d\bar{z}_j - \bar{z}_j dz_j\right),$$

where $\left[\delta_{S^{2\ell-1}}\right]$ denotes the 1-current on $\mathbb{C}^\ell$ defined by integration over

$$S^{2\ell-1} = \left\{ z \in \mathbb{C}^\ell \,\middle|\, |z| = \left(\sum_{j=1}^{\ell}|z_j|^2\right)^{\frac{1}{2}} = 1 \right\},$$

and $r = |z|$.

From the proof of Proposition 1 we deduce

Proposition 2. [Y-200811]. *Let $\Omega \subset \mathbb{C}$ be a domain and $f(z) = (f_1(z), f_2(z), \cdots, f_\ell(z)) : \Omega \to \mathbb{C}^\ell$ be a nonconstant holomorphic vector-valued function on Ω.*

Let C be the smooth (possibly non-closed) curve in Ω consisting of all points z of Ω such that $|f(z)| = 1$ and $f'(z) = (f_1'(z), f_2'(z), \cdots, f_\ell'(z)) \neq 0$, where $|f(z)| = (|f_1(z)|^2 + |f_2(z)|^2 + \cdots + |f_\ell(z)|^2)^{\frac{1}{2}}$. Then

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{k=0}^{n}|f(z)|^{2k}\right)\right) = \chi_{f^{-1}(\mathbb{C}^\ell - B^\ell)}\left(\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|f(z)|\right)$$
$$+\left[\delta_{S^{2\ell-1}}\right] \wedge \frac{\sqrt{-1}}{2}\sum_{j=1}^{\ell}\left(f_j(z)d\bar{f}_j(z) - \bar{f}_j(z)df_j(z)\right),$$

where $\left[\delta_{S^{2\ell-1}}\right]$ denotes the 1-current on $\mathbb{C}^\ell$ defined by integration over

$$S^{2\ell-1} = \left\{ f(z) \in \mathbb{C}^\ell \,\middle|\, |f(z)| = \left(\sum_{j=1}^{\ell}|f_j(z)|^2\right)^{\frac{1}{2}} = 1 \right\}.$$

3. The Main Theorem and It's Proof

In this section, we formulate and prove our main result of the paper. To start with, given a sequence $\left(g_\mu\right)_{\nu=0}^{\infty}$ of holomorphic functions on a plane domain Ω, we consider random polynomials in f_j and g_ν given by

$$G_n^*(z) = \sum_{\nu=0}^{n}\sum_{1\le j_1,\ldots,j_\nu\le\ell} a_{j_1,\ldots,j_\nu} g_\nu(z) f_{j_1}(z)\cdots f_{j_\nu}(z)$$

where each coefficient $a_{j_1,\ldots,j_\nu}$ is an indeterminates satisfying the Gaussian distribution exactly as in the context of Main Theorem. We consider the expectation of the normalized counting divisor $\mathbf{Z}(G_n^*)$ of $G_n^*(z)$ and look for conditions on the sequence $\left(g_\mu\right)_{\nu=0}^{\infty}$ of holomorphic functions such that the limit of the expectations $\mathbf{E}(\mathbf{Z}(G_n^*))$ exists and is equal to that of the expectation $\mathbf{E}(\mathbf{Z}(G_n))$ as in Main Theorem. In order to avoid artificial zeros of $G_n^*(z)$, $\left(f_j\right)_{j=1}^{\ell}$ and $\left(g_\mu\right)_{\nu=0}^{\infty}$ should have no common zeros, and for convenience this can be guaranteed by imposing the condition that g_0 is nowhere vanishing on Ω. We introduce

Definition 1. (1) Given a sequence of holomorphic functions $\left(h_\mu\right)_{\nu=0}^{\infty}$ on Ω, we say that $\left(h_\mu\right)_{\nu=0}^{\infty}$ is of slow growth if and only if for two sequences of nonnegative integers $\kappa_{z,n}$ and $\lambda_{z,n}$ depending on $z \in \Omega$ and satisfying $\lim\limits_{n\to\infty}\frac{\kappa_{z_0,n}}{n} = 0$ and $\lim\limits_{n\to\infty}\frac{\lambda_{z_0,n}}{n} = 0$ and for a positive function B_z the following holds true. For each $0 \le \nu < \infty$ and each point $z \in \Omega$, we have

$$(\dagger) \qquad \left|h_\nu(z)\right| \le (B_z)^{\kappa_{z,\nu}}\left(1 + \left|f(z)\right|\right)^{\lambda_{z,\nu}},$$

If furthermore B_z and $\kappa_{z,\nu}$, $\lambda_{z,\nu}$ can be chosen to be uniformly bounded on compact subsets of Ω we say that $\left(h_\mu\right)_{\nu=0}^{\infty}$ is uniformly of slow growth on compact subsets.
(2) Given a sequence of meromorphic functions $\left(h_\mu\right)_{\nu=0}^{\infty}$ on Ω we say that $\left(h_\mu\right)_{\nu=0}^{\infty}$ is of slow growth with respect to f if and only if at each $z \in \Omega$ there exists a nonnegative integer $N(z)$ such that (†) holds true whenever $\nu \geq N(z)$.

Remark 1. The slow growth condition (†) on $\left(h_\nu\right)_{\nu=0}^{\infty}$ is formulated in terms of $\left(f_j\right)_{j=1}^{\ell}$ since comparison with f is natural in the construction of examples for the Main Theorem. It should however be noted that the condition (†) is independent of f. In fact, (†) is satisfied at $z \in \Omega$ if and only if $\overline{\lim_{\nu\to\infty}}\left|h_\nu(z)\right|^{\frac{1}{\nu}} = 1$.

Main Theorem. *Let $\Omega \subset \mathbb{C}$ be a domain, ℓ be a positive integer and $f_1(z), \ldots, f_\ell(z)$ be holomorphic functions on Ω. Let $\left(g_\nu(z)\right)$ be a sequence of holomorphic function on Ω such that*

(i) *$g_0(z)$ is nowhere zero on Ω.*
(ii) *$\left(g_\nu(z)\right)_{\nu=0}^{\infty}$ is uniformly of slow growth on compact subsets.*
(iii) *$\left(\frac{1}{g_\nu(z)}\right)_{\nu=0}^{\infty}$ is of slow growth for any $z \in \Omega$.*

For any positive integer n let

$$G_n^*(z) = \sum_{\nu=0}^{n} \sum_{1\leq j_1,\ldots,j_\nu\leq \ell} a_{j_1,\ldots,j_\nu} g_\nu(z) f_{j_1}(z)\cdots f_{j_\nu}(z)$$

be a random polynomial, where each coefficient $a_{j_1,\ldots,j_\nu}$ for $1 \leq j_1, \ldots, j_\nu \leq \ell$ and $0 \leq \nu \leq n$ is an indeterminate which satisfies the Gaussian distribution $\frac{1}{\pi}e^{-|z|^2}$ on $\mathbb{C}$, with the convention that a_0 is the single indeterminate $a_{j_1,\ldots,j_\nu}$ when $\nu = 0$. Let $\mathbf{Z}(G_n^*)$ be the normalized counting divisor of $G_n^*(z)$ on Ω (in the sense of distribution) given by

$$\mathbf{Z}(G_n^*) = \frac{1}{n} \sum_{\substack{G_n^*(z)=0,\\ z\in\Omega}} \delta_z \ ,$$

where δ_z is the Dirac delta on $\mathbb{C}$ at the point z of $\mathbb{C}$. Let $\mathbf{E}(\mathbf{Z}(G_n^*))$ be the expectation of $\mathbf{Z}(G_n^*)$ with respect to the probability measure $d\mu_{\ell,n}$ on the Euclidean space $\mathbb{C}^{N_{\ell,n}}$ of random coefficients $\left(a_{j_1,\ldots,j_\nu}\right)_{1\leq j_1,\ldots,j_\nu\leq\ell;\, 0\leq\nu\leq n}$. Then $\mathbf{E}(\mathbf{Z}(G_n^*))$ agrees with $\mathbf{E}(\mathbf{Z}(G_n))$. In other words, $\mathbf{E}(\mathbf{Z}(G_n^*))$ is equal to the sum of

$$\chi_{f^{-1}(\mathbb{C}^\ell - B^\ell)}\left(\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|f(z)|\right)$$

and the measure on C defined by the 1-form

$$\frac{\sqrt{-1}}{2}\sum_{j=1}^{\ell}\left(f_j(z)\overline{df_j(z)} - \overline{f_j(z)}df_j(z)\right).$$

Note that in the formulation of the Main Theorem, by (iii) we impose at each point $z \in \Omega$ a pointwise condition on lower bounds of $|g_\nu(z)|$, a condition which nonetheless allows for the existence of zeros for the holomorphic functions g_ν. On the other hand, in the computation of mathematical expectations of normalized counting divisors, some uniformity on upper bounds of $|g_\nu(z)|$ on compact subsets is required in order to prove convergence of positive $(1,1)$-currents.

For the proof of Main Theorem we will need to formulate a lemma on limits of certain potential functions. To start with define on the domain $\Omega \subset \mathbb{C}$ the following subharmonic functions

$$\gamma_n = \frac{1}{n}\log(1 + |f|^2 + \cdots + |f|^{2n}).$$

Define $\varphi : \Omega \to \mathbb{R}$ by

$$\begin{cases} \varphi(z) = \log|f(z)|^2 & \text{if } \ |f(z)| \geq 1 \\ \varphi(z) = 0 & \text{if } \ |f(z)| \leq 1 \end{cases}.$$

In other words, $\varphi(z) = \max(0, \log|f|^2) = \log^+|f|^2$. Then, we have

Lemma 1. *$\gamma_n(z)$ converges uniformly to $\varphi(z)$ on Ω. As a consequence, $\sqrt{-1}\partial\overline{\partial}\gamma_n$ converges to $\sqrt{-1}\partial\overline{\partial}\varphi$ as positive $(1,1)$-currents on Ω.*

Proof. For each positive integer n define $\lambda_n : [0,\infty) \to \mathbb{R}$ by

$$\lambda_n(t) = \frac{1}{n}\log(1 + t + \cdots + t^n).$$

For $0 \leq t \leq 1$ we have

$$0 \leq \lambda_n(t) \leq \frac{1}{n}\log(n+1).$$

On the other hand, for $t \geq 1$ we have

$$\log t = \frac{1}{n}\log(t^n) \leq \lambda_n(t) \leq \frac{1}{n}\log\big((n+1)t^n\big) = \frac{1}{n}\log(n+1) + \log t.$$

Let $\lambda : [0,\infty) \to \mathbb{R}$ be the monotonically increasing continuous function defined by

$$\begin{cases} \lambda(t) = \log t & \text{for } \ t \geq 1; \\ \lambda(t) = 0 & \text{for } \ 0 \leq t \leq 1. \end{cases}$$

Then,

$$\lambda(t) \leq \lambda_n(t) \leq \frac{1}{n}\log(n+1) + \lambda(t).$$

Thus, over $[0,\infty)$, $\lambda_n(t)$ converges uniformly to $\lambda(t)$. For the map $f:\Omega\to\mathbb{C}$,

$$\gamma_n = \frac{1}{n}\log(1+|f|^2+\cdots+|f|^{2n}) = \lambda_n(|f|^2),$$

so that γ_n converges uniformly to $\lambda(|f|^2)=\varphi$, and it follows that $\sqrt{-1}\partial\overline{\partial}\gamma_n$ converges to $\sqrt{-1}\partial\overline{\partial}\varphi$ as positive $(1,1)$-currents on Ω, as desired. ∎

We proceed to give a proof of Main Theorem.

Proof of Main Theorem. In the language of probability theory,

$$(a_{j_1,\cdots,j_\nu})_{0\le\nu\le n,1\le j_1\le\ell,\cdots,1\le j_\nu\le\ell}$$

are independent complex Gaussian random variables of mean 0 and variance 1. Let $N_{\ell,n}$ be the number of elements in

$$(a_{j_1,\cdots,j_\nu})_{0\le\nu\le n,1\le j_1\le\ell,\cdots,1\le j_\nu\le\ell},$$

which is

$$N_{\ell,n} = 1+\ell+\ell^2+\cdots+\ell^n.$$

Let a_0 be the single indeterminate $a_{j_1,\cdots,j_\nu}$ when $\nu=0$. By Cauchy's integral formula (or the Poincaré-Lelong formula)

$$(*)\qquad \frac{1}{n}\sum_{\substack{G_n^*(z)=0,\\ z\in\Omega}}\delta_z = \frac{\sqrt{-1}}{n\pi}\partial\bar\partial\log|G_n^*(z)|$$

on Ω, where δ_z is the Dirac delta on $\mathbb{C}^\ell$ at the point z of $\mathbb{C}^\ell$. We now consider the normalized counting divisor $\mathbf{Z}(G_n^*)$ of $G_n^*(z)$ on Ω (in the sense of distribution) which is given by

$$\mathbf{Z}(G_n^*) = \frac{1}{n}\sum_{\substack{G_n^*(z)=0,\\ z\in\Omega}}\delta_z.$$

By $(*)$, the expectation $\mathbf{E}(\mathbf{Z}(G_n^*))$ of $\mathbf{Z}(G_n^*)$ is equal to

$$\int_{(a_{j_1,\cdots,j_\nu})\in\mathbb{C}^{N_{\ell,n}}}\left(\frac{\sqrt{-1}}{n\pi}\partial\bar\partial\log|G_n^*(z)|\right)$$
$$\times\prod_{(a_{j_1,\cdots,j_\nu})\in\mathbb{C}^{N_{\ell,n}}}\left(\frac{1}{\pi}\,e^{-|a_{j_1\cdots j_\nu}|^2}\frac{\sqrt{-1}}{2}da_{j_1\cdots j_\nu}\wedge d\overline{a_{j_1\cdots j_\nu}}\right).$$

We introduce two column vectors

$$\vec{\mathbf{a}} = [a_{j_1,\cdots,j_\nu}]_{0\le\nu\le n,1\le j_1\le\ell,\cdots,1\le j_\nu\le\ell}$$

and

$$\vec{\mathbf{v}}(z) = \left[g_\nu(z) f_{j_1}(z) \cdots f_{j_\nu}(z)\right]_{0\leq\nu\leq n, 1\leq j_1\leq\ell,\cdots,1\leq j_\nu\leq\ell}$$

of $N_{\ell,n}$ components each. Here we set $f_0(z) = 1$. Then $G_n(z)$ is equal to the inner product

$$\langle\vec{\mathbf{a}},\, \vec{\mathbf{v}}(z)\rangle = \sum_{0\leq\nu\leq n, 1\leq j_1\leq\ell,\cdots,1\leq j_\nu\leq\ell} a_{j_1,\cdots,j_\nu} g_\nu(z) f_{j_1}(z)\cdots f_{j_\nu}(z)$$

of the two $N_{\ell,n}$-vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{v}}(z)$. The length of the $N_{\ell,n}$-vector $\vec{\mathbf{v}}(z)$ is given by

$$\|\vec{\mathbf{v}}(z)\| = \left(\sum_{0\leq\nu\leq n, 1\leq j_1\leq\ell,\cdots,1\leq j_\nu\leq\ell} |g_\nu(z)|^2 |f_{j_1}(z)|^2\cdots|f_{j_\nu}(z)|^2\right)^{\frac{1}{2}}.$$

Introduce the unit $N_{\ell,n}$-vector

$$\vec{\mathbf{u}}(z) = \frac{1}{\|\vec{\mathbf{v}}(z)\|}\,\mathbf{v}(z) = \frac{1}{\left(\sum_{0\leq\nu\leq n, 1\leq j_1,\cdots,j_\nu\leq\ell} |g_\nu(z)|^2 |f_{j_1}(z)|^2\cdots|f_{j_\nu}(z)|^2\right)^{\frac{1}{2}}}$$
$$\left[g_\nu(z) f_{j_1}(z)\cdots f_{j_\nu}(z)\right]_{0\leq\nu\leq n, 1\leq j_1,\cdots,j_\nu\leq\ell}$$

in the same direction as $\vec{\mathbf{v}}(z)$. Then

$$\log|G_n^*(z)| = \log|\langle\vec{\mathbf{a}},\, \vec{\mathbf{v}}(z)\rangle| = \log|\langle\vec{\mathbf{a}},\, \|\vec{\mathbf{v}}(z)\|\,\vec{\mathbf{u}}(z)\rangle| = \log\|\vec{\mathbf{v}}(z)\| + \log|\langle\vec{\mathbf{a}},\, \vec{\mathbf{u}}(z)\rangle|.$$

Now $\mathbf{E}\left(\mathbf{Z}\left(G_n^*\right)\right)$ is equal to

$$\int_{\left(a_{j_1,\cdots,j_\nu}\right)\in\mathbb{C}^{N_{\ell,n}}} \left(\frac{\sqrt{-1}}{n\pi}\partial\bar\partial\left(\log\|\vec{\mathbf{v}}(z)\| + \log|\langle\vec{\mathbf{a}},\, \vec{\mathbf{u}}(z)\rangle|\right)\right)$$
$$\times\prod_{\left(a_{j_1,\cdots,j_\nu}\right)\in\mathbb{C}^{N_{\ell,n}}} \left(\frac{1}{\pi}\,e^{-|a_{j_1\cdots j_\nu}|^2}\frac{\sqrt{-1}}{2} da_{j_1\cdots j_\nu}\wedge d\overline{a_{j_1\cdots j_\nu}}\right).$$

Let $\vec{\mathbf{e}}_0$ be the $N_{\ell,n}$-vector

$$\left(e_{j_1,\cdots,j_\nu}\right)_{0\leq\nu\leq n, 1\leq j_1\leq\ell,\cdots,1\leq j_\nu\leq\ell}$$

whose only nonzero component is $e_0 = 1$. By exactly the same argument as in the proof of the Main Theorem the limit of $\mathbf{E}\left(\mathbf{Z}\left(G_n^*\right)\right)$ as $n\to\infty$ is equal to

$$\lim_{n\to\infty}\int_{\left(a_{j_1,\cdots,j_\nu}\right)\in\mathbb{C}^{N_{\ell,n}}} \left(\frac{\sqrt{-1}}{n\pi}\partial\bar\partial\log\|\vec{\mathbf{v}}(z)\|\right)\frac{1}{\pi^{N_{\ell,n}}}\,e^{-\|\vec{\mathbf{a}}\|^2},$$

which after integration over

$$\left(a_{j_1,\cdots,j_\nu}\right)_{0\leq\nu\leq n, 1\leq j_1\leq\ell,\cdots,1\leq j_\nu\leq\ell}$$

is simply equal to

$$\lim_{n\to\infty}\frac{\sqrt{-1}}{n\pi}\partial\bar{\partial}\log\|\vec{\mathbf{v}}(z)\| = \lim_{n\to\infty}\frac{1}{n}\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\sum_{k=0}^{n}|g_k(z)|^2\left(\sum_{j=1}^{\ell}|f_j(z)|^2\right)^k\right)$$

From Proposition 1, we have

$$\lim_{n\to\infty}\frac{1}{n}\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\sum_{k=0}^{n}\left(\sum_{j=1}^{\ell}|f_j(z)|^2\right)^k\right)$$

is equal to the pullback by f of

$$\chi_{\mathbb{C}^\ell - B^\ell}\left(\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|w|\right) + \left[\delta_{S^{2\ell-1}}\right]\wedge\frac{\sqrt{-1}}{2}\sum_{j=1}^{\ell}(w_j d\bar{w}_j - \bar{w}_j dw_j),$$

where $w \in \mathbb{C}^\ell = (w_1, \cdots, w_\ell)$ is variable in the target space of the map $f = (f_1, \cdots, f_\ell) : \Omega \to \mathbb{C}^\ell$. By computation

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{k=0}^{n}|g_k(z)|^2\left(\sum_{j=1}^{\ell}|f_j(z)|^2\right)^k\right)\\
&-\lim_{n\to\infty}\frac{1}{n}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{k=0}^{n}\left(\sum_{j=1}^{\ell}|f_j(z)|^2\right)^k\right)\\
=&\lim_{n\to\infty}\frac{1}{n}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(\log\sum_{k=0}^{n}|g_k(z)|^2\left(\sum_{j=1}^{\ell}|f_j(z)|^2\right)^k - \log\sum_{k=0}^{n}\left(\sum_{j=1}^{\ell}|f_j(z)|^2\right)^k\right)\\
=&\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\lim_{n\to\infty}\frac{1}{n}\left(\log\sum_{k=0}^{n}|g_k(z)|^2\left(\sum_{j=1}^{\ell}|f_j(z)|^2\right)^k - \log\sum_{k=0}^{n}\left(\sum_{j=1}^{\ell}|f_j(z)|^2\right)^k\right)\\
=&\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{\sum_{k=0}^{n}|g_k(z)|^2\left(\sum_{j=1}^{\ell}|f_j(z)|^2\right)^k}{\sum_{k=0}^{n}\left(\sum_{j=1}^{\ell}|f_j(z)|^2\right)^k}\right).
\end{aligned}$$

The hypotheses (i)-(iii) on the sequence $(g_\nu)_{\nu=0}^{\infty}$ of holomorphic functions on Ω as stated in Theorem 1 can be formulated more explicitly as follows. For each $z \in \Omega$, there are four sequences of non-negative integers, $0 \le n < \infty$.

$$\kappa_{z,n}, \lambda_{z,n}, \xi_{z,n}, \eta_{z,n} \ ,$$

satisfying

$$\lim_{n\to\infty} \frac{\kappa_{z,n}}{n} = 0 \ , \lim_{n\to\infty} \frac{\lambda_{z,n}}{n} = 0 \ , \lim_{n\to\infty} \frac{\xi_{z,n}}{n} = 0 \ , \lim_{n\to\infty} \frac{\eta_{z,n}}{n} = 0 \ .$$

and positive numbers A_z and B_z such that

(i) $g_0(z)$ is nowhere zero on Ω.

(ii) For each $0 \le \nu < \infty$ and each point $z \in \Omega$

$$\left|g_\nu(z)\right| \le (B_z)^{\kappa_{z,\nu}} \left(1 + \left|f(z)\right|\right)^{\lambda_{z,\nu}} \ .$$

Furthermore for each compact subset $K \subset \Omega$ there exists $C_K > 0$ such that for each $z \in K$ and for each positive integer ν we have

$$B_z \le C_K; \quad \frac{\kappa_{z,\nu}}{\nu}, \frac{\lambda_{z,\nu}}{\nu} \le C_K.$$

(iii) For each point $z \in \Omega$

$$\liminf_{\nu\to\infty} (A_z)^{\xi_{z,\nu}} \left(1 + \left|f(z)\right|\right)^{\eta_{z,\nu}} \left|g_\nu(z)\right| > 0 \ .$$

Without loss of generality, we may assume that for any $z \in \Omega$, A_z, $B_z \ge 1$. Granted this, replacing $\kappa_{z,n}$ by $\max\left\{\kappa_{z,0}, \ldots, \kappa_{z,n}\right\}$, etc., without loss of generality we may assume that the four sequences $\kappa_{z,n}$, $\lambda_{z,n}$, $\xi_{z,n}$ and $\eta_{z,n}$ are non-decreasing sequences. By (iii) for every $z \in \Omega$ there exists a positive constant c_z and a positive integer $N(z)$ such that whenever $\nu \ge N(z)$ we have

$$A_z^{\xi_{z,\nu}} \left(1 + |f(z)|\right)^{\eta_{z,\nu}} |g_\nu(z)| \ge c_z.$$

(Here and in what follows to streamline the notations we will write $A_z^{\xi_{z,\nu}}$ to mean $(A_z)^{\xi_{z,\nu}}$, etc.) For every $z \in \Omega$ we have

$$\begin{aligned} &\sum_{k=0}^{n} |g_k(z)|^2 |f(z)|^{2k} \ge \max\left(|g_0(z)|^2, |g_n(z)|^2 |f(z)|^{2n}\right) \\ \ge \quad &\max\left(|g_0(z)|^2, c_z^2 A_z^{-2\xi_{z,n}} \left(1 + |f(z)|\right)^{-2\eta_{z,n}} |f(z)|^{2n}\right) \ . \end{aligned}$$

On the other hand, when $|f(z)| \le 1$ we have

$$(\dagger) \qquad \sum_{k=0}^{n} |g_k(z)|^2 |f(z)|^{2k} \le (n+1) B_z^{2\kappa_{z,n}} \cdot 4^{\lambda_{z,n}} \ ;$$

and, when $|f(z)| \ge 1$ we have

$$\sum_{k=0}^{n} |g_k(z)|^2 |f(z)|^{2k} \le (n+1) B_z^{2\kappa_{z,n}} \left(1 + |f(z)|\right)^{2\lambda_{z,n}} |f(z)|^{2n} \ ,$$

so that

$$\sum_{k=0}^{n} |g_k(z)|^2 |f(z)|^{2k} \leq \max \Big((n+1) B_z^{2\kappa_{z,n}} \cdot 4^{\lambda_{z,n}} \ , \\ (n+1) B_z^{2\kappa_{z,n}} \left(1 + |f(z)|\right)^{2\lambda_{z,n}} |f(z)|^{2n} \Big) \ .$$

Similarly for the function $\left(\sum_{k=0}^{n} |f(z)|^{2k} \right)^{\frac{1}{n}}$ we have

$$\max\left(1, |f(z)|^2\right) \leq \left(\sum_{k=0}^{n} |f(z)|^{2k} \right)^{\frac{1}{n}} \leq (n+1)^{\frac{1}{n}} \max\left(1, |f(z)|^2\right) \ .$$

Finally, recalling that

$$\left(h_n(z)\right)^{\frac{1}{n}} = \left(\frac{\sum_{k=0}^{n} |g_k(z)|^2 |f(z)|^{2k}}{\sum_{k=0}^{n} |f(z)|^{2k}} \right)^{\frac{1}{n}} \ ,$$

we have, for $z \in \Omega$,

$$\frac{\max\left(|g_0(z)|^2, c_z^2 A_z^{-2\xi_{z,n}} \left(1+|f(z)|\right)^{-2\eta_{z,n}}\right)^{\frac{1}{n}}}{(n+1)^{\frac{1}{n}} \max\left(1, |f(z)|^2\right)} \leq \left(h_n(z)\right)^{\frac{1}{n}} \\ \leq \frac{\max\left((n+1) B_z^{2\kappa_{z,n}} \cdot 4^{\lambda_{z,n}}, (n+1) B_z^{2\kappa_{z,n}} \left(1+|f(z)|\right)^{2\lambda_{z,n}} |f(z)|^{2n}\right)^{\frac{1}{n}}}{\max\left(1, |f(z)|^2\right)} \ .$$

For the lower bound of $\left(h(z)\right)^{\frac{1}{n}}$ we note that

$$\lim_{n\to\infty} |g_0(z)|^{\frac{2}{n}} = 1 \ ; \\ \lim_{n\to\infty} \left(c_z^2 A_z^{-2\xi_{z,n}} \left(1+|f(z)|\right)^{-2\eta_{z,n}}\right)^{\frac{1}{n}} \left|f(z)\right|^{2n} \\ = \lim_{n\to\infty} c_z^{\frac{2}{n}} A_z^{-\frac{2\xi_{z,n}}{n}} \left(1+|f(z)|\right)^{-\frac{2\eta_{z,n}}{n}} \left|f(z)\right|^2 = \left|f(z)\right|^2$$

where we have used the assumptions $\lim_{n\to\infty} \frac{\xi_{z,n}}{n} = \lim_{n\to\infty} \frac{\eta_{z,n}}{n} = 0$. For the upper bound of $\left(h_n(z)\right)^{\frac{1}{n}}$ we note that

$$\lim_{n\to\infty} \left((n+1) B_z^{2\kappa_{z,n}} \cdot 4^{\lambda_{z,n}}\right)^{\frac{1}{n}} = \lim_{n\to\infty} (n+1)^{\frac{1}{n}} B_z^{\frac{2\kappa_{z,n}}{n}} 4^{\frac{\lambda_{z,n}}{n}} = 1 \ ; \\ \lim_{n\to\infty} \left((n+1) B_z^{2\kappa_{z,n}} \left(1+|f(z)|\right)^{2\lambda_{z,n}} \left|f(z)\right|^{2n}\right)^{\frac{1}{n}} \\ = \lim_{n\to\infty} (n+1)^{\frac{1}{n}} B_z^{\frac{2\kappa_{z,n}}{n}} \left(1+|f(z)|\right)^{\frac{2\lambda_{z,n}}{n}} \left|f(z)\right|^2 = \left|f(z)\right|^2$$

where we have used the assumptions $\lim_{n\to\infty} \frac{\kappa_{z,n}}{n} = \lim_{n\to\infty} \frac{\lambda_{z,n}}{n} = 0$. Thus, for any $z \in \Omega$ we have

$$1 = \frac{\max\left(1, |f(z)|^2\right)}{\max\left(1, |f(z)|^2\right)} \le \varliminf_{n\to\infty} h_n(z)^{\frac{1}{n}} \le \varlimsup_{n\to\infty} h_n(z) \le \frac{\max\left(1, |f(z)|^2\right)}{\max\left(1, |f(z)|^2\right)} = 1 \ .$$

so that

$$\lim_{n\to\infty} h_n(z)^{\frac{1}{n}} = 1 \ ; \quad \lim_{n\to\infty} \log\left(h_n(z)^{\frac{1}{n}}\right) = 0 \ .$$

Under the assumptions of Main Theorem write

$$\varphi_n = \log\Big(\sum_{k=0}^{n} \big|g_k(z)\big|^2 \big|f(z)\big|^{2k}\Big)^{\frac{1}{n}} \ .$$

Then, $\log h_n{}^{\frac{1}{n}} = \varphi_n - \gamma_n$. Since γ_n converges to $\varphi = \log^+ |f|^2$ by Lemma 6 and $\log h_n{}^{\frac{1}{n}}$ converges pointwise to 0, we conclude that $\varphi_n(z)$ converges to $\varphi(z)$ for every $z \in \Omega$. Clearly φ_n and φ are continuous subharmonic functions on Ω. Moreover from $(\dagger)$ we have for every $z \in \Omega$

$$\varphi_n(z) \le \frac{1}{n}\log(n+1) + \frac{2\kappa_{z,n}}{n}\log B(z) + \frac{\lambda_{z,n}}{n}\log 4,$$

and by assumption on any compact subset $K \subset \Omega$, B_z and the sequence of functions $\dfrac{\kappa_{z,n}}{n}$ and $\dfrac{\lambda_{z,n}}{n}$ are uniformly bounded from above by some constant C_K for $z \in K$, and we conclude that the sequence of subharmonic functions $\big(\varphi_n(z)\big)_{n=0}^{\infty}$ are uniformly bounded from above on compact subsets. Finally, we make use of Lemma 2 below on the convergence of positive $(1,1)$ currents. Granting Lemma 2, the Main Theorem follows readily. ■

The discussion below involves distributions on a domain in $\mathbb{C}$. Denote by $d\lambda$ the Lebesgue measure on $\mathbb{C}$. Any locally integrable function s on Ω defines a distribution T_s on Ω given by $T_s(\rho) = \displaystyle\int_\Omega s\rho \, d\lambda$ for any smooth function ρ on Ω of compact support, and in what follows we will identify s with the distribution T_s it defines. There is a standard procedure for smoothing distributions, as follows. Let χ be a nonnegative smooth function on $\mathbb{C}$ of support lying on the unit disk Δ such that $\chi(e^{i\theta}z) = \chi(z)$ for any $z \in \mathbb{C}$ and any $\theta \in \mathbb{R}$, and for any $\epsilon > 0$ write $\chi_\epsilon(z) = \chi\Big(\frac{z}{\epsilon}\Big)$. For a distribution Q defined on some domain in $\mathbb{C}$ and for $\epsilon > 0$ we write $Q_\epsilon := Q * \chi_\epsilon$ wherever the convolution is defined. We have the following elementary lemma on positive currents associated to subharmonic functions.

Lemma 2. *Let $\Omega \subset \mathbb{C}$ be a plane domain. Suppose $(\varphi_n)_{n=0}^{\infty}$ is a sequence of subharmonic functions on Ω such that $\varphi_n(z)$ are uniformly bounded from above on* *each compact subset K of Ω. Assume that φ_n converges pointwise to some continuous (subharmonic) function φ. Then, $\lim_{n\to\infty} \varphi_n = \varphi$ in $L^1_{\mathrm{loc}}(\Omega)$. As a consequence, $\sqrt{-1}\,\partial\bar\partial\varphi_n$ converges to $\sqrt{-1}\,\partial\bar\partial\varphi$ in the sense of currents.*

Proof of Lemma 2. Let $D = \Delta(a;r)$ be any disk centered at $a \in \Omega$ of radius $r > 0$ such that $\overline{D} \subset \Omega$. We claim that the Lebesgue integrals $\int_{\Delta(a;r)} |\varphi_n|\, d\lambda$ are bounded independent of n. Without loss of generality, we may assume that $\varphi \le 0$ on $\overline{D}$. By the sub-mean-value inequality for subharmonic functions we have

$$\varphi_n(a) \le \frac{1}{\pi r^2}\int_{\Delta(a;r)} \varphi_n(\zeta)\; d\xi\; d\eta$$

where $\zeta = \xi + \sqrt{-1}\eta$ is the Euclidean coordinate of the variable of integration ζ, showing that the integral of $-\varphi_n$ over $\Delta(a;r)$ are bounded independent of n. Covering Ω by a countable and locally finite family of relatively compact open disks D, it follows that on any compact subset $K \subset \Omega$ the L^1-norms of $\varphi_n|_K$ are bounded independent of n. As a consequence, given any subsequence $\varphi_{\sigma(n)}$ of φ_n, some subsequence $\psi_n := \varphi_{\sigma(\tau(n))}$ of $\varphi_{\sigma(n)}$ must converge to a distribution S on Ω. We claim that any such a limit must be given by the (continuous) subharmonic function φ. As a consequence, φ_n converges to φ in $L^1_{\mathrm{loc}}(\Omega)$. Since ψ_n converges to the distribution S, for any $\epsilon > 0$, $\varphi_{n,\epsilon}$ converges to the smooth function S_ϵ as n tends to ∞. Since ψ_n is subharmonic, $\psi_{n,\epsilon}$ is monotonically decreasing as $\epsilon \mapsto 0$ for each nonnegative integer n, and it follows readily that S_ϵ is also monotonically decreasing as $\epsilon \mapsto 0$. Hence, S is the limit as a distribution of the smooth functions S_ϵ. Writing $\psi(z) := \lim_{\epsilon\mapsto 0} S_\epsilon(z)$, by the Monotone Convergence Theorem the distribution S is nothing other than the function ψ, which is in particular locally integrable. Since ψ_n converges to S as distributions, we conclude that $\varphi_{\sigma(n)} = \psi_n$ converges to ψ in $L^1_{\mathrm{loc}}(\Omega)$, implying that ψ_n converges pointwise to ψ almost everywhere on D. However, by assumption $\psi_n = \varphi_{\sigma(n)}$ converges pointwise to φ, hence φ and ψ must agree almost everywhere on Ω. In particular, φ_n must converge to φ in $L^1_{\mathrm{loc}}(\Omega)$. The proof of Lemma 2 is complete. ∎

As a consequence of the proof of Main Theorem we deduce

Corollary 1. *Let $\left(b_\nu\right)_{\nu=0}^{\infty}$ be a sequence of complex numbers such that $b_0 \ne 1$ and $\lim_{\nu\to\infty} b_\nu = 0$. For $\nu \ge 0$ and $1 \le j_1, \ldots, j_\nu \le \ell$ denote by $a_{j_1,\ldots,j_\nu}$ the same random coefficient as in the Main Theorem (with the same convention when $\nu = 0$). Define*

$$H_n(z) = \sum_{\nu=1}^{n} \sum_{1\le j_1,\ldots,j_\nu\le \ell} a_{j_1,\ldots,j_\nu} b_\nu f_{j_1}(z)\cdots f_{j_\nu}(z)\,.$$

Let $\mathbf{Z}\,(r, G_n - H_n)$ be the normalized counting divisor of $G_n(z) - H_n(z)$ on the disk $|z| < r$, $z \in \mathbb{C}$ (in the sense of distribution) given by

$$\mathbf{Z}\left(r, G_n - H_n\right) = \frac{1}{n} \sum_{\substack{G_n(z)=H_n(z), \\ |z|<r}} \delta_z,$$

where δ_z is the Dirac delta on $\mathbb{C}$ at the point z of $\mathbb{C}$. Let $\mathbf{E}\left(\mathbf{Z}\left(r, G_n - H_n\right)\right)$ be the expectation of $\mathbf{Z}\left(r, G_n - H_n\right)$ with respect to the probability measure $d\mu_{\ell,n}$ on the Euclidean space $\mathbb{C}^{N_{\ell,n}}$ of random coefficients $\left(a_{j_1,\dots,j_\nu}\right)_{1\le j_1,\dots,j_\nu\le\ell;\,0\le\nu\le n}$. Let C be the smooth (possibly non-closed) curve in $\mathbb{C}$ consisting of all points z such that $|f(z)| = \left(\sum_{j=1}^{\ell} |f_j(z)|^2\right)^{\frac{1}{2}} = 1$ and such that $f'(z) = (f_1'(z), \dots, f_\ell'(z)) \neq 0$. Then $\mathbf{E}\left(\mathbf{Z}\left(r, G_n - H_n\right)\right)$ agrees with $\mathbf{E}\left(\mathbf{Z}\left(r, G_n\right)\right)$. In other words, $\mathbf{E}\left(\mathbf{Z}\left(r, G_n - H_n\right)\right)$ is equal to the sum of

$$\chi_{f^{-1}(\mathbb{C}^\ell - B^\ell)}\left(\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|f(z)|\right)$$

and the measure on C defined by the 1-form

$$\frac{\sqrt{-1}}{2}\sum_{j=1}^{\ell}\left(f_j(z)d\overline{f_j(z)} - \overline{f_j(z)}df_j(z)\right),$$

In the very special case where ($b_0 \neq 1$ and) $b_\nu = 0$ for any $\nu > 0$, Corollary 1 gives the expectation of $\mathbf{Z}(G_n - H_n)$, where H_n is a randomized constant function, showing that $\mathbf{E}\big(\mathbf{Z}(G_n - H_n)\big) = \mathbf{E}\big(\mathbf{Z}(G_n)\big)$. As such this very special case resembles a form of First Main Theorem in classical Nevanlinna Theory.

Proof of Corollary 1. In the notations of the statement of the Main Theorem and Corollary 1 we have $G_n - H_n = G_n^*$, where G_n^* is defined as in the statement of the Main Theorem for the sequence of holomorphic functions $(g_\nu(z))_{\nu=0}^{\infty}$ where $g_\nu(z)$ is the constant function $1 - b_\nu$ for $0 \le \nu \le \infty$. It suffices to check that the hypothesis of the Main Theorem is satisfied for $(g_\nu(z))_{\nu=0}^{\infty}$. By the hypothesis of Corollary 1 we have $b_0 \neq 1$, so that $g_0(z) = 1 - b_0$ is nowhere zero, hence condition (i) in the statement of the Main Theorem is satisfied. On the other hand, by assumption b_ν tends to 0 as ν tends to ∞, so that $g_\nu(z) = 1 - b_\nu$ tends to 1. Obviously the pointwise upper bounds and lower bounds imposed on $(g_\nu(z))_{\nu=0}^{\infty}$ in conditions (ii) and (iii) are satisfied. Moreover, all the integers $\kappa_{z,\nu}$, $\lambda_{z,\nu}$, $\xi_{z,\nu}$ and $\eta_{z,\nu}$ can be taken to be 0 for $z \in \Omega$ and for any nonnegative integer ν, and the function B_z in condition (ii) can be taken to be a constant. In particular, the hypothesis that B_z, $\kappa_{z,\nu}$ and $\lambda_{z,\nu}$ be uniformly bounded on compact subsets is trivially satisfied. Corollary 1 follows. ■

Finally we conclude the article with a question.

Open Question. Can our Main Theorem be generalized to the case of meromorphic functions?

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