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HEAT KERNELS FOR DIFFERENTIAL OPERATORS WITH RADICAL FUNCTION COEFFICIENTS

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Dedicated to Professor Stere Ianus on the occasion of his seventieth birthday

Abstract. The first part of the paper deals with finding the heat kernel by probabilistic methods for the 1-dimensional elliptic differential operator $\frac{1}{2}(1+x^2)\frac{d^2}{dx^2}+\big(\sqrt{1+x^2}+\frac{x}{2}\big)\frac{d}{dx}$. In the second part we apply the same method to the 2-dimensional operator $\frac{1}{2}\Big(\frac{\partial^2}{\partial x_1^2}+2\sqrt{1+x_2^2}\,\frac{\partial^2}{\partial x_1\partial x_2}+(1+x_2^2)\frac{\partial^2}{\partial x_2^2}\Big)+\frac{1}{2}x_2\partial_{x_2}$ and provide explicit formulas for its heat kernel.

1. Introduction

The use of probabilistic methods to find heat kernels for second order differential operators was initiated by Kolmogorov [6] in early 1930s, who investigated the heat kernel for the following degenerated operator on $\mathbb{R}^2$

$$\partial_x^2 - x\partial_y.$$

In late 1970s Hulanicki [5] and Gaveau [4] used probabilistic methods and Brownian motion to determine the heat kernel for the Heisenberg-Laplacian, i.e. the subelliptic operator on $\mathbb{R}^3$ given by

$$\frac{1}{2}(\partial_x - 2y\partial_z)^2 + \frac{1}{2}(\partial_y - 2z\partial_z)^2.$$

The first part of the present paper deals with the heat kernel $\mathcal{K}(x_0, x; t)$ of the 1-dimensional elliptic operator

$$L = \frac{1}{2}(1+x^2)\frac{d^2}{dx^2} + \big(\sqrt{1+x^2} + \frac{x}{2}\big)\frac{d}{dx}, \tag{1.1}$$

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i.e., finding the solution of

$$\begin{aligned} \partial_t \mathcal{K} &= L\mathcal{K}, \quad t>0, x\in\mathbb{R}, \\ \lim_{t\searrow 0}\mathcal{K}(x,x_0;t) &= \delta_{x_0}(x), \end{aligned}$$

where δ_{x_0} stands for the Dirac distribution centered at x_0.

There are quite a few methods for computing heat kernels that can be found in the literature, sometimes several of them being applicable to the same operator. However, for the operator (1.1) most of the methods do not yield immediate results. The radical coefficients make the method of path integrals difficult to apply. Other methods like Van Vleck's formula of Feynman-Kac's formula are also not applicable for this operator. The Fourier transform method cannot be successfully applied because of the nonconstant coefficients. The "geometric method" discussed for instance in [3], chap.10, is not feasible here since the associated bicharacteristics system is nonlinear and hard to solve here and the associated action is difficult to produce. The interesting feature of the operator (1.1) is that it represents an example where the probabilistic method is the most appropriate.

The probabilistic method associates a stochastic differential equation (SDE) with the aforementioned operator. The next step is to show that the SDE has a unique solution, given an initial condition, and to solve the SDE explicitly. Then one needs to find the probability density function of the solution stochastic process. This will provide us with the deserved heat kernel. It is worth to note that in this case all the above steps can be done with no much difficulty. The main results of this paper are given below.

Theorem 1.1. *The heat kernel of the operator* (*1.1*) *is given by*

$$K(x_0,x;t)=\frac{1}{\sqrt{2\pi t}}\,\frac{x+\sqrt{1+x^2}}{x_0+\sqrt{1+x_0^2}}\,e^{-\frac{1}{2t}\ln^2\frac{x+\sqrt{1+x^2}}{x_0+\sqrt{1+x_0^2}}-\frac{t}{2}}, \qquad t>0. \tag{1.2}$$

Theorem 1.2. *The heat kernel of the operator*

$$A=\frac{1}{2}\Big(\frac{\partial^2}{\partial x_1^2}+2\sqrt{1+x_2^2}\,\frac{\partial^2}{\partial x_1\partial x_2}+(1+x_2^2)\frac{\partial^2}{\partial x_2^2}\Big)+\frac{1}{2}x_2\partial_{x_2}$$

is equal to

$$\begin{aligned} &K(x_0,x;t) \\ &= \frac{1}{2\pi t}\frac{1}{\sqrt{1+x_2^2}}\left[\frac{x_2+\sqrt{1+x_2^2}}{x_2^0+\sqrt{1+(x_2^0)^2}}\right]^{\frac{x_1-x_1^0}{t}} e^{-\frac{1}{2t}(x_1-x_1^0)^2-\frac{1}{t}\left(\sinh^{-1}(x_2)-\sinh^{-1}(x_2^0)\right)^2}, \end{aligned}$$

with $t>0$, *where* $x=(x_1,x_2)$, $x_0=(x_1^0,x_2^0)$.

2. The Proof of Theorem 1.1

Consider the following time-homogeneous Ito diffusion equation

$$(2.1) \qquad dX_t = \Big(\sqrt{1+X_t^2} + \frac{1}{2}X_t\Big)\,dt + \sqrt{1+X_t^2}\,dW_t, \qquad X_0 = x_0,$$

where W_t denotes the Brownian motion starting at $W_0 = 0$. In order to show the existence and uniqueness of a stochastic process $(X_t)_{t\geq 0}$ satisfying the equation with $X_0 = x_0$, we shall use the following result (see Theorem 5.2.1 and Definition 7.1.1. of [8]):

Theorem 2.1. *If the time-homogeneous Ito diffusion*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x_0$$

satisfies the Lipschitz condition

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x-y|, \qquad \forall x, y \in \mathbb{R},$$

then there is a unique solution $(X_t)_{t\geq 0}$ with $X_0 = x_0$.

In our case the drift and diffusion coefficients are given by $b(x) = \sqrt{1+x^2} + \frac{1}{2}x$, and $\sigma(x) = \sqrt{1+x^2}$. Since

$$\begin{aligned}\frac{|x+y|}{\sqrt{1+x^2}+\sqrt{1+y^2}} &\leq \frac{|x|}{\sqrt{1+x^2}+\sqrt{1+y^2}} + \frac{|y|}{\sqrt{1+x^2}+\sqrt{1+y^2}} \\ &< \frac{|x|}{\sqrt{1+x^2}} + \frac{|y|}{\sqrt{1+x^2}} < 2,\end{aligned}$$

then the next estimation holds

$$\begin{aligned}|\sigma(x) - \sigma(y)| &= |\sqrt{1+x^2} - \sqrt{1+y^2}| = \frac{|x^2-y^2|}{\sqrt{1+x^2}+\sqrt{1+y^2}} \\ &= \frac{|x+y|}{\sqrt{1+x^2}+\sqrt{1+y^2}} \cdot |x-y| < 2|x-y|,\end{aligned}$$

and then

$$\begin{aligned}|b(x) - b(y)| &\leq |\sqrt{1+x^2} - \sqrt{1+y^2}| + \frac{1}{2}|x-y| \\ &< 2|x-y| + \frac{1}{2}|x-y| = \frac{5}{2}|x-y|.\end{aligned}$$

Using the previous estimations we obtain the Lipschitz condition with $D = 9/2$

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq \frac{9}{2}|x-y|, \qquad \forall x, y \in \mathbb{R}.$$

By Theorem 2.1, there is a unique solution X_t for the SDE (2.1). In order to find the solution X_t, invoking the uniqueness, it suffices to guess a solution. We shall construct in the following the solution explicitly.

Consider the process $U_t = c + t + W_t$, with W_t Brownian motion and constant $c = \sinh^{-1}(x_0)$ and construct the process $X_t = \sinh(U_t)$. It is easy to verify that $X_0 = x_0$. Applying Ito's formula we obtain

$$\begin{aligned} dX_t &= \cosh(U_t)\,dU_t + \frac{1}{2}\sinh U_t\,(dU_t)^2 \\ &= \cosh(U_t)\,(dt + dW_t) + \frac{1}{2}\sinh(U_t)\,dt \\ &= \big(\cosh(U_t) + \frac{1}{2}\sinh(U_t)\big)\,dt + \cosh(U_t)\,dW_t \\ &= \Big(\sqrt{1+\sinh^2(U_t)} + \frac{1}{2}\sinh(U_t)\Big)\,dt + \sqrt{1+\sinh^2(U_t)}\,dW_t \\ &= \Big(\sqrt{1+X_t^2} + \frac{1}{2}X_t\Big)\,dt + \sqrt{1+X_t^2}\,dW_t. \end{aligned}$$

Hence

$$X_t = \sinh(c + t + W_t), \qquad c = \sinh^{-1}(x_0) \tag{2.2}$$

is the solution of the equation (2.1).

The generator of the aforementioned process X_t is the second partial differential operator A defined by

$$Af(x) = \lim_{t \searrow 0} \frac{E^x[f(X_t)] - f(x)}{t}, \qquad f \in C_0^2(\mathbb{R}),$$

where E^x is the conditional expectation operator, given $X_0 = x$, see [8], p.121. Using standard properties of generators, we obtain that

$$A = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} = L,$$

with L given by (1.1). In order to find the transition density of X_t, which is the heat kernel of L, one needs to compute first the the conditional distribution function of X_t

$$\begin{aligned} F_{X|X_0}(x\,|\,x_0) &= P(X_t \le x\,|\,X_0 = x_0) = P(\sinh(U_t) \le x\,|\,U_0 = \sinh^{-1} x_0) \\ &= P(c + t + W_t \le \sinh^{-1} x\,|\,c = \sinh^{-1} x_0) \\ &= P(W_t \le \sinh^{-1} x - c - t\,|\,c = \sinh^{-1} x_0) \\ &= P(W_t \le \sinh^{-1} x - \sinh^{-1} x_0 - t) \\ &= \int_{-\infty}^{\sinh^{-1} x - \sinh^{-1} x_0 - t} \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{u^2}{2t}}\,du, \end{aligned}$$

so the transition density of X_t is

(2.3) $$p_t(x_0, x) = \frac{d}{dx} F_{X|X_0}(x|x_0) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(\sinh^{-1} x - \sinh^{-1} x_0 - t)^2}{2t}}, \quad t > 0.$$

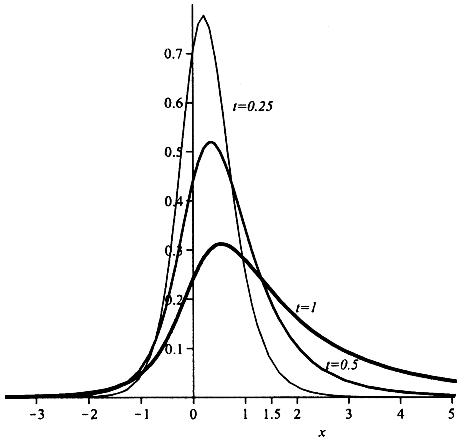

Fig. 1. The transition density as a function of x, with $x_0 = 0$ in the cases: $t = 1$, $t = 0.5$ and $t = 0.25$. For t small the graph tends to the Dirac distribution centered at $x_0 = 0$.

Elementary algebraic computations show that

$$\sinh^{-1} x - \sinh^{-1} x_0 = \ln \frac{x + \sqrt{1+x^2}}{x_0 + \sqrt{1+x_0^2}},$$

$$(\sinh^{-1} x - \sinh^{-1} x_0 - t)^2 = \ln^2 \frac{x + \sqrt{1+x^2}}{x_0 + \sqrt{1+x_0^2}} - 2t \ln \frac{x + \sqrt{1+x^2}}{x_0 + \sqrt{1+x_0^2}} + t^2.$$

Substituting in (2.3) yields the transition density

(2.4) $$p_t(x_0, x) = \frac{1}{\sqrt{2\pi t}} \frac{x + \sqrt{1+x^2}}{x_0 + \sqrt{1+x_0^2}} e^{-\frac{1}{2t} \ln^2 \frac{x+\sqrt{1+x^2}}{x_0+\sqrt{1+x_0^2}} - \frac{t}{2}}, \qquad t > 0.$$

Standard properties of stochastic processes state that (2.4) is the heat kernel of the operator (1.1). The proof of Theorem 2.4 is thus finished. The density function is represented in Fig. 1 for $x_0 = 0$ at the instances $t = 1$, $t = 0.5$ and $t = 0.25$. The distribution is skewed and has fatter tails to the right. This means that more heat propagates from x_0 to the right rather than to the left.

3. The Proof of Theorem 1.2

Let $W_1(t)$ and $W_2(t)$ be two independent Brownian motions starting at 0, and consider the following system of SDE in the matrix form

$$
(3.1)\quad \begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}X_2(t) \end{pmatrix} dt + \begin{pmatrix} 1 & 1 \\ 0 & \sqrt{1+X_2(t)^2} \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix},
$$

with the coefficients

$$
b(x) = \begin{pmatrix} 0 \\ \frac{1}{2}x_2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 1 \\ 0 & \sqrt{1+x_2^2} \end{pmatrix}, \quad \sigma\sigma^T = \begin{pmatrix} 1 & \sqrt{1+x_2^2} \\ \sqrt{1+x_2^2} & 1+x_2^2 \end{pmatrix}.
$$

Since we have the estimations

$$
\begin{aligned}
|b(x)-b(y)| + |\sigma(x)-\sigma(y)| &\le \frac{1}{2}|x_2-y_2| + \left|\sqrt{1+x_2^2} - \sqrt{1+y_2^2}\right| \\
&\le \frac{5}{2}|x_2-y_2| \le \frac{5}{2}|x-y|,
\end{aligned}
$$

where $|\sigma|^2 = \sum |\sigma_{ij}|^2$, the Lipschitz condition holds, and hence there is a unique process X_t which satisfies the SDE system (3.1) starting at $X_0 = x_0 = (x_1^0, x_2^0)$, for any given point $x_0 \in \mathbb{R}^2$.

In order to find the solution, invoking the uniqueness, it suffices to construct a solution. Consider the 2-dimensional stochastic process $X_t = (X_1(t), X_2(t))$ defined by

$$
\begin{aligned}
X_1(t) &= W_1(t) + W_2(t) + x_1^0 \\
X_2(t) &= \sinh\big(W_2(t) + c\big), \quad c = \sinh^{-1}(x_2^0).
\end{aligned}
$$

It is easy to check that the process starts at $X_0 = (x_1^0, x_2^0) = x_0$ and satisfies the following stochastic differential system of equations

$$
\begin{aligned}
dX_1(t) &= dW_1(t) + dW_2(t) \\
dX_2(t) &= \cosh\big(W_2(t)+c\big)\, dW_2(t) + \frac{1}{2}X_2(t)\, dt \\
&= \sqrt{1+X_2(t)^2}\, dW_2(t) + \frac{1}{2}X_2(t)\, dt.
\end{aligned}
$$

Standard results of stochastic processes yield the following generator for X_t

$$
\begin{aligned}
A &= \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_j b_j(x)\frac{\partial}{\partial x_j} \\
&= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{1+x_2^2} \\ \sqrt{1+x_2^2} & 1+x_2^2 \end{pmatrix}\frac{\partial^2}{\partial x_i \partial x_j} + \begin{pmatrix} 0 \\ \frac{1}{2}x_2 \end{pmatrix} \cdot (\partial_{x_1}, \partial_{x_2}) \\
&= \frac{1}{2}\Big(\frac{\partial^2}{\partial x_1^2} + 2\sqrt{1+x_2^2}\,\frac{\partial^2}{\partial x_1 \partial x_2} + (1+x_2^2)\frac{\partial^2}{\partial x_2^2}\Big) + \frac{1}{2}x_2\partial_{x_2}.
\end{aligned}
$$

The heat kernel for the aforementioned operator A is given by the transition density of the process $X_t = (X_1(t), X_2(t))$, which will be obtained in terms of the transition density of the 2-dimensional Brownian motion $\big(W_1(t), W_2(t)\big)$.

The transformation

$$x_1 = u_1 + u_2 + x_1^0$$
$$x_2 = \sinh(u_2 + c), \quad c = \sinh^{-1}(x_2^0)$$

has the inverse

(3.2) $$u_1 = (x_1 - x_1^0) - \big(\sinh^{-1}(x_2) - \sinh^{-1}(x_2^0)\big)$$

(3.3) $$u_2 = \sinh^{-1}(x_2) - \sinh^{-1}(x_2^0),$$

with the nonsingular Jacobian

$$\det \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{1}{\sqrt{1 + x_2^2}} \neq 0.$$

Standard results of transformations of distributions (see for instance [7], p.111) yield the density function of X_t

$$\begin{aligned} p_t(x_0, x) &= f_{X|X_0}(x|x_0) = f_{W_1,W_2}\big(u_1(x_0, x), u_2(x_0, x)\big) \det \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} \\ &= f_{W_1}\big(u_1(x_0, x)\big) f_{W_2}\big(u_2(x_0, x)\big) \det \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} \qquad (3.4) \\ &= \frac{1}{\sqrt{2\pi t}} e^{-\frac{u_1^2}{2t}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{u_2^2}{2t}} \frac{1}{\sqrt{1 + x_2^2}} = \frac{1}{2\pi t} \frac{1}{\sqrt{1 + x_2^2}} e^{-\frac{|u|^2}{2t}}, \end{aligned}$$

with u given by $(3.2) - (3.3)$. A computation provides

$$\begin{aligned} |u|^2 &= u_1^2 + u_2^2 \\ &= (x_1 - x_1^0)^2 - 2(x_1 - x_1^0)\big(\sinh^{-1}(x_2) - \sinh^{-1}(x_2^0)\big) \\ &\quad + 2\big(\sinh^{-1}(x_2) - \sinh^{-1}(x_2^0)\big)^2 \\ &= (x_1 - x_1^0)^2 - 2(x_1 - x_1^0) \ln \frac{x_2 + \sqrt{1 + x_2^2}}{x_2^0 + \sqrt{1 + (x_2^0)^2}} + 2\big(\sinh^{-1}(x_2) - \sinh^{-1}(x_2^0)\big)^2 \end{aligned}$$

and hence

$$e^{-\frac{|u|^2}{2t}} = \left[\frac{x_2 + \sqrt{1 + x_2^2}}{x_2^0 + \sqrt{1 + (x_2^0)^2}}\right]^{\frac{x_1 - x_1^0}{t}} e^{-\frac{1}{2t}(x_1 - x_1^0)^2 - \frac{1}{t}\left(\sinh^{-1}(x_2) - \sinh^{-1}(x_2^0)\right)^2}.$$

Substituting in (3.4) leads to the desired result.

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