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EXISTENCE OF NONZERO, LINEAR AND CONTINUOUS OPERATOR BETWEEN TWO MUSIELAK-ORLICZ SPACES

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Abstract. Let X, Y be linear topological subspaces of the space of all measurable functions over a σ-finite, atomless and complete measure space Ω. The question of existence of a nonzero, linear continuous operator $T : X \to Y$ is a natural extension of the question ”does X admit a nonzero, linear continuous functional?”. In case of $X = L^{\Phi}(\Omega), Y = L^{\Psi}(\Omega)$ being Orlicz spaces Ph.Turpin ([11]) gave a criterion telling when there is no nonzero, linear and continuous operator between $L^{\Phi}(\Omega)$ and $L^{\Psi}(\Omega)$. That result was extended to the case of Musielak-Orlicz spaces by A.K.Kalindé, R.Płuciennik and M.Wisl /a ([2, 8]) but only necessary conditions have been presented by them - in fact conditions assuring that there is no nonzero, linear and continuous operator between $L^{\Phi}(\Omega)$ and $L^{\Psi}(\Omega)$. In this paper we give inverse theorems to the Kalindé and Płuciennik-Wisła theorems. Generally speaking we state that there exists a nonzero, linear and continuous operator if and only if it there exists a set $A \subset \Omega$ of positive and finite measure such that the inclusion operator $i : L^{\Phi}(A) \to L^{\Psi}(A)$ is nonzero and continuous.

1. Introduction

Orlicz functions were defined by W. Orlicz in [6]. Since that time a lot of mathematicians made some researches and there were received some generalizations. In 1959 J. Musielak and W. Orlicz in the paper [5] introduced the modular spaces generated by functions depended on parameter. In 1961 W. Orlicz in [7] introduced s-convex modulars, where $0 < s \leq 1$. In this section we remind some facts and definitions.

A map $\varphi : \mathbb{R} \to [0, \infty]$ is said to be a pregenfunction ([10]), if it is even, nondecreasing on $[0, \infty]$, $\varphi(0) = 0$ and φ is identically equal to neither 0 nor ∞.

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A pregenfunction $\varphi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if it is left-continuous on $[0, \infty)$, continuous at 0 and $\lim\limits_{u\to\infty} \varphi(u) = \infty$. We will say that the Orlicz function φ takes only finite values whenever $\varphi(u) < \infty$ for all $u \in \mathbb{R}$. Let us note that if φ is concave on $(0, \infty)$ and takes finite values, then the condition $\lim\limits_{u\to\infty} \varphi(u) = \infty$ is equivalent to the fact that φ is strictly increasing on $(0, \infty)$.

Let (Ω, Σ, μ) be a measure space with a σ-finite nonatomic and complete measure μ and $L^o(\mu)$ be the set of all μ-equivalence classes of real and Σ-measurable functions defined on Ω. A map $\Phi : \mathbb{R} \times \Omega \to [0, \infty]$ is called a Musielak-Orlicz function whenever the following conditions are satisfied:

(i) for every $t \in \Omega$, the function $u \to \Phi(u, t)$ is an Orlicz function,

(ii) for every $u \in \mathbb{R}$, the function $t \to \Phi(u, t)$ is Σ-measurable.

A Musielak-Orlicz function Φ is called locally integrable if $\int_A \Phi(u, t)d\mu < \infty$ for all $u \in \mathbb{R}$ and for all measurable subsets $A \subset \Omega$ of finite measure. We shall say that a Musielak-Orlicz function Φ is continuous if the function $u \to \Phi(u, t)$ takes only finite values and it is continuous on $\mathbb{R}$ for μ-a.e. $t \in \Omega$.

We say that a Musielak-Orlicz function Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$ in short) if there exist a constant $K > 0$ and an non-negative integrable function h such that

$$\Phi(2u, t) \le K\Phi(u, t) + h(t) \tag{1.1}$$

for all $u \in \mathbb{R}$ and μ-a.e. $t \in \Omega$. Note that $\Phi \in \Delta_2$ implies that Φ takes finite values for μ-a.e. $t \in \Omega$.

We will say that a Musielak-Orlicz function Φ is concave on an interval $(a, b) \subset (0, \infty)$, if the functions $u \to \Phi(u, t)$ are concave on (a, b) for μ-a.e. $t \in \Omega$. Note that if Φ takes finite values and it is concave on $(0, \infty)$ then Φ satisfies the Δ_2-condition with the function $h = 0$.

In the case of Orlicz function φ, the function h in (1.1) can be replaced by a nonnegative constant. We say that φ satisfies the the Δ_2-condition at infinity (resp. for all $u \in \mathbb{R}$) whenever $h > 0$ (resp. $h = 0$).

For all $f \in L^0(\mu)$ the function $t \to \Phi(f(t), t)$ is Σ-measurable and the functional $m_\Phi(f) = \int_\Omega \Phi(f(t), t)d\mu$ is a modular on $L^0(\mu)$. Moreover, if $u \to \Phi(u, t)$ is an s-convex function for all $t \in \Omega$, then m_Φ is an s-convex modular.

By $L^\Phi = L^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu)$ we define the class of all functions $f \in L^0(\mu)$ such that $m_\Phi(\lambda f) < \infty$ for some $\lambda > 0$. Analogously, by $E^\Phi = E^\Phi(\Omega) = E^\Phi(\Omega, \Sigma, \mu)$ we denote the class of all functions $f \in L^0(\mu)$ such that $m_\Phi(\lambda f) < \infty$ for every $\lambda > 0$. The classes L^Φ and E^Φ are linear subspaces of the space $L^0(\mu)$ and $E^\Phi \subset L^\Phi$. The class L^Φ is called the Musielak-Orlicz space and E^Φ is called the space of finite elements. For an arbitrary measurable subset $A \subset \Omega$ define $L^\Phi(A) = \{f \in L^\Phi(\Omega) : \operatorname{supp} f \subset A\}$ and $E^\Phi(A) = \{f \in E^\Phi(\Omega) : \operatorname{supp} f \subset A\}$.

where $\operatorname{supp} f = \{t \in \Omega : f(t) \neq 0\}$. Let us note that $\Phi \in \Delta_2$ implies the equality $L^\Phi(A) = E^\Phi(A)$ for any subset $A \subset \Omega$.

On the Musielak-Orlicz space $L^\Phi(\Omega)$ we define an F-norm by the formula

$$||f||_\Phi = \inf\{\lambda > 0 : m_\Phi(f/\lambda) \leq \lambda\}.$$

It is known that the space $L^\Phi(\Omega)$ with this F-norm is complete. Let us note that $\|f_n - f\|_\Phi \to 0$ if and only if $m_\Phi(\lambda(f_n - f)) \to 0$ for every $\lambda > 0$. For more information on F-normed spaces (and general metric linear spaces) we refer to [3, 9].

Let X and Y be two linear-topological spaces. By $\mathcal{L}(X, Y)$ we denote the space of all linear and continuous operators from X to Y. One of the commonly used linear operators are inclusions. We say that $\Psi \prec \Phi$, if we can find constants $K_1, K_2 > 0$ and an nonnegative integrable function h such that

$$\Psi(u, t) \leq K_1 \Phi(K_2 u, t) + h(t)$$

for all $u \in \mathbb{R}$ and μ-a.e. $t \in \Omega$.

Theorem 1.1. ([4], 8.5). *Let Φ and Ψ be Musielak-Orlicz functions. The inclusion operator $i : L^\Phi(\Omega) \to L^\Psi(\Omega)$ is well defined if and only if $\Psi \prec \Phi$.*

Note that inclusion operator need not to be continuous. In 1984 M. Wisła received the necessary condition for the continuity of the inclusion operator between two Musielak-Orlicz spaces.

Theorem 1.2. ([12], 3.2.2). *Assume that Φ and Ψ are Musielak-Orlicz functions such that $\Psi \prec \Phi$. If the function $u \to \Phi(u, t)$ is continuous for μ-a.e. $t \in \Omega$ and the function $u \to \Psi(u, t)$ is continuous at $u = 0$ for μ-a.e. $t \in \Omega$ then the inclusion operator $i : L^\Phi(\Omega) \to L^\Psi(\Omega)$ is continuous.*

Corollary 1.3. *Under assumptions of Theorem 1.2, the inclusion operator $i : E^\Phi(\Omega) \to E^\Psi(\Omega)$ is also continuous.*

Proof. Since $\Psi \prec \Phi$, $E^\Phi(\Omega) \subset E^\Psi(\Omega)$. Let $f_n \in E^\Phi(\Omega)$, $n \in \mathbb{N}$, be such a sequence that $\|f_n\|_\Phi \to 0$. Since $f_n \in L^\Phi(\Omega)$ as well, by Theorem 1.2,

$$\|f_n\|_\Phi \to 0 \text{ in } E^\Phi(\Omega) \Leftrightarrow \|f_n\|_\Phi \to 0 \text{ in } L^\Phi(\Omega)$$
$$\Rightarrow \|f_n\|_\Psi \to 0 \text{ in } L^\Psi(\Omega) \Leftrightarrow \|f_n\|_\Psi \to 0 \text{ in } E^\Psi(\Omega). \qquad \blacksquare$$

Evidently, if there exists a nonzero, linear and continuous operator between the spaces X and Y it does not mean that this operator has to be an inclusion. But in the case of Musielak-Orlicz spaces the inclusion operator is nearby every time. In 1973 P. Turpin proved the following theorem.

Theorem 1.4. ([11], 2.2.1). *Let φ and ψ be Orlicz functions with finite values such that φ satisfies the Δ_2-condition for all $u \in \mathbb{R}$ and ψ is concave on $(0, \infty)$. Then $\mathcal{L}(L^\varphi(\Omega), L^\psi(\Omega)) \neq \{0\}$ if and only if $\limsup_{u\to\infty} \frac{\psi(u)}{\varphi(u)} < \infty$.*

As an immediate consequence of Turpin result we get that, if the assumptions of Theorem 1.4 are satisfied and, moreover, $\mu(\Omega) < \infty$, then $\mathcal{L}(L^\varphi(\Omega), L^\psi(\Omega)) \neq \{0\}$ if and only if the Orlicz space $L^\varphi(\Omega)$ is continuously embedded into the Orlicz space $L^\psi(\Omega)$ (see Corollary 1.5 below). In the case of infinite measure space Ω the above conclusion does not hold true.

Corollary 1.5. *Let φ and ψ be Orlicz functions with finite values such that φ satisfies the Δ_2-condition for all $u \in \mathbb{R}$ and ψ is concave on $(0, \infty)$. Then $\mathcal{L}(L^\varphi(\Omega), L^\psi(\Omega)) = \mathcal{L}(E^\varphi(\Omega), E^\psi(\Omega)) \neq \{0\}$ if and only if there exist a measurable subset $A \subset \Omega$ such that the inclusion operator $i : L^\varphi(A) \to L^\psi(A)$ is continuous.*

Proof. Since ψ is concave, $\psi \in \Delta_2$. Since $\varphi \in \Delta_2$ as well, $L^\varphi(\Omega) = E^\varphi(\Omega)$ and $L^\psi(\Omega) = E^\psi(\Omega)$ up to set and topology.

If $\mathcal{L}(L^\varphi(\Omega), L^\psi(\Omega)) \neq \{0\}$ then, by Theorem 1.4, we can find constants $K, u_0 > 0$ such that $\psi(u) \leq K\varphi(u)$ for all $u \geq u_0$. Let f be a measurable function, $\lambda > 0$ and put $B = \{t \in \Omega : \lambda|f(t)| < u_0\}$. Then

$$m_\psi(\lambda f \chi_A) \leq \psi(u_0)\mu(A \cap B) + m_\varphi(\lambda f \chi_A)$$

for every measurable subset $A \subset \Omega$, so $L^\varphi(A) \subset L^\psi(A)$ as long as $\mu(A) < \infty$.

We shall show that the inclusion operator $i : L^\varphi(A) \to L^\psi(A)$ is continuous. Let $f_n \in L^\varphi(A)$ be such that $\|f_n\|_\varphi \to 0$, i.e., $m_\varphi(\lambda f_n) \to 0$ for every $\lambda > 0$. Fix $\lambda > 0$ and let $\varepsilon > 0$. Since ψ is continuous at 0, we can find $0 < u_1 \leq u_0$ such that $\psi(u_1)\mu(A) < \varepsilon/2$. Since φ satisfies Δ_2 for all $u \in \mathbb{R}$, φ can vanish only at 0, so $\varphi(u_1) > 0$. Put $K_1 = \max\{K, \psi(u_0)/\varphi(u_1)\}$. Then

$$\psi(u) \leq K_1\varphi(u) + \psi(u_1)$$

for all $u \in \mathbb{R}$. Take $n_0 \in \mathbb{N}$ such that $m_\varphi(\lambda f_n) < \varepsilon/2$ for all $n \geq n_0$. Then

$$m_\psi(\lambda f_n) \leq K_1 m_\varphi(\lambda f_n) + \psi(u_1)\mu(A) < \varepsilon$$

for all $n \geq n_0$, i.e., $m_\psi(\lambda f_n) \to 0$. By arbitrariness of $\lambda > 0$, $\|f_n\|_\psi \to 0$, and we have proved that the inclusion $i : L^\varphi(A) \to L^\psi(A)$ is continuous.

Conversely, let $i : L^\varphi(A) \to L^\psi(A), i(f) = f$ be well defined and continuous. Let $T_1 : L^\varphi(\Omega) \to L^\varphi(A)$ and $T_2 : L^\psi(A) \to L^\psi(\Omega)$ be the operators defined by the formulas: $T_1(f) = f\chi_A$, $T_2(f) = f$. Since the F-norm $\|\cdot\|_\varphi$ is monotone, the operator T_1 is continuous. Further, since $Ker(T_2) = \{0\}$, the operator $T = T_2 \circ i \circ T_1$ is a nonzero, linear and continuous operator from $L^{\varphi}(\Omega)$ to $L^{\psi}(\Omega)$. ∎

A. Kalindé generalized Turpin's result to the case of Musielak-Orlicz function Φ and the finite measure space.

Theorem 1.6. ([2], p. 34). *Let $\mu(\Omega) < \infty$, Φ be a continuous Musielak-Orlicz function satisfying the Δ_2-condition and let ψ be a concave and strictly increasing Orlicz function with finite values. If, for all $t \in \Omega$, the limit $\lim\limits_{u\to\infty} \frac{\psi(u)}{\Phi(u,t)}$ exists and it is equal to ∞, then $\mathcal{L}(L^{\Phi}(\Omega), L^{\psi}(\Omega)) = \{0\}$.*

2. Kalindé Inverse Theorem

Let X and Y be some linear-topological spaces. It is obvious that if $X^* \neq \{0\}$, then $\mathcal{L}(X, Y) \neq \{0\}$, but the inverse implication is not true. Hence the most interesting case is the one when $(L^{\Phi})^* = \{0\}$. In the paper [1] L.Drewnowski showed that if Φ is a continuous Musielak-Orlicz function satisfying the Δ_2-condition, then there exists an linear functional on L^{Φ} if and only if there exists a measurable set $A \in \Sigma$ with $\mu(A) > 0$ such that

$$\liminf_{u\to\infty} \frac{\Phi(u,t)}{u} > 0$$

for all $t \in A$.

Taking into account that the function $\psi(u) = |u|$ is an Orlicz function, the Drewnowski condition can be written as follows

$$\limsup_{u\to\infty} \frac{\psi(u)}{\Phi(u,t)} < \infty$$

for all $t \in A$ and we get a similar condition that is used in Kalindé Theorem 1.6. Note that this condition implies that the inclusion operator $i : L^{\Phi}(A) \to L^1(A)$ is continuous. Hence, evidently, $(L^{\Phi}(A))^* \neq \{0\}$.

We start with a few auxiliary lemmas.

Lemma 2.1. *If there is a measurable set $A \subset \Omega$ with $0 < \mu(A) < \infty$ such that $L^{\Phi}(A) \neq \{0\}$ and the inclusion operator $i : L^{\Phi}(A) \to L^{\Psi}(A)$, $i(f) = f$ is well defined and continuous, then $\mathcal{L}(L^{\Phi}(\Omega), L^{\Psi}(\Omega)) \neq \{0\}$.*

Proof. Define the operators $T_1 : L^{\Phi}(\Omega) \to L^{\Phi}(A)$ and $T_2 : L^{\Psi}(A) \to L^{\Psi}(\Omega)$ by $T_1(f) = f\chi_A$ and $T_2(f) = f$. Then $T = T_2 \circ i \circ T_1$ is a linear and continuous operator from $L^{\Phi}(\Omega)$ to $L^{\Psi}(\Omega)$. Take $f \in L^{\Phi}(A) \setminus \{0\}$. Then $f\chi_A \neq 0$, so $T(f) = f\chi_A \neq 0$ as well. ∎

Analogously, the following lemma can be proved.

Lemma 2.2. *If there is a measurable set* $A \subset \Omega$ *with* $0 < \mu(A) < \infty$ *such that* $E^\Phi(A) \neq \{0\}$ *and the inclusion operator* $i : E^\Phi(A) \to E^\Psi(A)$, $i(f) = f$ *is well defined and continuous, then* $\mathcal{L}(E^\Phi(\Omega), E^\Psi(\Omega)) \neq \{0\}$.

Lemma 2.3. *Let* Φ *be a continuous locally integrable Musielak-Orlicz function and let* ψ *be an Orlicz function continuous at* 0. *If the set* $B = \{t \in \Omega : \lim\limits_{v\to\infty} \frac{\psi(v)}{\Phi(v,t)} < \infty\}$ *contains a subset of positive measure, then* B *contains a subset* A *with positive and finite measure such that the inclusion operator* $i : L^\Phi(A) \to L^\psi(A)$ *is nonzero and continuous.*

Proof. Let $A_0 \subset B$ be a set of positive and finite measure and define

$$A_p = \{t \in A_0 : \lim_{v\to\infty} \frac{\psi(v)}{\Phi(v,t)} < 2^p\}.$$

Then $A_0 = \bigcup_{p\in\mathbb{N}} A_p$ and $A_p \uparrow A$. Since $0 < \mu(A_0) < \infty$, we have $0 < \mu(A_p) < \infty$ for some (fixed from now on) $p \in \mathbb{N}$. Define the sets $A_{p,r}$ by the formula

$$A_{p,r} = \Big\{t \in A_p : \frac{\psi(v)}{\Phi(v,t)} < 2^p\big(1 - \frac{1}{r}\big) \text{ for all } v \geq r\Big\},$$

where $r \in \mathbb{N}$. Then $\psi(v) < 2^p(1 - \frac{1}{r})\Phi(v,t)$ for all $v \geq r$ and $t \in A_{p,r}$. From $A_p = \bigcup_{r=1}^\infty A_{p,r}$, we infer that $0 < \mu(A_{p,r}) < \infty$ for some $r \in \mathbb{N}$. Therefore $L^\Phi(A_{p,r}) \subset L^\psi(A_{p,r})$ for all $r \in \mathbb{N}$. Finally, by Theorem 1.2, we conclude that the inclusion operator $i : L^\Phi(A_{p,r}) \to L^\psi(A_{p,r})$ is continuous. Since Φ is locally integrable, $L^\Phi(A_{p,r}) \neq \{0\}$, so the operator i is nonzero as well. ∎

Theorem 2.4. *Let* Φ *be a continuous locally integrable Musielak-Orlicz function satisfying the* Δ_2*-condition and let* ψ *be a concave and strictly increasing Orlicz function with finite values. Assume that* $\lim_{u\to\infty} \frac{\psi(u)}{\Phi(u,t)}$ *exists for all* $t \in \Omega$. *Then* $\mathcal{L}(L^\Phi(\Omega), L^\psi(\Omega)) = \mathcal{L}(E^\Phi(\Omega), E^\psi(\Omega)) \neq \{0\}$ *if and only if the inclusion operator* $i : L^\Phi(A) \to L^\psi(A)$ *is continuous for some measurable set* $0 < \mu(A) < \infty$.

Proof. The sufficiency part of the prof follows from Lemma 2.1. Note, that the local integrability of Φ implies that $L^\Phi(A) \neq \{0\}$, so the inclusion operator i and the operator T defined in Lemma 2.1 are nonzero operators.

Assume that $T : L^\Phi(\Omega) \to L^\psi(\Omega)$ is a nonzero, linear and continuous operator. From Theorem 1.6 it follows that the set $\{t \in \Omega : \lim_{v\to\infty} \frac{\psi(v)}{\Phi(v,t)} < \infty\}$ contains a subset of positive measure. Hence, by Lemma 2.3, the inclusion operator $i : L^\Phi(A) \to L^\psi(A)$ is continuous for some measurable set $A \subset \Omega$ with $0 < \mu(A) < \infty$. ∎

3. Weighted Orlicz Spaces

In this section we will consider a special subclass of Musielak-Orlicz functions called weighted Orlicz functions. We say that a Musielak-Orlicz function Φ is weighted Orlicz function if there exists an Orlicz function φ and a nonnegative measurable function ω with finite values such that $\Phi(u,t) = \varphi(u)\omega(t)$ for all $u \in \mathbb{R}$ and $t \in \Omega$. Analogously, the space $L^{\Phi}(\Omega)$ is called weighted Orlicz space in that case.

Theorem 3.1. *Let Φ and Ψ be weighted Orlicz functions defined by $\Phi(u,t) = \varphi(u)\omega(t)$, $\Psi(u,t) = \psi(u)q(t)$, where φ and ψ are Orlicz functions with finite values such that φ satisfies the Δ_2-condition for all $u \in \mathbb{R}$, ψ is concave on $(0,\infty)$ and ω and q are nonnegative measurable functions. The following conditions are equivalent:*

(i) $\mathcal{L}(L^{\Phi}(\Omega), L^{\Psi}(\Omega)) = \mathcal{L}(E^{\Phi}(\Omega), E^{\Psi}(\Omega)) \neq \{0\}$,

(ii) $\mathcal{L}(L^{\varphi}(\Omega), L^{\psi}(\Omega)) = \mathcal{L}(E^{\varphi}(\Omega), E^{\psi}(\Omega)) \neq \{0\}$,

(iii) $\limsup\limits_{u\to\infty} \frac{\psi(u)}{\varphi(u)} < \infty$,

(iv) *there exist a measurable subset $A \subset \Omega$ such that the inclusion operator $i : L^{\varphi}(A) \to L^{\psi}(A)$ is continuous.*

Proof. Let $B_n = \{t \in \Omega : \frac{1}{n} \le \omega(t) \le n, \frac{1}{n} \le q(t) \le n\}$. Since the weights ω and q are bounded on the set B_n, we have $L^{\Phi}(B_n) = L^{\varphi}(B_n)$ and $L^{\Psi}(B_n) = L^{\psi}(B_n)$. Moreover, the F-norms $\|\cdot\|_{\Phi}$ and $\|\cdot\|_{\varphi}$ (respectively, $\|\cdot\|_{\Psi}$ and $\|\cdot\|_{\psi}$) reduced to $L^{\Phi}(B_n)$ (respectively, to $L^{\Psi}(B_n)$) are equivalent. Thus

$$\mathcal{L}(L^{\varphi}(B_n), L^{\psi}(B_n) = \mathcal{L}(L^{\Phi}(B_n), L^{\Psi}(B_n)) \tag{3.1}$$

for all $n \in \mathbb{N}$. Evidently $\bigcup_{n\in\mathbb{N}} B_n = \Omega$ and $B_n \uparrow \Omega$.

(i)⇒(ii). Let $T \in \mathcal{L}(L^{\varphi}(\Omega), L^{\psi}(\Omega)) \setminus \{0\}$. Take $f_0 \in L^{\varphi}(\Omega) \setminus \{0\}$ with $T(f_0) \neq 0$. By Δ_2-condition, $m_{\varphi}(f) < \infty$.

Since $B_n \uparrow \Omega$ and $|f\chi_{B_n}(t) - f(t)| \le |f(t)|$ for all $t \in \Omega$, by Lebesgue Theorem we get $m_{\varphi}(f\chi_{B_n} - f) \to 0$. Again, by the Δ_2-condition, we have $||f\chi_{B_n} - f||_{\varphi} \to 0$. By the continuity of T, $||T(f\chi_{B_n}) - T(f)||_{\Psi} \to 0$. Since $T(f) \neq 0$, we have $T(f\chi_{B_n}) \neq 0$ for sufficiently large n. By definition of the sets B_n, passing to a larger index if necessary, we can assume that $T(f_0\chi_{B_p})\chi_{B_p} \neq 0$ as well.

Define $T_0 : L^{\Phi}(B_p) \to L^{\Psi}(B_p)$ by $T_0(f) = T(f)\chi_{B_p}$. Applying (3.1), it is easy to verify that T_0 is continuous. Let $T_1 : L^{\Phi}(\Omega) \to L^{\Phi}(B_p)$, $T_2 : L^{\Psi}(B_p) \to L^{\Psi}(\Omega)$ be defined by $T_1(f) = f\chi_{B_p}$, $T_2(f) = f$. Then the operator $S : L^{\Phi}(\Omega) \to L^{\Psi}(\Omega)$ defined by $S = T_2 \circ T_0 \circ T_1$ is continuous and nonzero, because

$$S(f_0) = T_2(T_0(T_1(f_0))) = T(f_0\chi_{B_p})\chi_{B_p} \neq 0.$$

The implication (ii)⇒(i) can be proved analogously. The equivalence (ii)⇔(iii) follows from Theorem 1.4 and (iii)⇔(iv) follows from Corollary 1.5. ∎

Theorem 3.2. *Let Φ be a locally integrable continuous Musielak-Orlicz function satisfying the Δ_2-condition and let Ψ be a weighted Orlicz function defined by $\Psi(u,t) = \psi(u)q(t)$ such that the Orlicz function ψ takes finite values and it is concave on $(0,\infty)$. Assume that, for all $t \in \Omega$, there exists the limit $\lim\limits_{u\to\infty} \frac{\psi(u)}{\Phi(u,t)}$. The following conditions are equivalent:*

(*i*) $\mathcal{L}(L^\Phi(\Omega), L^\Psi(\Omega)) = \mathcal{L}(E^\Phi(\Omega), E^\Psi(\Omega)) \neq \{0\}$,

(*ii*) $\mathcal{L}(L^\Phi(\Omega), L^\psi(\Omega)) = \mathcal{L}(E^\Phi(\Omega), E^\psi(\Omega)) \neq \{0\}$,

(*iii*) *there exist a measurable subset $A \subset \Omega$ such that the inclusion operator $i : L^\Phi(A) \to L^\psi(A)$ is continuous.*

Proof. Let $B_n = \{t \in \Omega : \frac{1}{n} \leq q(t) \leq n\}$. Then $B_n \uparrow \Omega$ and there exists $n_0 \in \mathbb{N}$ such that $\mu(B_{n_0}) > 0$. Since on the set B_n the weight q is bounded, we have $L^\Psi(B_n) = L^\psi(B_n)$ and the norms $\|\cdot\|_\Psi$, $\|\cdot\|_\psi$ restricted to $L^\Psi(B_n)$ are equivalent for all $n \geq n_0$.

(i)⇒(ii). Let $T \in \mathcal{L}(L^\Phi(\Omega), L^\Psi(\Omega)) \setminus \{0\}$ and take $f \in L^\Phi(\Omega)$ with $T(f) \neq 0$. Since $B_n \uparrow \Omega$, $T(f)\chi_{B_n} \neq 0$ for some $n \geq 0$. Note that

$$T(f)\chi_{B_n} \in L^\Psi(B_n) = L^\psi(B_n) \subset L^\psi(\Omega).$$

Thus the operator $T_0 : L^\Phi(\Omega) \to L^\psi(\Omega)$ defined by $T_0(f) = T(f)\chi_{B_n}$ is nonzero. Since $\|T_0(f)\|_\psi \approx \|T(f)\chi_{B_n}\|_\Psi \leq \|T(f)\|_\Psi$, T_0 is continuous as well.

(ii)⇒(i). This implication can be proved in an analogous way as the implication (i)⇒(ii) - it suffices to replace Ψ with ψ and ψ with Ψ, respectively.

(ii)⇒(iii). Let $T \in \mathcal{L}(L^\Phi(\Omega), L^\psi(\Omega)) \setminus \{0\}$. and take $f \in L^\Phi(\Omega)$ with $T(f) \neq 0$. By Δ_2-condition, $m_\Phi(f) < \infty$. Since $B_n \uparrow \Omega$ and $|f\chi_{B_n}(t) - f(t)| \leq |f(t)|$ for all $t \in \Omega$, by Lebesgue Theorem we get $m_\Phi(f\chi_{B_n} - f) \to 0$. Again, by the Δ_2-condition, we have $||f\chi_{B_n} - f||_\Phi \to 0$. By the continuity of T it follows that $||T(f\chi_{B_n}) - T(f)||_\Psi \to 0$. Since $T(f) \neq 0$, we have $T(f\chi_{B_p}) \neq 0$ for sufficiently large $p \in \mathbb{N}$. Since $B_n \uparrow \Omega$, we can assume that $T(f\chi_{B_p})\chi_{B_p} \neq 0$ as well.

Hence the operator $T_0 : L^\Phi(B_p) \to L^\psi(B_p)$, $\quad T_0(f) = T(f)\chi_{B_p}$ is nonzero, linear and continuous. By Theorem 1.6 there exists a set $A \in \Sigma$, $A \subset B_p$, $0 < \mu(A) < \infty$ such that $\lim\limits_{v\to\infty} \frac{\psi(v)}{\Phi(v,t)} < \infty$ for every $t \in A$. By Lemma 2.3, $\mathcal{L}(L^\Phi(\Omega), L^\psi(\Omega)) \neq \{0\}$.

(iii)⇒(ii). Since Φ is locally integrable, $L^\Phi(A) \neq \{0\}$. Thus (ii) follows from Lemma 2.1. ∎

4. General Case of Musielak-orlicz Spaces

In this section we look for conditions on the functions Φ and Ψ that will imply the inverse theorem to Theorem 4.1 in the real valued Musielak-Orlicz function case. We will consider the spaces of finite elements E^Φ instead of the whole spaces L^Φ. Although the Theorem 2.4 provides nice equivalent conditions for existence of nonzero, linear and continuous operator, the assumption that the limit $\lim_{u\to\infty} \frac{\psi(u)}{\Phi(u,t)}$ exists for all $t \in \Omega$ is very strong. We will give a theorem that will use a weaker condition than the above one.

R. Płuciennik and M. Wisła presented the necessary condition for a nonexistence of any linear continuous operators from $E^\Phi(T,\Sigma,\mu)$ into $E^\Psi(S,\Xi,\nu)$, where (T,Σ,μ) and (S,Ξ,ν) are two measure spaces with the σ-finite and atomless measures, and $\Phi : X \times T \to [0,\infty]$ and $\Psi : Y \times S \to [0,\infty]$ are Musielak-Orlicz vector valued functions, where X is p-normed space, and Y is q-normed space. In order to simplify the notation, we will recall that theorem in the case of real-valued Musielak-Orlicz functions only. Let A, C be arbitrary Σ-measurable subsets of Ω. Define

$$\widetilde{\varphi}_A(u) = \begin{cases} [\mathrm{supess}_{t\in A}\,\Phi(\frac{1}{u},t)]^{-1} & \text{for } u \neq 0 \\ 0 & \text{for } u = 0, \end{cases}$$

$$\widetilde{\psi}_C(u) = \mathrm{supess}_{t\in C} \sup_{v\neq 0} \frac{\min\{1,|uv|\}}{\Psi(v,t)}.$$

In the case when Ψ is an Orlicz function we have $\widetilde{\psi}_C(u) = [\Psi(\frac{1}{u})]^{-1}$ for $u > 0$. Both functions $\widetilde{\varphi}_A$ and $\widetilde{\psi}_C$ are even, nondecreasing on $(0,\infty)$ and vanish at 0. But it can happen that they are not pregenfunctions. By Lemma 2.2 and 2.3 in [8] $\widetilde{\varphi}_A$ and $\widetilde{\psi}_C$ are pregenfunctions if and only if there exist $a, b, c > 0$ such that

$$0 < \mathrm{supess}_{t\in A}\,\Phi(c,t) \;<\; \infty$$

$$\mathrm{supess}_{t\in C} \sup_{0<|v|<a} \frac{|v|}{\Psi(v,t)} \;<\; \infty \quad \textit{and} \quad \mathrm{infess}_{t\in C}\,\Psi(b,t) > 0.$$

Note that if $\widetilde{\varphi}_A$ is a pregenfunction, then $L^\Phi(A) \neq \{0\}$. But in order to assure that $E^\Phi(A) \neq \{0\}$ we have to assume that Φ is locally integrable.

Theorem 4.1. ([8], 3.1). *Assume that for every $\delta > 0$ there exist divisions: $A_0, A_1 \ldots$ and $C_0, C_1, \ldots$ of the set Ω such that the families $\{A_i\}$ and $\{C_j\}$ consist of pairwise disjoint sets, $\mu(A_0) < \delta$, $\mu(C_0) < \delta$ and $\widetilde{\varphi}_{A_i}$, $\widetilde{\psi}_{C_j}$ are pregenfunctions for $i, j = 1, 2, \ldots$. If $\liminf_{u\to 0} \frac{\widetilde{\psi}_{C_j}(u)}{\widetilde{\varphi}_{A_i}(du)} = 0$ for all $d > 0$ and $i, j = 1, 2, \ldots$, then $\mathcal{L}(E^\Phi(\Omega), E^\Psi(\Omega)) = \{0\}$.*

Let Ψ is a weighted Orlicz function, say $\Psi(u,t) = \psi(u)q(t)$. If the function q is bounded on a set C then $\widetilde{\psi}_C$ a pregenfunction if and only if there exist $a > 0$ and $K > 0$ such that $|v| \leq K\psi(v)$ for all $0 < v < a$. This condition is satisfied if, for example, the function ψ is concave on the interval $(0, a)$.

We shall say that a Musielak-Orlicz function Φ is of bounded growth on Ω at infinity, if for every $\varepsilon > 0$ we can find a measurable set T_ε such that $\mu(\Omega \setminus T_\varepsilon) < \varepsilon$ and

$$\limsup_{u\to\infty} \frac{\operatorname{supess}_{t\in T_\varepsilon} \Phi(u,t)}{\operatorname{infess}_{t\in T_\varepsilon} \Phi(u,t)} < \infty. \tag{4.1}$$

Lemma 4.2. *If a Musielak-Orlicz function Φ is of bounded growth on Ω at infinity, then $\widetilde{\varphi}_A$ is a pregenfunction for every $A \subset T_\varepsilon$.*

Proof. By (4.1), we can find a set T_ε and constants $M > 0$ and $a > 0$ such that

$$0 < \operatorname{supess}_{t\in A} \Phi(a,t) \leq \operatorname{supess}_{t\in T_\varepsilon} \Phi(a,t) \leq M \operatorname{infess}_{t\in T_\varepsilon} \Phi(a,t) < \infty$$

for every $A \subset T_\varepsilon$. ■

Theorem 4.3. *Let Φ be a locally integrable continuous Musielak-Orlicz function of bounded growth on Ω at infinity and let Ψ be a weighted Orlicz space, $\Psi(u,t) = \psi(u)q(t)$, where ψ is a concave Orlicz function on $(0,\infty)$. The following conditions are equivalent:*

(i) $\mathcal{L}(E^\Phi(\Omega), E^\Psi(\Omega)) \neq \{0\}$,

(ii) $\mathcal{L}(E^\Phi(\Omega), E^\psi(\Omega)) \neq \{0\}$,

(iii) *there exists a set $A \in \Sigma$ of positive and finite measure such that*

$$\limsup_{u\to\infty} \frac{\psi(u)}{\operatorname{infess}_{t\in A} \Phi(u,t)} < \infty, \tag{4.2}$$

(iv) *there exists a set $A \in \Sigma$ of positive and finite measure such that the inclusion operator $i : E^\Phi(A) \to E^\psi(A)$ is well defined and continuous,*

Proof. The proof of the equivalence (i)⇔(ii) goes analogously to the proof of the part (i)⇔(ii) of the Theorem 3.2.

(ii)⇒(iii) Let $\varepsilon > 0$ and put $M = \limsup\limits_{u\to\infty} \frac{\operatorname{supess}_{t\in T_\varepsilon} \Phi(u,t)}{\operatorname{infess}_{t\in T_\varepsilon} \Phi(u,t)} < \infty$. Since the measure μ is σ-finite, we can find a countable family (A_j) of pairwise disjoint measurable subsets of T_ε of finite measures such that $\bigcup_{j=1}^\infty A_j = T_\varepsilon$. By Lemma 4.2 and by concavity of ψ, each function $\widetilde{\varphi}_{A_j}$ and $\widetilde{\psi}_{A_k}$ is a pregenfunction.

Suppose that $\limsup_{u\to\infty} \frac{\psi(u)}{\operatorname{infess}_{t\in A_j} \Phi(u,t)} = \infty$ for each $j \in \mathbb{N}$. Fix any $1 \le j, k < \infty$. Then there exists a sequence (u_n) such that $u_n \to \infty$ and

$$\psi(u_n) \ge 2^n \operatorname{infess}_{t\in A_k} \Phi(u_n, t).$$

For any $0 < d < \infty$ put $d_1 = \min\{d, 1\}$. Applying the concavity of ψ, we obtain

$$\begin{aligned}
\liminf_{v\to 0} \frac{\widetilde{\psi}_{A_k}(v)}{\widetilde{\varphi}_{A_j}(dv)} &\le \liminf_{v\to 0} \frac{\widetilde{\psi}_{A_k}(v)}{\widetilde{\varphi}_{A_j}(d_1 v)} \le \lim_{n\to\infty} \frac{\widetilde{\psi}_{A_k}(\frac{1}{d_1 u_n})}{\widetilde{\varphi}_{A_j}(\frac{1}{u_n})} \\
&= \lim_{n\to\infty} \frac{\operatorname{supess}_{t\in A_j} \Phi(u_n, t)}{\psi(d_1 u_n)} \\
&\le \lim_{n\to\infty} \frac{1}{2^n d_1} \cdot \frac{\operatorname{supess}_{t\in A_j} \Phi(u_n, t)}{\operatorname{infess}_{t\in A_k} \Phi(u_n, t)} \le \lim_{n\to\infty} \frac{M}{2^n d_1} = 0.
\end{aligned}$$

By Theorem 4.1 we get $\mathcal{L}(E^\Phi(\Omega), E^\psi(\Omega)) = \{0\}$, and we arrived at a contradiction. Thus (ii) holds true.

(iii)⇒(iv) By (4.2), we can find $A \in \Sigma$, $K > 0$, $u_0 > 0$ such that $0 < \mu(A) < \infty$ and

$$\psi(u) \le K \cdot \operatorname{infess}_{t\in A} \Phi(u, t) \le K\Phi(u, t)$$

for all $u \ge u_0$ and $t \in A$. Then, $\psi(u) \le K\Phi(u, t) + \psi(u_0)$ for every $u \in \mathbb{R}$ and $t \in A$, so $L^\Phi(A) \subset L^\psi(A)$ and $E^\Phi(A) \subset E^\psi(A)$. By continuity of Φ and by Theorem 1.2, the inclusion operator $i : L^\Phi(A) \to L^\psi(A)$ is continuous. Hence the inclusion operator $i_0 : E^\Phi(A) \to E^\psi(A)$ is continuous as well.

(iv)⇒(ii) By local integrability of Φ, $E^\Phi(A) \neq \{0\}$. Thus, by Lemma 2.2, $\mathcal{L}(E^\Phi(\Omega), E^\psi(\Omega)) \neq \{0\}$. ■

Remark. The boundedness growth at infinity of the function Φ and the concavity of ψ was used in the proof of the implication (i)⇒(ii) only.

Example 4.4. Let $\Omega = (0, 1)$, $\Phi(u, t) = |u|^t$ and $\psi(u) = \sqrt{|u|}$. Then Φ is a locally integrable Musielak-Orlicz function and ψ is concave on $(0, \infty)$. Moreover, $\operatorname{infess}_{t\in(0,1)} \Phi(u, t) = u$ and $\limsup_{u\to\infty} \frac{\psi(u)}{\operatorname{infess}_{t\in(0,1)} \Phi(u,t)} = \lim_{u\to\infty} \frac{\sqrt{u}}{u} = 0 < \infty$. Hence, by Theorem 4.3, $\mathcal{L}(E^\Phi(\Omega), E^\psi(\Omega)) \neq \{0\}$. Moreover, since $\Phi, \psi \in \Delta_2$, we have $\mathcal{L}(L^\Phi(\Omega), L^\psi(\Omega)) \neq \{0\}$ as well.

Example 4.5. Let $\Omega = [0, 1]$, $\Phi(u, t) = u^\alpha t$ and $\psi(u) = u^\beta$, where $0 < \alpha < \infty$ and $0 < \beta \le 1$. Then Φ is a locally integrable, continuous Musielak-Orlicz function and ψ is concave. Moreover, $\Phi, \psi \in \Delta_2$ and Φ is of bounded growth on Ω at ∞. Indeed, for any $\varepsilon > 0$, putting $T_\varepsilon = (\varepsilon/2, 1]$, we have $\mu(\Omega \setminus T_\varepsilon) < \varepsilon$ and

$$\limsup_{u\to\infty} \frac{\operatorname{supess}_{t\in(\varepsilon/2,1]} u^\alpha t}{\operatorname{infess}_{t\in(\varepsilon/2,1]} u^\alpha t} = \lim_{u\to\infty} \frac{u^\alpha}{u^\alpha \varepsilon/2} = \frac{2}{\varepsilon} < \infty.$$

Thus, by Theorem 4.3 $\mathcal{L}(E^{\Phi}(\Omega), E^{\psi}(\Omega)) \neq \{0\}$ if and only if there exist $a > 0$ such that

$$\limsup_{u\to\infty} \frac{\psi(u)}{\operatorname{infess}_{t\geq a} \Phi(u,t)} = \lim_{u\to\infty} \frac{u^{\beta}}{u^{\alpha} a} = \lim_{u\to\infty} a^{-1} u^{\beta-\alpha} < \infty$$

and this is true if and only if $0 < \beta \leq \alpha < \infty$.

REFERENCES

1. L. Drewnowski, Compact operators on Musielak-Orlicz spaces, *Comment. Math.*, **27** (1988), 225-232.
2. A. Kalindé, Operateurs lineaires continus entre des espaces de Musielak-Orlicz et d'Orlicz non localement convexes, *Bull. Soc. Math. de Belgique*, **37(1)** Ser. B, (1985), 27-36.
3. N. Kalton, *An F-Space Sampler*, Cambridge University Press, 1984.
4. J. Musielak, *Orlicz Spaces and Modular Spaces*, Vol. 1034, Springer-Verlag, 1983.
5. J. Musielak and W. Orlicz, On modular spaces, *Studia Math.*, **18** (1959), 49-65.
6. W. Orlicz, Über eine gewisse Klasse von Räumen von Typus B, *Bull. Acad. Polon. A*, (1932), 207-220.
7. W. Orlicz, A note on modular spaces I, *Bull. Acad. Sci. Math.*, **9** (1961), 157-162.
8. R. Płuciennik and M. Wisła, Linear operators between Musielak-Orlicz spaces of vector valued functions, *Bull. Soc. Math. Belg.*, **40(1)** (1988), Ser. B, 95-109.
9. S. Rolewicz, *Metric Linear Spaces*, PWN, Warsaw, 1972.
10. I. V. Sragin, Conditions for the inbedding of classes of sequences and their consequences, *Mat. Zamietki*, **20(5)** (1976), 681-692, (in Russian).
11. P. Turpin, Operateurs lineaires entre espaces d'Orlicz non localement convexes, *Studia Math.*, **46** (1973), 153-165.
12. M. Wisła, Continuity of the identity embedding of some Orlicz spaces, *Comment. Math.*, **24** (1984), 343-356.

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