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PARA-EISENSTEIN SERIES FOR THE MODULAR GROUP $\mathrm{GL}(2, \mathbb{F}_q[T])$

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Abstract. We introduce para-Eisenstein series as a second analogue of classical elliptic Eisenstein series in the framework of Drinfeld modular forms and show that they share many properties with ordinary Eisenstein series.

1. Introduction

In the well-known analogy between the respective arithmetics of the rational number field $\mathbb{Q}$ and the rational function field $K = \mathbb{F}_q(T)$ over a finite field $\mathbb{F}_q$, the part of classical modular forms is played by Drinfeld modular forms, certain rigid-analytic functions on Drinfeld's upper half-plane. See e.g. [4, 6, 7] for some results as well as for a discussion of similarities and differences of both theories.

On both sides Eisenstein series are crucial in that they generate the rings of modular forms for the modular groups $\Gamma_{\text{class}} := \mathrm{SL}(2, \mathbb{Z})$ or $\Gamma := \mathrm{GL}(2, \mathbb{F}_q[T])$, respectively. The occurrence of Eisenstein series in the classical theory is (at least) twofold: As coordinates of elliptic curves (e.g., the coefficients g_2, g_3 in a Weierstrass equation) and as coefficients in the Weierstrass $\wp$-function ([16] p. 157), where these data depend on a lattice $\Lambda = \mathbb{Z}\omega + \mathbb{Z}$ in $\mathbb{C}$.

All the named objects have their function field counterparts: elliptic curves E correspond to rank-two Drinfeld modules ϕ, which, like $E = \mathbb{C}/\Lambda$, are uniformized by a lattice Λ in the function field version C_∞ of $\mathbb{C}$; the quantities g_2, g_3 correspond to coefficients of ϕ, and the complex $\wp$-function $\wp_\Lambda$ to the rigid analytic function e_Λ of the lattice Λ in C_∞.

However, the two roles of classical Eisenstein series break up on the function field side into two different families of modular forms. While the coefficients of ϕ are still described by Eisenstein series of the classical shape (introduced in the Drinfeld module context by David Goss [10, 12]), the coefficients of e_Λ are of different nature. We baptize them *para-Eisenstein* series, since they still share many features with ordinary (or "ortho-") Eisenstein series as studied by Goss.

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The aim of the present note is to develop some properties of these: elementary identities, congruence properties modulo primes $\mathfrak{p}$ of $\mathbb{F}_q[T]$ and balancedness properties (invariance properties under automorphims of $\mathbb{F}_q[T]$), and to parallel them with similar properties of ortho-Eisenstein series. The principal results are Theorems 3.5 and 4.3.

Notation.

$\mathbb{F}_q$	=	finite field with q elements
A	=	$\mathbb{F}_q[T]$, the polynomial ring over $\mathbb{F}_q$ in an indeterminate T, with quotient field $K = \mathbb{F}_q(T)$
K_∞	=	$\mathbb{F}_q((T^{-1}))$, the completion of K with respect to the ∞-adic valuation
$\lvert\ \rvert$	=	the absolute value on K_∞, normalized by $\lvert T\rvert = q$
C_∞	=	completed algebraic closure of K_∞ with respect to the unique extension of $\lvert\ \rvert$ to a fixed algebraic closure $\overline{K}_\infty$
Ω	=	$\mathbb{P}^1(C_\infty) - \mathbb{P}^1(K_\infty) = C_\infty - K_\infty$ the Drinfeld upper half-plane
$\lvert\ \rvert_i :$		$\Omega \longrightarrow \mathbb{R}$ the "imaginary part" function; $\lvert z\rvert_i = \inf_{x\in K_\infty} \lvert z-x\rvert$
Γ	=	$\mathrm{GL}(2, A)$, the Drinfeld modular group, which acts on Ω through fractional linear transformations
$C_\infty\{\tau\}$		(resp. $C_\infty\{\{\tau\}\}$) the non-commutative polynomial ring (resp. power series ring) over C_∞ with commutation rule $\tau x = x^q\tau$ for constants $x \in C_\infty$. We identify $C_\infty\{\tau\}$ (resp. $C_\infty\{\{\tau\}\}$) with the ring (multiplication defined by insertion) of $\mathbb{F}_q$-linear polynomials (resp. power series) in a variable X through $\sum_i a_i\tau^i = \sum_i a_i X^{q^i}$.
$[k]$	=	$T^{q^k} - T \in A$, the product of the monic irreducibles in A of degree d dividing k, if $k \in \mathbb{N}$, and $[0] = 0$
L_k	=	$\prod_{1\le i\le k} [i]$, $D_k = \prod_{1\le i\le k} [i]^{q^{k-i}}$ for $k \geq 1$, $L_0 = D_0 = 1$

1. Background on Drinfeld Modular Forms (see [3] for more details)

A *lattice* Λ in C_∞ is a finitely generated (hence free) discrete A-submodule of C_∞. With such a Λ, we associate its *exponential function* $e_\Lambda : C_\infty \longrightarrow C_\infty$, which is defined as the everywhere and locally uniformly convergent product

$$e_\Lambda(z) := z \prod_{0\neq\lambda\in\Lambda} (1 - z/\lambda).$$

It has an additive (also everywhere convergent) expansion

$$e_\Lambda(z) = \sum_{k\geq 0} \alpha_k z^{q^k} = \sum \alpha_k \tau^k \tag{1.1}$$

with coefficients $\alpha_k \in C_\infty$, and satisfies a functional equation

$$e_\Lambda(Tz) = \phi_T(e_\Lambda(z)) \tag{1.2}$$

with some $\phi_T = \phi_T^\Lambda \in C_\infty\{\tau\}$ of shape

$$\phi_T(X) = TX + g_1 X^q + \cdots + g_r X^{q^r} = T\tau^0 + \cdots + g_r\tau^r, \tag{1.3}$$

where $g_r \neq 0$ and r is the rank of Λ as an A-module.

The rule $\Lambda \longmapsto \phi_T^\Lambda$ establishes a bijective correspondence between A-lattices Λ of rank r and Drinfeld A-modules of rank r over C_∞. We will only need the two special cases: (a) $\Lambda = L := \overline{\pi}A$ has rank one, and is scaled (through the choice of the constant $\overline{\pi}$) such that the associated Drinfeld module ϕ^Λ is the *Carlitz module* ρ, defined by

$$\rho_T = TX + X^q = T + \tau. \tag{1.4}$$

Here the exponential function is

$$e_L(z) = \sum_{k\geq 0} D_k^{-1} z^{q^k}, \tag{1.5}$$

as is immediate from (1.2). Note that $\overline{\pi}$ is defined up to a $(q-1)$-th root of unity; hence only $\overline{\pi}^{q-1}$ is well-defined through (1.4). Many explicit formulas for $\overline{\pi}^{q-1} = -T^q + T - T^{-(q^2-2q)} + \cdots$ are available, see [4] (4.9), (4.10), (4.11).

(b) $\Lambda = \overline{\pi}(A\omega + A)$ with some $\omega \in \Omega$ and the constant $\overline{\pi}$ above. Here $\phi = \phi^\Lambda = \phi^\omega$ has rank two, and is given by

$$\phi_T = TX + gX^q + \Delta X^{q^2} = T + g\tau + \Delta\tau^2 \tag{1.6}$$

with $0 \neq \Delta \in C_\infty$.

A *Drinfeld modular form* for Γ of weight $k \in \mathbb{N} \cup \{0\}$ (and type zero: there will be no other "types" in this paper) is a holomorphic function $f : \Omega \longrightarrow C_\infty$ subject to

(1.7)

(i) $f(\gamma(z)) = (cz+d)^k f(z)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $z \in \Omega$;

(ii) $f(z) = \sum_{i\geq 0} a_i s^i(z)$ with some power series $\sum a_i s^i$ with positive convergence radius in $s(z) = e_L^{1-q}(\overline{\pi}z)$, the identity being valid for sufficiently large imaginary parts $|z|_i$.

By abuse of notation, we often identify $f = \sum a_i s^i \in C_\infty[[s]]$, since f is uniquely determined by its s-expansion. Letting ω vary over Ω, the quantities $g = g(\omega)$ and $\Delta = \Delta(\omega)$ in (1.6) become functions from Ω to C_∞, and in fact, modular forms of respective weights $q-1$ and q^2-1. The forms g and Δ are algebraically independent and generate the C_∞-algebra

$$M := \bigoplus_{k\geq 0} M_k = C_\infty[g,\Delta] \tag{1.8}$$

of all modular forms (D. Goss [11]), where M_k is the C_∞-vector space of modular forms of weight k, which vanishes for $k \not\equiv 0 \pmod{q-1}$. Our normalization is such that

$$g(z) = \sum_{i\geq 0} a_i s^i,\ \Delta(z) = \sum_{i\geq 1} b_i s^i \tag{1.9}$$

with $a_0 = 1$, $b_1 = -1$, and all the coefficients a_i, b_i lie in A. Moreover, g and Δ generate the A-algebra M_A of modular forms which have their s-expansion coefficients in A.

Other instances of modular forms are:

(1.10) Consider the $\mathbb{F}_q$-algebra homomorphism

$$\begin{array}{cccc} \phi^\omega: & A & \longrightarrow & C_\infty\{\tau\} \\ & a & \longmapsto & \phi^\omega_a \end{array}$$

uniquely determined by $\phi^\omega_T = T + g(\omega)\tau + \Delta(\omega)\tau^2$, and write

$$\phi^\omega_a = \sum_{0\leq i\leq 2\deg\ a} \ell_i(a,\omega)\tau^i.$$

Then $\ell_i(a,\cdot)$ defines a modular form of weight $k = q^i - 1$, a so-called coefficient form.

(1.11) The *Eisenstein series*

$$E_k(\omega) := \overline{\pi}^{-k} \sum_{(0,0)\neq(a,b)\in A\times A} \frac{1}{(a\omega+b)^k}$$

defines a non-zero element of M_k, provided that $0 < k \equiv 0 (\operatorname{mod} q-1)$.

(1.12) For $\Lambda = \overline{\pi}(A\omega + A)$ as before, write

$$e_\Lambda(z) = \sum_{i\geq 0} \alpha_i(\omega) z^{q^i}.$$

Once again, α_i is a modular form of weight $k = q^i - 1$, a *para-Eisenstein series*. We will study some of its arithmetical properties.

(1.13) **Remark.** Recall that classical Eisenstein series

$$E_k(\omega) = (2\pi i)^{-k} \sum_{(0,0)\neq(a,b)\in\mathbb{Z}\times\mathbb{Z}} \frac{1}{(a\omega+b)^k}$$

occur both as coefficients forms attached to elliptic curves (e.g. the coefficients g_2, g_3 in a Weierstrass equation) and as the coefficients of the Weierstrass $\wp$-function $\wp_\Lambda$ of $\Lambda = \mathbb{Z}\omega+\mathbb{Z}$. Since the exponential function e_Λ through its functional equation uniformizes the Drinfeld module ϕ^Λ in the same way as $\wp_\Lambda$ uniformizes the elliptic curve $E = \mathbb{C}/\Lambda$, the α_i provide a function field analogue of classical Eisenstein series different from the one described in (1.11). This explains the terminology used.

From (1.2) and (1.6) we get the formula

$$[k]\alpha_k = g\alpha_{k-1}^q + \Delta\alpha_{k-2}^{q^2}, \tag{1.14}$$

valid for $k \geq 1$, where $\alpha_k = 0$ for $k < 0$ and $\alpha_0 = 1$. (Recall that $[k] = T^{q^k} - T$.)

The *modular invariant* is the function

$$j := \frac{g^{q+1}}{\Delta} : \Omega \longrightarrow C_\infty. \tag{1.15}$$

It is Γ-invariant and identifies the quotient space $\Gamma \setminus \Omega$ biholomorphically with the affine line over C_∞. Accordingly, if $f \in M_k$ is a modular form of weight k, where $k = a(q^2-1) + b(q-1)$ with $a \in \mathbb{N}_0$ and $0 \leq b \leq q$, there exists a unique polynomial $\varphi(X) = \varphi_f(X) \in C_\infty[X]$ such that

$$f = \varphi(j)\Delta^a g^b. \tag{1.16}$$

We call it the *companion polynomial* of f and its zeroes the j-*zeroes* of f.

2. The Para-eisenstein Series m_k.

The Eisenstein series of weight $q^k - 1$ are particularly important. We normalize them as follows:

$$g_k(z) := (-1)^{k+1} L_k E_{q^k-1}(z) \quad (L_k = [k][k-1]\cdots[1]). \tag{2.1}$$

Then g_k has its s-coefficients in A, with absolute term 1, and satisfies the recursion

$$g_k = -[k-1]g_{k-2}\Delta^{q^{k-2}} + g_{k-1}g^{q^{k-1}} \quad (k \geq 2) \tag{2.2}$$

with $g_0 = 1$, $g_1 = g$ (see [4] 6.9). It is easily checked that its companion polynomial $\gamma_k = \varphi_{g_k}$ satisfies

$$\gamma_k(X) = X^{\lambda(k)}\gamma_{k-1}(X) - [k-1]\gamma_{k-2}(X) \quad (k \geq 2), \tag{2.3}$$

$\gamma_0 = \gamma_1 = 1$, γ_k is monic of degree $\nu(k)$, $g_k = \gamma_k(j)\Delta^{\nu(k)}g^{\chi(k)}$, where

$$\lambda(k) = \frac{q^{k-1} + (-1)^k}{q+1},\ \nu(k) = \frac{q^k - q^{\chi(k)}}{q^2-1},\ \chi(k) = 0(1)$$

if k is even (odd), respectively.

Since the lattice $\overline{\pi}(A\omega + A)$ degenerates to $\overline{\pi}A = L$ for $|\omega|_i \longrightarrow \infty$, i.e., $s(\omega) \longrightarrow 0$, the constant term of $\alpha_k(z)$ equals the k-coefficient D_k^{-1} of $e_L(z)$. We therefore normalize the *para-Eisenstein series*

$$m_k(z) := D_k \alpha_k(z), \tag{2.4}$$

which has constant term 1. In analogy with (2.2) and (2.3), we have the recursions derived from (1.14):

$$m_k = g m_{k-1}^q + [k-1]^q \Delta m_{k-2}^{q^2} \tag{2.5}$$

and for the companion polynomials $\mu_k = \varphi_{m_k}$:

$$\mu_k(X) = X^{\chi(k-1)} \mu_{k-1}^q(X) + [k-1]^q X^{(q-1)\chi(k)} \mu_{k-2}^{q^2}(X), \tag{2.6}$$

both valid for $k \geq 2$, with $m_0 = 1$, $m_1 = g$, $\mu_0 = \mu_1 = 1$. Since g and Δ have their s-coefficients in A, the same holds for the m_k. Again, μ_k is monic of degree $\nu(k) = \deg \gamma_k$, and $m_k = \mu_k(j)\Delta^{\nu(k)}g^{\chi(k)}$.

The first few of the polynomials γ_k, μ_k are as follows.

2.7 Table.

k	$\gamma_k(X)$	$\mu_k(X)$
2	$X - [1]$	$X + [1]^q$
3	$X^q - [1]X^{q-1} - [2]$	$X^q + [2]^q X^{q-1} + [1]^{q^2}$
4	$X^{q^2+1} - [1]X^{q^2} - [2]X^{q^2-q+1}$	$X^{q^2+1} + [3]^q X^{q^2} + [2]^{q^2} X^{q^2-q+1}$
	$-[3]X + [1][3]$	$+[1]^{q^3}X + [3]^q[1]^{q^3}$

Defining the *support* $\mathrm{supp}(f)$ of a polynomial (or power series) f as the set of exponents with nonvanishing coefficients, we observe that $\mathrm{supp}(\gamma_k)$ and $\mathrm{supp}(\mu_k)$ agree, although the recursions (2.3) and (2.6) are rather different. This is not accidental, and will result from Theorem 3.5. A first step towards the proof of this fact is:

2.8 Proposition. *Let $S(k) = \mathrm{supp}(\gamma_k)$ and $T(k) = \mathrm{supp}(\mu_k)$ be the supports. We have for $k \geq 2$*

(*i*) $S(k) = S(k-1) + \lambda(k) \,\dot{\cup}\, S(k-2)$;

(*ii*) $T(k) = qT(k-1) + \chi(k-1) \dot{\cup} q^2T(k-2) + (q-1)\chi(k)$;

(*iii*) $|S(k)| = |T(k)| = F_k :=$ *the k-th Fibonacci number defined by* $F_1 = 1$, $F_2 = 2$, $F_k = F_{k-1} + F_{k-2}$ $(k \geq 3)$.

Proof. Consider the recursions (2.3) and (2.6).

(i) Since $\lambda(k) > \deg \gamma_{k-2} = \nu(k-2)$, there is no cancellation of terms in (2.3), hence $S(k-1) + \lambda(k)$ and $S(k-2)$ are disjoint.

(ii) Similarly, the two sets in the right hand side of eq. (ii) are disjoint since they belong to different residue classes modulo q.

(iii) is immediate from (i) and (ii). ∎

2.9 **Proposition.** *All the zeroes of g_k and m_k are simple. Equivalently, the polynomials γ_k and μ_k are separable.*

Proof. The equivalence of the two statements results from the fact that the canonical mapping from Ω to $\Gamma \backslash \Omega$ is unramified off elliptic points (i.e., those where $j(z) \neq 0$) and g has simple zeroes at elliptic points ([4] 5.15).

The separability of γ_k is shown in [6] 7.7, 7.8. That of μ_k will result in a similar way from Theorem 3.5. However, there is a simple direct proof as follows. The derivative of $\mu_k(X)$ is $\mu_{k-1}^q(X)$ if k is even and $-[k-1]^q X^{q-2} \mu_{k-2}^{q^2}(X)$ if k is odd. Assume that $\mu_k(x) = 0 = \mu_k'(x)$, where $x \neq 0$ and $k \geq 3$. In both cases (k even / k odd) we conclude from (2.6) that $0 = \mu_k(x) = \mu_{k-1}(x) = \mu_{k-2}(x)$. Again by (2.6), this implies $0 = \mu_{k-3}(x) = \cdots = \mu_1(x)$, which however is not the case. Therefore there are no multiple roots of μ_k. ∎

2.10 **Remark.** There are two important qualitative differences between the ortho-Eisenstein series g_k and the para-Eisenstein series m_k. The non-elliptic j-zeroes x of g_k (i.e., zeroes of γ_k) all satisfy $|x| = q^q$, or, what amounts to the same, $g_k(z) = 0$ with non-elliptic z in the standard fundamental domain $\mathcal{F} \subset \Omega$ of Γ implies $|z| = |z|_i = 1$ (see [6] 6.7). This is similar to the corresponding property of classical Eisenstein series for $\mathrm{SL}(2, \mathbb{Z})$ (see [15]). However, the j-zeroes x of m_k (which have been determined in [7]) are in general larger than q^q in absolute value, and $\max\{|x| \mid x \text{ zero of } \mu_k\} \longrightarrow \infty$ as $k \longrightarrow \infty$.

A second difference is the behavior under Hecke operators. While g_k is always an eigenform with simple eigenvalues ([4] 7.2), m_k is in general not an eigenform, as can be seen e.g. from

$$m_2 = [2]\Delta + g_2,$$

where Δ and g_2 are eigenforms with different eigenvalues.

3. $\mathfrak{p}$-ADIC CONGRUENCES

Let $\mathfrak{p}$ be an irreducible monic prime polynomial in A of degree d. We use the same symbol for the prime ideal generated by $\mathfrak{p}$ and write $\mathbb{F}_\mathfrak{p}$ for the finite A-field $A/\mathfrak{p}$, $\overline{\mathbb{F}}_\mathfrak{p}$ for its algebraic closure, and $\mathbb{F}_\mathfrak{p}^{(2)}$ for the quadratic extension of $\mathbb{F}_\mathfrak{p}$ in $\overline{\mathbb{F}}_\mathfrak{p}$.

For the reader's convenience, we recall some of the relevant facts about supersingularity of Drinfeld modules, which are strikingly similar to the corresponding facts about supersingularity of elliptic curves. Missing definitions and more details can be found in [5]. Let L be a field subextension of $\overline{\mathbb{F}}_\mathfrak{p}/\mathbb{F}_\mathfrak{p}$. A Drinfeld module ϕ over L is supersingular if and only if one of the following equivalent conditions is satisfied:

(i) ϕ has no $\mathfrak{p}$-torsion over $\overline{\mathbb{F}}_\mathfrak{p}$;

(ii) the "multiplication-by-$\mathfrak{p}$" map $\phi_\mathfrak{p}$ is purely inseparable;

(iii) the ring $\mathrm{End}_{\overline{\mathbb{F}}_\mathfrak{p}}(\phi)$ of endomorphisms of ϕ over $\overline{\mathbb{F}}_\mathfrak{p}$ is non-commutative. (In this case it is an order, in fact a maximal A-order, in a certain quaternion algebra over K.)

Supersingularity of ϕ depends only on the $\overline{\mathbb{F}}_\mathfrak{p}$-isomorphism class of ϕ, and therefore on its j-invariant $j(\phi) \in \overline{\mathbb{F}}_\mathfrak{p}$. The set $\Sigma(\mathfrak{p}) \subset \overline{\mathbb{F}}_\mathfrak{p}$ of supersingular invariants is finite, contained in $\mathbb{F}_\mathfrak{p}^{(2)}$ and stable under the Galois conjugation of $\mathbb{F}_\mathfrak{p}^{(2)}$ over $\mathbb{F}_\mathfrak{p}$. Moreover, we have

$$0 \in \Sigma(\mathfrak{p}) \Leftrightarrow d \text{ odd } \Leftrightarrow \chi(d) = 1 \quad \text{and}$$
$$|\Sigma(\mathfrak{p}) - \{0\}| = \nu(d) = \deg \gamma_d = \deg \mu_d.$$

Let $ss_\mathfrak{p}(X)$ be the polynomial $\prod_{0 \neq j \in \Sigma(\mathfrak{p})}(X - j) \in \mathbb{F}_\mathfrak{p}[X]$. Then in fact

$$\gamma_d(X) \equiv ss_\mathfrak{p}(X) \pmod{\mathfrak{p}}. \tag{3.1}$$

This is implicit in [4] Cor. 12.3 and explicit in [1] and [6]. A similar congruence in the framework of classical modular forms and elliptic curves is due to Swinnerton-Dyer [17], see also [14]. A crucial step in the proof of (3.1) is the congruence, valid for $k \geq 0$ (see [4] 6.11):

$$g_{k+d}(s) \equiv g_k(s)^{q^d} \equiv g_k(s^{q^d}) \pmod{\mathfrak{p}}, \tag{3.2}$$

where we abuse notation to write $g_k(s)$ for the power series expansion of g_k in s. In particular,

$$g_d \equiv 1 \pmod{\mathfrak{p}}. \tag{3.3}$$

In view of $[d] = \prod_{(\deg \mathfrak{p})|d} \mathfrak{p}$, the two preceding congruences hold in fact modulo $[d]$. Finally, we have the periodicity relation

$$\gamma_{k+d}(X) \equiv X^{\chi(k)\lambda(d+1)}\gamma_k^{q^d}(X)\gamma_d(X) \pmod{\mathfrak{p}}, \tag{3.4}$$

valid for $k \geq 0$ ([6] 7.6). We show similar $\mathfrak{p}$-adic properties of the para-Eisenstein series m_k and their companions $\mu_k(X)$.

3.5 **Theorem.** *Let $\mathfrak{p}$ be a prime of A of degree d.*

(i) *The polynomial $\mu_d(X)$ satisfies*

$$\mu_d(X) \equiv \gamma_d(X) \equiv ss_{\mathfrak{p}}(X) \pmod{\mathfrak{p}}.$$

We further have for $k \geq 0$ the following congruences $\pmod{\mathfrak{p}}$:

(ii) $m_{k+d}(s) \equiv m_k(s)$;

(iii) $\mu_{k+d}(X) \equiv X^{\chi(d)\lambda(k+1)}\mu_d^{q^k}(X)\mu_k(X)$.

Proof. It is known ([4] (2.6)-(2.9)) that the series $\sum_{k\geq 0}\alpha_k(\omega)\tau^k$ and $-\sum_{k\geq 0} E_{q^k-1}(\omega)\tau^k$ are inverses of each other in $C_\infty\{\{\tau\}\}$. Since up to the normalizations (2.1) and (2.4) E_{q^k-1} and α_k agree with g_k and m_k, respectively, we get

$$\sum_{\substack{i,j\geq 0\\ i+j=k}} (-1)^{j+1}\frac{D_k}{D_i L_j^{q^i}} m_i g_j^{q^i} = 0 \qquad \text{for } k \geq 1.$$

The coefficients $D_k/(D_i L_{k-i}^{q^i})$ always belong to A and are divisible by each prime $\mathfrak{q}$ of A of degree k if $0 < i < k$, as is easily seen from their definitions, see [13] Theorem 3.1.5. In particular,

$$m_d = \sum_{0\leq i<d} (-1)^{d-i+1}\frac{D_d}{D_i L_{d-i}^{q^i}} m_i g_{d-i}^{q^i}.$$

As m_i, g_j belong to $A[[s]]$ and $D_d/L_d \equiv (-1)^{d+1} \pmod{\mathfrak{p}}$ ([4] 11.4), we find

$$m_d \equiv g_d \equiv 1 \pmod{\mathfrak{p}} \tag{3.6}$$

as a power series. Assertion (i) is now a formal consequence of (3.6) and Theorem 12.1 of [4]. Viz, let $G_d(X,Y)$ and $M_d(X,Y)$ be the polynomials in $A[X,Y]$ defined by $G_d(g,\Delta) = g_d$, $M_d(g,\Delta) = m_d$, and $\tilde{G}_d$, $\tilde{M}_d$ their respective reductions in $\mathbb{F}_{\mathfrak{p}}[X,Y]$. Then $\tilde{M}_d - 1$ lies in the kernel of the homomorphism $\epsilon : \mathbb{F}_{\mathfrak{p}}[X,Y] \longrightarrow \mathbb{F}_{\mathfrak{p}}[[s]]$ which to X resp. Y associates the s-expansion $\pmod{\mathfrak{p}}$ of g resp. Δ. However, $\ker(\epsilon)$ is generated by $\tilde{G}_d - 1$ ([4] (12.1)+(12.2)). Comparing leading coefficients in X, we find $\tilde{M}_d = \tilde{G}_d$. In view of $g_d = G_d(g,\Delta) = \gamma_d(j)\Delta^{\nu(d)}g^{\chi(d)}$ and similarly, $m_d = M_d(g,\Delta) = \mu_d(j)\Delta^{\nu(d)}g^{\chi(d)}$, (i) follows.

From (2.5), (3.6), $[d] \equiv 0$, $[d+k] \equiv [k]$ (all the congruences are $(\mathrm{mod}\mathfrak{p})$), we find $m_{d+1} \equiv g \equiv m_1$, which finally implies $m_{d+2} \equiv m_2$, $m_{d+3} \equiv m_3$ ..., hence (ii). The last assertion (iii) now follows from (2.6) through a standard induction on k (distinguishing the 4 cases k/d even/odd, respectively), which we omit. ∎

3.7 **Corollary.** $m_d \equiv 1 \ (\mathrm{mod}[d])$

Proof. By the theorem, the congruence holds modulo all primes $\mathfrak{p}$ of degree dividing d. ∎

3.8 **Corollary.** *The supports $S(k)$ of γ_k and $T(k)$ of μ_k agree.*

Proof. The two sets have the same size by (2.8), hence it suffices to show an inclusion between them. A view on (2.3) reveals that the non-vanishing coefficients of γ_k are products of terms $[i]$ with $0 < i < k$, which are incongruent to zero modulo each prime $\mathfrak{q}$ of degree k. The assertion now results from $\gamma_k \equiv \mu_k$ modulo $\mathfrak{q}$ and the fact that for each k there exists a prime $\mathfrak{q}$ of degree k. ∎

4. Balancedness

Let $\mathrm{Aff}(\mathbb{F}_q)$ be the group of matrices of shape $\binom{u,\,v}{0,\,1}$ over $\mathbb{F}_q$ ($u \neq v$). It acts naturally on A by $T \longmapsto uT + v$.

4.1 **Definition** [8]. A polynomial or power series $f(s) = \sum c_k s^k$ in $A[[s]]$ is *balanced* if the (visibly equivalent) conditions hold:

- (a) each c_k as a polynomial in T satisfies $c_k(uT+v) = u^k c_k(T)$ for $\binom{u,\,v}{0,\,1} \in \mathrm{Aff}(\mathbb{F}_q)$;
- (b) (i) $c_k \in \mathbb{F}_q[T^q - T]$, the ring of invariants under shifts $T \longmapsto T+v$;
  (ii) if $c_k = \sum_j c_{k,j}T^j$ with $c_{k,j} \in \mathbb{F}_q$ then $c_{k,j} \neq 0$ implies $j \equiv k \ (\bmod \ q-1)$;
- (c) f is invariant under the action of $\mathrm{Aff}(\mathbb{F}_q)$ on $A[[s]]$ that extends the natural action on "constants" in A and satisfies $\binom{u,\,v}{0,\,1}(s) = u^{-1}s$.

In particular, the set of balanced power series is a subring of $A[[s]]$. It will turn out that most of our modular forms are balanced as power series in s.

Fix an element $\binom{u,\,v}{0,\,1}$ of $\mathrm{Aff}(\mathbb{F}_q)$, and let $T' = uT + v$, another uniformizer for our ring A. We may calculate all the relevant quantities, labelled by a prime $(\ )'$ with respect to the coordinate T' and relate them to the corresponding quantity w.r.t. T. Let $\alpha \in \overline{\mathbb{F}}_q$ satisfy $\alpha^{q-1} = u$. Then

(4.2)

(i) $[k]' = u \cdot [k]$

(ii) $\rho' = \alpha \circ \rho \circ \alpha^{-1}$ (i.e., $\rho'_T = \alpha \circ \rho_T \circ \alpha^{-1}$)
(iii) $(\overline{\pi}^{q-1})' = u \cdot \overline{\pi}^{q-1}$.

Here (i) is obvious, (ii) is $\rho'_{T'} = (uT+v)+\tau = \alpha(uT+v+u\tau)\alpha^{-1} = \alpha \circ \rho_{T'} \circ \alpha^{-1}$, and (iii) follows from either one of the formulas (4.9), (4.10), (4.11) for $\overline{\pi}^{q-1}$ of [4].

4.3 **Theorem.** *The following modular forms are balanced as power series in* s*:* (i) g, (ii) $[k]\Delta$ *(*$k \in \mathbb{N}$*)*, (iii) g_k, (iv) m_k. *Proof.* We have $g = g_1$ and

$$g_k^{(\omega)} = (-1)^{k+1} L_k \overline{\pi}^{1-q^k} \sum_{(0,0) \neq (a,b) \in A^2} \frac{1}{(a\omega + b)^{q^k - 1}}.$$

Here the lattice sum is intrinsic (independent of the choice of the coordinate T of A). In view of (4.2), we have $L'_k = u^k L_k$ and $(\overline{\pi}^{q^k-1})' = u^k(\overline{\pi}^{q^k-1})$, which together gives (iii) and thus (i). Now from (2.2) we have

$$\Delta = \frac{1}{[1]}(g^{q+1} - g_2),$$

which implies $\Delta' = u^{-1}\Delta$. Together with (4.2)(i), the balancedness of $[k]\Delta$ results. Finally, (iv) comes out from (2.5) and the fact that the balanced elements of $A[[s]]$ form a subring. ∎

Remark. The present argument is essentially from [8]; however, due to different normalizations, some formulas differ from the corresponding ones given there.

There are similar invariance properties for the polynomials γ_k and μ_k.

4.4 **Proposition.** *If we endow the ring* $A[X] = \mathbb{F}_q[T, X]$ *with the degree graduation modulo* $q-1$*, the polynomials* γ_k *and* μ_k *are homogeneous of degree* $[k/2]$ $= k/2$ *resp.* $(k-1)/2$ *if* k *is even resp. odd. That is, regarded as polynomials in* T *and* X*,* $\gamma_k(uT, uX) = u^{[k/2]}\gamma_k(T, X)$ *for* $u \in \mathbb{F}_q^*$*, and similarly for* μ_k*.*

Proof. The assertion could be formally derived from (4.3); however it is much easier to refer to the recursions (2.2) and (2.5), from which it comes out in a straightforward fashion. ∎

4.5 **Remark.** The functions $f = \Delta$ and $f = g_k$ satisfy $k \in \mathrm{supp}(f) \Rightarrow k \equiv 0, 1 (\bmod\ q)$, see [4] 6.10. That property also holds for the forms m_k, as is evident from (2.5). Hence the results (4.3) and (4.4) imply strong restrictions on the nature of the coefficients of $g_k, m_k, \gamma_k, \mu_k$. These, together with bounds on the degrees of the coefficients, suffice to determine some of the coefficients e.g. of the g_k [2]. A similar study of coefficients of the m_k would be desirable.

Resumé. We tabulate properties of classical Eisenstein series E_k for $\mathrm{SL}(2, \mathbb{Z})$ and the corresponding properties of the ortho-Eisenstein series g_k and para-Eisenstein series m_k for $\Gamma = \mathrm{GL}(2, A)$, along with properties of the respective companion polynomials. Here $p \geq 5$ is a natural prime and $\mathfrak{p}$ a prime of A of degree d. We don't give precise statements in the classical case (which would require more notation), but content ourselves with giving references.

	E_k	g_k	m_k
recursion of Eisenstein series	[17] p. 19	(2.2)	(2.5)
of companion polynomials	to be worked out	(2.3)	(2.6)
simplicity of non-elliptic zeroes	yes	yes	yes
location of zeroes z in standard fundamental domain $\mathcal{F}$	$\lvert z\rvert = 1, [15]$	$\lvert z\rvert = 1$, [6]	different, [7]
Hecke eigenform property	yes	yes	no
congruences mod p resp. mod $\mathfrak{p}$	$E_{p-1} \equiv 1$, [17]	$g_d \equiv 1$	$m_d \equiv 1$
of companions	[9] Thm. 2.2	$\gamma_d \equiv ss_{\wp}$	$\mu_d \equiv ss_{\wp}$
periodicity of companions reduced mod p, mod $\mathfrak{p}$	[9] Thm. 2.4	(3.4)	(3.5)(iii)

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References

1. G. Cornelissen, *Geometric properties of modular forms over rational function fields*, Thesis Universiteit Gent, 1997.
2. J. Gallardo and B. Lopez, "Weak" congruences for coefficients of the Eisenstein series for $\mathbb{F}_q[T]$ of weight $q^k - 1$, *J. Number Theory*, **102** (2003), 107-117.
3. E.-U. Gekeler, *Drinfeld Modular Curves*, Lect. Notes Math. Vol. 1231, Springer, 1986.
4. E.-U. Gekeler, On the coefficients of Drinfeld modular forms, *Invent. Math.*, **93** (1988), 667-700.
5. E.-U. Gekeler, On finite Drinfeld modules, *J. Algebra*, **141** (1991), 187-203.
6. E.-U. Gekeler, Some new results on modular forms for $\mathrm{GL}(2, \mathbb{F}_q[T])$, *Contemp. Math.*, **224** (1999), 111-141.
7. E.-U. Gekeler, A survey on Drinfeld modular forms, *Turk. J. Math.* **23** (1999), 485-518.

8. E.-U. Gekeler, Growth order and congruences of coefficients of the Drinfeld discriminant function, *J. Number Theory*, **77** (1999), 314-325.

9. E.-U. Gekeler, Some observations on the arithmetic of Eisenstein series for the modular group $\mathrm{SL}(2,\mathbb{Z})$, *Arch. Math.*, **77** (2001), 5-21.

10. D. Goss, π-adic Eisenstein series for function fields, *Comp. Math.*, **41** (1980), 3-38.

11. D. Goss, Modular forms for $\mathbb{F}_r[T]$, *J. Reine Angew. Math.*, **317** (1980), 16-39.

12. D. Goss, The algebraist's upper half-plane, *Bull. AMS* $($*n.S.*$)$, **2** (1980), 391-415.

13. D. Goss, *Basic Structures of Function Field Arithmetic*, Springer, 1996.

14. M. Kaneko and D. Zagier, Supersingular j-invariants, hypergeometric series, and Atkin's orthogonal polynomials, in: *Studies in Advanced Mathematics*, Vol. 7, 1998, pp. 97-126.

15. F. C. K. Rankin and H. P. F. Swinnerton-Dyer, On the zeros of Eisenstein series, *Bull. London Math. Soc.*, **2** (1970), 169-170.

16. J. Silverman, *The Arithmetic of Elliptic Curves*, Springer, 1986.

17. H. P. F. Swinnerton-Dyer, *On ℓ-adic representations and congruences for coefficients of modular forms*, Lect. Notes Math. Vol. 350, Springer, 1973, pp. 1-55.

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