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PERIODIC SOLUTIONS FOR AN ORDINARY P-LAPLACIAN SYSTEM

Xingyong Zhang and Xianhua Tang*

Abstract. In this paper, some existence theorems are obtained for multiple solutions of an ordinary p-Laplacian system by using the minimax principle. Our results generalize and improve those in the literatures.

1. Introduction and Main Results

Consider the second order Hamiltonian system

$$\begin{cases} \ddot{u}(t) + \nabla F(t, u(t)) = e(t), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $e \in L^1(0, T; \mathbf{R}^N)$ and $F : [0, T] \times \mathbf{R}^N \to \mathbf{R}$ satisfies the following assumption:

(A) *$F(t, x)$ is measurable in t for every $x \in \mathbf{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbf{R}^+, \mathbf{R}^+)$ and $b \in L^1([0, T], \mathbf{R}^+)$ such that*

$$|F(t, x)| \le a(|x|)b(t), \quad |\nabla F(t, x)| \le a(|x|)b(t)$$

for all $x \in \mathbf{R}^N$ and a.e. $t \in [0, T]$.

Moreover, we suppose that $F(t, x)$ is T_i-periodic in x_i, $1 \le i \le r$, that is

$$F\left(t, x + \sum_{i=1}^{r} k_i T_i e_i\right) = F(t, x) \tag{1.2}$$

for a.e.$t \in [0, T]$, all integers $k_i, 1 \le i \le r$, where e_i is the canonical basis of $\mathbf{R}^N$ for $1 \le i \le N$.

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*Corresponding author.

When $e(t) \equiv 0$, it has been proved that problem (1.1) has at least one solution by the least action principle and the minimax methods [1,4-18]. Many solvability conditions are given, such as the coercive condition [1], the periodicity condition [14]; the convexity condition [4]; the subadditive condition [9]. Especially, Tang and Wu [13] obtained the following result.

Theorem A. Suppose that (1.2) holds and

$$\int_0^T e(t)dt = 0. \tag{1.3}$$

Assume that there exist $f, g \in L^1(0,T;\mathbf{R}^+)$ and $\alpha \in [0,1)$ such that

$$|\nabla F(t,x)| \le f(t)|x|^\alpha + g(t) \tag{1.4}$$

and

$$|x|^{-2\alpha} \int_0^T F(t,x)dt \to +\infty \quad (\text{ or } \quad -\infty), \tag{1.5}$$

as x tends to infinity in $0 \times \mathbf{R}^{N-r}$. Then problem (1.1) has at least $r+1$ geometrically distinct solutions in $W_T^{1,2}$.

Let

$$\begin{aligned} F(t,x) = (0.5T-t) & \left(r+1+\sin^2 x_1 + \cdots + \sin^2 x_r + \frac{1}{2}\sum_{j=r+1}^{N} x_j^2 \right)^{7/8} \\ & + \frac{T^4}{16}\left(r+1+\sin^2 x_1 + \cdots + \sin^2 x_r + \frac{1}{2}\sum_{j=r+1}^{N} x_j^2 \right)^{3/4}, \end{aligned} \tag{1.6}$$

where $x = (x_1, x_2, \cdots, x_N)^T \in \mathbf{R}^N$. We can prove F satisfies (1.2) and (1.4) with $\alpha = 3/4$. However, $F(t,x)$ does not satisfy (1.5) neither for the case $+\infty$ nor the case $-\infty$. In fact,

$$\lim_{|x|\to\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt = \frac{T^5}{2^{19/4}}$$

as $x \in 0 \times \mathbf{R}^{N-r}$. The details can be seen in Example 4.3 in Section 4. Hence, the above example shows that it is valuable to improve (1.5).

In this paper, by using the minimax principle, we study more general ordinary p-Laplacian system

$$\begin{cases} (|u'(t)|^{p-2}u'(t))' + \nabla F(t,u(t)) = e(t), \quad \text{a.e. } t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.7}$$

where $p > 1, q > 1$ with p and q satisfying $1/p + 1/q = 1$, and $T,\ e(t)$ and $F(t, x)$ are the same as problem (1.1). Our main results are the following theorems.

Theorem 1.1. *Suppose that (1.2) and (1.3) hold. Assume that there exist $f, g \in L^1(0, T; \mathbf{R}^+)$ and $\alpha \in (0, p-1)$ such that*

$$|\nabla F(t, x)| \le f(t)|x|^\alpha + g(t) \tag{1.8}$$

for a.e. $t \in [0, T]$ and all $x \in \mathbf{R}^N$ and

$$\liminf_{|x|\to\infty} |x|^{-q\alpha} \int_0^T F(t, x)dt > \frac{(p+1)T}{p(q+1)} \left(\int_0^T f(t)dt\right)^q \tag{1.9}$$

as $x \in 0 \times \mathbf{R}^{N-r}$. Then problem (1.7) has at least $r + 1$ geometrically distinct solutions in $W_T^{1,p}$.

Theorem 1.2. *Assume that (1.2), (1.3) and (1.8) hold, and that*

$$\limsup_{|x|\to\infty} |x|^{-q\alpha} \int_0^T F(t, x)dt < -\frac{T}{q(q+1)} \left(\int_0^T f(t)dt\right)^q \tag{1.10}$$

as $x \in 0 \times \mathbf{R}^{N-r}$. Then problem (1.7) has at least $r + 1$ geometrically distinct solutions in $W_T^{1,p}$.

By Theorem 1.1 and Theorem 1.2, it is easy to obtain the following Corollaries.

Corollary 1.1. *Suppose that (1.2), (1.3) and (1.8) hold. If*

$$|x|^{-q\alpha} \int_0^T F(t, x)dt \to +\infty \tag{1.11}$$

as x tends to infinity in $0 \times \mathbf{R}^{N-r}$, then problem (1.7) has at least $r+1$ geometrically distinct solutions in $W_T^{1,p}$.

Corollary 1.2.
Suppose that (1.2), (1.3) and (1.8) hold. If

$$|x|^{-q\alpha} \int_0^T F(t, x)dt \to -\infty \tag{1.12}$$

as x tends to infinity in $0 \times \mathbf{R}^{N-r}$, then problem (1.7) has at least $r+1$ geometrically distinct solutions in $W_T^{1,p}$.

Corollary 1.3. *Assume that* (1.2), (1.3) *and* (1.4) *with* $\alpha \in (0,1)$ *hold, and that*

$$\liminf_{|x|\to\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt > \frac{T}{2}\left(\int_0^T f(t)dt\right)^2 \tag{1.13}$$

as $x \in 0 \times \mathbf{R}^{N-r}$. *Then problem (1.1) has at least* $r+1$ *geometrically distinct solutions in* $W_T^{1,2}$.

Corollary 1.4. *Assume that* (1.2), (1.3) *and* (1.4) *with* $\alpha \in (0,1)$ *hold, and that*

$$\limsup_{|x|\to\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt < -\frac{T}{6}\left(\int_0^T f(t)dt\right)^2 \tag{1.14}$$

as $x \in 0 \times \mathbf{R}^{N-r}$. *Then problem* (1.1) *has at least* $r+1$ *geometrically distinct solutions in* $W_T^{1,2}$.

Remark 1.1. When $\alpha = 0$, Corollary 1.1 and Corollary 1.2 still hold. The reasons can be easily found in the process of proofs of Theorem 1.1 and Theorem 1.2. Therefore, Theorem A is the special case of Corollary 1.1 and Corollary 1.2.

For the Sobolev space $\tilde{W}_T^{1,2}$, we have the following sharp estimates(see Proposition 1.3 in [5]):

$$\int_0^T |u(t)|^2 dt \le \frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2 dt \tag{1.15}$$

(Wirtinger's inequality) and

$$\|u\|_\infty^2 \le \frac{T}{12}\int_0^T |\dot{u}(t)|^2 dt \tag{1.16}$$

(Sobolev's inequality). By the above two inequality, we can prove the following better results than Corollary 1.3 and Corollary 1.4.

Theorem 1.3. *In Corollary 1.3, (1.13) is replaced by*

$$\liminf_{|x|\to\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt > \frac{T}{8}\left(\int_0^T f(t)dt\right)^2 \tag{1.17}$$

as $x \in 0 \times \mathbf{R}^{N-r}$, *the conclusion in Corollary 1.3 still holds.*

Theorem 1.4. *In Corollary 1.4, (1.14) is replaced by*

$$\limsup_{|x|\to\infty} |x|^{-2\alpha}\int_0^T F(t,x)dt < -\frac{T}{24}\left(\int_0^T f(t)dt\right)^2 \tag{1.18}$$

as $x \in 0\times \mathbf{R}^{N-r}$*, the conclusion in Corollary 1.4 still holds.*

Remark 1.2. Obviously, when $\alpha \in (0,1)$, Theorem 1.3 and Theorem 1.4 improve Theorem A.

2. Preliminaries

Let

$$W_T^{1,p} = \{u : [0,T] \to \mathbf{R}^N \mid u(t) \text{ is absolutely continuous on } [0,T],\ u(0)=u(T) \text{ and } \dot{u} \in L^p(0,T;\mathbf{R}^N)\}.$$

Then $W_T^{1,p}$ is a Banach space with the norm defined by

$$\|u\| = \left[\int_0^T |u(t)|^p dt + \int_0^T |\dot{u}(t)|^p dt\right]^{1/p}, \qquad u \in W_T^{1,p}.$$

It follows from [5] that $W_T^{1,p}$ is also reflexive and uniformly convex $Banach$ space. From [2], one can know that a locally uniformly convex Banach space has the Kadec-Klee property, that is for any sequence $\{u_n\}$ satisfying $u_n \rightharpoonup u$ weakly in Banach space X and $\|u_n\| \to \|u\|$, one has $u_n \to u$ strongly in X. This property will be used later.

Lemma 2.1. *Let* $a > 0,\ b,\ c \geq 0,\ \varepsilon > 0.$

(i) *If* $\alpha \in (0,1]$*, then*

$$(a+b+c)^\alpha \leq a^\alpha + b^\alpha + c^\alpha; \tag{2.1}$$

(ii) *If* $\alpha \in (1,+\infty)$*, then there exists* $B(\varepsilon) > 1$ *such that*

$$(a+b+c)^\alpha \leq (1+\varepsilon)a^\alpha + B(\varepsilon)b^\alpha + B(\varepsilon)c^\alpha. \tag{2.2}$$

Proof. It is easy to verify (i). In the sequel, we only prove (ii). Since

$$\lim_{x\to+\infty} \frac{x^{\alpha/(\alpha-1)}-1}{\left[x^{1/(\alpha-1)}-1\right]^\alpha} = 1,$$

it follows that there exists a constant $M = M(\varepsilon) > 1$ such that

$$\frac{M^{\alpha/(\alpha-1)} - 1}{\left[M^{1/(\alpha-1)} - 1\right]^{\alpha}} < 1 + \varepsilon.$$

Set

$$f(t) = (1+t)^{\alpha} - Mt^{\alpha}, \qquad t \in [0,1].$$

Then

$$f(t) \le \frac{M^{\alpha/(\alpha-1)} - 1}{\left[M^{1/(\alpha-1)} - 1\right]^{\alpha}} < 1 + \varepsilon, \qquad t \in [0,1].$$

It follows that

$$(1+t)^{\alpha} \le 1 + \varepsilon + Mt^{\alpha}, \qquad t \in [0,1]. \tag{2.3}$$

If $a \le b + c$, then

$$(a+b+c)^{\alpha} \le 2^{\alpha}(b+c)^{\alpha} \le 2^{2\alpha-1}b^{\alpha} + 2^{2\alpha-1}c^{\alpha}.$$

This shows that (2.2) holds. If $a > b + c$, then by (2.3), we have

$$\begin{aligned}(a+b+c)^{\alpha} &\le a^{\alpha}\left(1 + \frac{b+c}{a}\right)^{\alpha} \le a^{\alpha}\left(1 + \varepsilon + M\frac{(b+c)^{\alpha}}{a^{\alpha}}\right) \\ &\le (1+\varepsilon)a^{\alpha} + 2^{\alpha-1}Mb^{\alpha} + 2^{\alpha-1}Mc^{\alpha}.\end{aligned}$$

This shows that (2.2) also holds. The proof is complete.

Lemma 2.2. *Let* $u \in W_T^{1,p}$ *and* $\int_0^T u(t)dt = 0$. *Then*

$$\|u\|_{\infty} \le \left(\frac{T}{q+1}\right)^{1/q} \left(\int_0^T |\dot{u}(s)|^p ds\right)^{1/p}, \tag{2.4}$$

and

$$\int_0^T |u(s)|^p ds \le \frac{T^p \Theta(p,q)}{(q+1)^{p/q}} \int_0^T |\dot{u}(s)|^p ds, \tag{2.5}$$

where

$$\Theta(p,q) = \int_0^1 \left[s^{q+1} + (1-s)^{q+1}\right]^{p/q} ds.$$

Proof. Fix $t \in [0,T]$. For every $\tau \in [0,T]$, we have

$$u(t) = u(\tau) + \int_{\tau}^{t} \dot{u}(s)ds. \tag{2.6}$$

Set

$$\phi(s) = \begin{cases} s, & 0 \le s \le t, \\ T - s, & t \le s \le T. \end{cases}$$

Integrating (2.6) over $[0, T]$ and using the Hölder inequality, we obtain

$$\begin{aligned} T|u(t)| &= \left| \int_0^T u(\tau) d\tau + \int_0^T \int_\tau^t \dot{u}(s) ds d\tau \right| \\ &\le \int_0^t \int_\tau^t |\dot{u}(s)| ds d\tau + \int_t^T \int_t^\tau |\dot{u}(s)| ds d\tau \\ &= \int_0^t s|\dot{u}(s)| ds + \int_t^T (T-s)|\dot{u}(s)| ds \\ &= \int_0^T \phi(s)|\dot{u}(s)| ds \\ &\le \left(\int_0^T [\phi(s)]^q ds \right)^{1/q} \left(\int_0^T |\dot{u}(s)|^p ds \right)^{1/p} \\ &= \frac{1}{(q+1)^{1/q}} \left[t^{q+1} + (T-t)^{q+1} \right]^{1/q} \left(\int_0^T |\dot{u}(s)|^p ds \right)^{1/p}. \end{aligned} \tag{2.7}$$

Since $t^{q+1} + (T-t)^{q+1} \le T^{q+1}$ for $t \in [0, T]$, it follows from (2.7) that (2.4) holds. On the other hand, from (2.7), we have

$$\begin{aligned} T^p \int_0^T |u(t)|^p dt &\le \frac{1}{(q+1)^{p/q}} \left(\int_0^T |\dot{u}(s)|^p ds \right) \int_0^T \left[t^{q+1} + (T-t)^{q+1} \right]^{p/q} dt \\ &\le \frac{T^{1+p(q+1)/q}}{(q+1)^{p/q}} \left(\int_0^T |\dot{u}(s)|^p ds \right) \int_0^1 \left[s^{q+1} + (1-s)^{q+1} \right]^{p/q} ds \\ &= \frac{T^{2p}\Theta(p,q)}{(q+1)^{p/q}} \int_0^T |\dot{u}(s)|^p ds. \end{aligned}$$

It follows that (2.5) holds. The proof is complete.

Let

$$\tilde{W}_T^{1,p} = \left\{ u \in W_T^{1,p} \;\middle|\; \int_0^T u(t) dt = 0 \right\}.$$

It is easy to know that $\tilde{W}_T^{1,p}$ is a subset of $W_T^{1,p}$ and $W_T^{1,p} = \mathbf{R}^N \oplus \tilde{W}_T^{1,p}$. For $u \in W_T^{1,p}$, let $\bar{u} = \frac{1}{T} \int_0^T u(t) dt$ and $\tilde{u}(t) = u(t) - \bar{u}$. By Lemma 2.2, we have

$$\int_0^T |\tilde{u}(t)|^p dt \le \frac{T^p \Theta(p,q)}{(q+1)^{p/q}} \int_0^T |\dot{u}(t)|^p dt \quad \text{for every } u \in W_T^{1,p}, \tag{2.8}$$

and

$$\|\tilde{u}\|_\infty^p \leq \left(\frac{T}{q+1}\right)^{p/q} \int_0^T |\dot{u}(t)|^p dt \quad \text{for every } \ u \in W_T^{1,p}. \tag{2.9}$$

Hence,

$$\|\tilde{u}\|^p \leq \left[\frac{T^p \Theta(p,q)}{(q+1)^{p/q}} + 1\right] \int_0^T |\dot{u}(t)|^p dt \qquad \text{for every } u \in W_T^{1,p}. \tag{2.10}$$

We will use the following two lemmas to obtain the critical points of φ:

Lemma 2.3. ([3]). Let X be a Banach space and have a decomposition: $X = Y + Z$ where Y and Z are two subspaces of X with dim$Y < +\infty$. Let V be a finite-dimensional, compact C^2-manifold without boundary. Let $f : X \times V \to \mathbf{R}$ be a C^1-function and satisfy the (PS)-condition. Suppose that f satisfies

$$\inf_{u \in Z \times V} f(u) \geq a, \qquad \sup_{u \in S \times V} f(u) \leq b < a,$$

where $S = \partial D$, $D = \{u \in Y \mid \|u\| \leq R\}$, R, a and b are constants. Then the function f has at least cuplength(V)+1 critical points.

Lemma 2.4. (see Theorem 4.12 in [5]). Let $\varphi \in C^1(X, \mathbf{R})$ be a $G-$invariant functional satisfying the (PS)$_G$-condition. If φ is bounded from below and if the dimension r of the space generated by G is finite, then φ has at least $r+1$ critical orbits.

Let $\varphi : W_T^{1,p} \to \mathbf{R}$ be defined by

$$\varphi(u) = \frac{1}{p} \int_0^T |\dot{u}(t)|^p dt - \int_0^T F(t, u(t)) dt + \int_0^T (e(t), u(t)) dt. \tag{2.11}$$

Then φ is continuously differentiable and weakly lower semicontinuous in $W_T^{1,p}$ (see [5]). Moreover,

$$\begin{aligned} \langle \varphi'(u), v \rangle = & \int_0^T \left(|\dot{u}(t)|^{p-2} \dot{u}(t), \dot{v}(t)\right) dt \\ & - \int_0^T (\nabla F(t, u(t)), v(t)) dt + \int_0^T (e(t), v(t)) dt \end{aligned} \tag{2.12}$$

for $u, v \in W_T^{1,p}$. It is well known that the solutions of problem (1.7) correspond to the critical points of φ (see [5]).

Let

$$\hat{u}(t) = P\bar{u} + Q\bar{u} + \tilde{u}(t),$$

where

$$P\bar{u} = \sum_{i=r+1}^{N} (\bar{u}, e_i)e_i, \qquad Q\bar{u} = \sum_{i=1}^{r} [(\bar{u}, e_i) - k_i T_i]e_i,$$

and $k_i(1 \leq i \leq r)$ is the unique integer such that

$$0 \leq (\bar{u}, e_i) - k_i T_i < T_i. \tag{2.13}$$

Let

$$G = \left\{ \sum_{i=1}^{r} k_i T_i e_i \;\middle|\; k_i \text{ is an interger, } 1 \leq i \leq r \right\}, \tag{2.14}$$

$Y = \text{span}\,\{e_{r+1}, \cdots, e_N\}$, $Z = \tilde{W}_T^{1,p}$, $X = Y + Z$ and $V = \text{span}\,\{e_1, \cdots, e_r\}/G$ be isomorphic to the torus T^r, which is nothing but the torus T^r. Now define $\Psi : X \times T^r \to \mathbf{R}$ by

$$\Psi((y + z(t), v)) = \varphi(y + v + z(t)), \qquad \forall\ (y, z, v) \in Y \times Z \times T^r. \tag{2.15}$$

It is easy to verify that Ψ is continuously differentiable and that

$$\begin{aligned} &\langle \Psi'((y_1 + z_1(t), v_1)), (y_2 + z_2(t), v_2)\rangle \\ &= \langle \varphi'(y_1 + v_1 + z_1(t)), y_2 + v_2 + z_2(t)\rangle, \\ &\quad \forall\ (y_i, z_i, v_i) \in Y \times Z \times T^r, \quad i = 1, 2. \end{aligned} \tag{2.16}$$

By (1.2) and (1.3), we have

$$\begin{aligned} F(t, u(t)) &= F\left(t, \hat{u}(t) + \sum_{i=1}^{r} k_i T_i e_i\right) = F(t, \hat{u}(t)), \\ \nabla F(t, u(t)) &= \nabla F\left(t, \hat{u}(t) + \sum_{i=1}^{r} k_i T_i e_i\right) = \nabla F(t, \hat{u}(t)) \end{aligned}$$

and

$$\int_0^T (e(t), u(t))dt = \int_0^T \left(e(t), \hat{u}(t) + \sum_{i=1}^{r} k_i T_i e_i\right) dt = \int_0^T (e(t), \hat{u}(t))dt.$$

Hence $\varphi(u) = \varphi(\hat{u})$ and $\varphi'(u) = \varphi'(\hat{u})$.

3. Proofs of Theorems

For the sake of convenience, we denote that

$$M_1 = \int_0^T f(t)dt, \quad M_2 = \int_0^T g(t)dt, \quad M_3 = \left(\sum_{i=1}^r T_i^2\right)^{1/2}, \quad M_4 = \int_0^T |e(t)|dt.$$

Proof of Theorem 1.1. We divide our proof into two steps.

Step 1. we prove Ψ defined by (2.15) satisfies (PS)- condition.

First we assume that $\{(y_n + z_n, v_n)\} \subset X \times T^r$ is a (PS)- sequence for Ψ, that is $\{\Psi((y_n + z_n, v_n))\}$ is bounded and $\Psi'((y_n + z_n, v_n)) \to 0$, where $y_n \in Y,\ z_n = z_n(t) \in Z,\ v_n \in T^r$ for $n = 1, 2, \ldots$. Set

$$u_n = y_n + v_n + z_n(t), \qquad n = 1, 2, \ldots.$$

Then it is easy to see that

$$y_n = P\bar{u}_n, \quad v_n = Q\bar{u}_n, \quad z_n(t) = \tilde{u}_n(t), \qquad n = 1, 2, \ldots.$$

It follows from (2.15) and (2.16) that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$.

By (1.9), we can choose constants $\varepsilon > 0$ and $a_1 > [T/(q+1)]^{1/q}$ such that

$$\liminf_{|x|\to\infty} |x|^{-q\alpha} \int_0^T F(t,x)dt > \left[\frac{1}{p} + \frac{1}{a_1}\left(\frac{T}{q+1}\right)^{1/q}\right] [(1+\varepsilon)a_1 M_1]^q \tag{3.1}$$

as $x \in 0 \times \mathbf{R}^{N-r}$. It follows from (1.8), (2.9), (2.11) and Young's inequality that for all $u \in W_T^{1,p}$

$$\begin{aligned}
&\left|\int_0^T (\nabla F(t,u(t)), \tilde{u}(t))dt\right| \\
&= \left|\int_0^T (\nabla F(t, P\bar{u} + Q\bar{u} + \tilde{u}(t)), \tilde{u}(t))dt\right| \\
&\le \int_0^T f(t)|P\bar{u} + Q\bar{u} + \tilde{u}(t)|^\alpha |\tilde{u}(t)|dt + \int_0^T g(t)|\tilde{u}(t)|dt \\
&\le (|P\bar{u}| + |Q\bar{u}| + \|\tilde{u}\|_\infty)^\alpha \|\tilde{u}\|_\infty \int_0^T f(t)dt + \|\tilde{u}\|_\infty \int_0^T g(t)dt \qquad (3.2)\\
&\le M_1 \left[(1+\varepsilon)|P\bar{u}|^\alpha + B(\varepsilon)|Q\bar{u}|^\alpha + B(\varepsilon)\|\tilde{u}\|_\infty^\alpha\right] \|\tilde{u}\|_\infty + M_2\|\tilde{u}\|_\infty \\
&\le \frac{1}{pa_1^p}\|\tilde{u}\|_\infty^p + \frac{[(1+\varepsilon)a_1M_1]^q}{q}|P\bar{u}|^{q\alpha} + B(\varepsilon)M_1\left(\sum_{i=1}^r T_i^2\right)^{\alpha/2} \|\tilde{u}\|_\infty \\
&\quad + B(\varepsilon)M_1\|\tilde{u}\|_\infty^{\alpha+1} + M_2\|\tilde{u}\|_\infty
\end{aligned}$$

$$\le \frac{1}{pa_1^p}\left(\frac{T}{q+1}\right)^{p/q}\|\dot{u}\|_{L^p}^p+\frac{[(1+\varepsilon)a_1M_1]^q}{q}|P\bar{u}|^{q\alpha}$$
$$+B(\varepsilon)\left(\frac{T}{q+1}\right)^{1/q}M_1M_3^{\alpha}\|\dot{u}\|_{L^p}$$
$$+B(\varepsilon)\left(\frac{T}{q+1}\right)^{(\alpha+1)/q}M_1\|\dot{u}\|_{L^p}^{\alpha+1}+\left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}\|_{L^p}.$$

Hence we have

$$\|\tilde{u}_n\| \ge |\langle \varphi'(u_n), \tilde{u}_n\rangle|$$
$$= \left|\int_0^T |\dot{u}_n(t)|^p dt - \int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))dt + \int_0^T (e(t), \tilde{u}_n(t))dt\right|$$
$$\ge \left[1-\frac{1}{pa_1^p}\left(\frac{T}{q+1}\right)^{p/q}\right]\|\dot{u}_n\|_{L^p}^p - \frac{[(1+\varepsilon)a_1M_1]^q}{q}|P\bar{u}_n|^{q\alpha} \tag{3.3}$$
$$-B(\varepsilon)\left(\frac{T}{q+1}\right)^{1/q}M_1M_3^{\alpha}\|\dot{u}_n\|_{L^p}-B(\varepsilon)\left(\frac{T}{q+1}\right)^{(\alpha+1)/q}M_1\|\dot{u}_n\|_{L^p}^{\alpha+1}$$
$$-\left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}_n\|_{L^p}-\left(\frac{T}{q+1}\right)^{1/q}M_4\|\dot{u}_n\|_{L^p}$$

for large n by the fact that $\varphi'(u_n) \to 0$. It follows from (2.10) and (3.3) that

$$\left[\frac{T^p\Theta(p,q)}{(q+1)^{p/q}}+1\right]^{1/p}\|\dot{u}_n\|_{L^p} \ge \|\tilde{u}_n\|$$
$$\ge \left[1-\frac{1}{pa_1^p}\left(\frac{T}{q+1}\right)^{p/q}\right]\|\dot{u}_n\|_{L^p}^p - \frac{[(1+\varepsilon)a_1M_1]^q}{q}|P\bar{u}_n|^{q\alpha}$$
$$-B(\varepsilon)\left(\frac{T}{q+1}\right)^{1/q}M_1M_3^{\alpha}\|\dot{u}_n\|_{L^p}-B(\varepsilon)\left(\frac{T}{q+1}\right)^{(\alpha+1)/q}M_1\|\dot{u}_n\|_{L^p}^{\alpha+1}$$
$$-\left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}_n\|_{L^p}-\left(\frac{T}{q+1}\right)^{1/q}M_4\|\dot{u}_n\|_{L^p}$$

for large n, which implies that

$$\frac{[(1+\varepsilon)a_1M_1]^q}{q}|P\bar{u}_n|^{q\alpha} \ge \frac{1}{q}\|\dot{u}_n\|_{L^p}^p + C_1, \tag{3.4}$$

where

$$C_1 = \min_{s\in[0,+\infty)} \left\{ \left(\frac{1}{p} - \frac{1}{pa_1^p} \left(\frac{T}{q+1} \right)^{p/q} \right) s^p - B(\varepsilon) \left(\frac{T}{q+1} \right)^{(\alpha+1)/q} M_1 s^{\alpha+1} - \left[B(\varepsilon) \left(\frac{T}{q+1} \right)^{1/q} M_1 M_3^\alpha + \left(\frac{T}{q+1} \right)^{1/q} (M_2 + M_4) + \left(\frac{T^p \Theta(p,q)}{(q+1)^{p/q}} + 1 \right)^{1/p} \right] s \right\}.$$

The fact $a_1 > (T/(q+1))^{1/q}$ implies that $-\infty < C_1 < 0$. By (3.4), we get

$$\|\dot{u}_n\|_{L^p}^p \leq [(1+\varepsilon)a_1 M_1]^q |P\bar{u}_n|^{q\alpha} + C_2 \tag{3.5}$$

and so

$$\|\dot{u}_n\|_{L^p} \leq [(1+\varepsilon)a_1 M_1]^{q/p} |P\bar{u}_n|^{q\alpha/p} + C_3, \tag{3.6}$$

where $C_2, C_3 > 0$. It follows from (1.8), (2.9), Lemma 2.1 and Young's inequality that

$$\begin{aligned} & \left| \int_0^T [F(t, P\bar{u} + Q\bar{u} + \tilde{u}(t)) - F(t, P\bar{u})] dt \right| \\ = & \left| \int_0^T \int_0^1 (\nabla F(t, P\bar{u} + s(Q\bar{u} + \tilde{u}(t))), Q\bar{u} + \tilde{u}(t)) ds dt \right| \\ \leq & \int_0^T \int_0^1 f(t) |P\bar{u} + s(Q\bar{u} + \tilde{u}(t))|^\alpha |Q\bar{u} + \tilde{u}(t)| ds dt \\ & + \int_0^T g(t) |Q\bar{u} + \tilde{u}(t)| dt \\ \leq & M_1 (|P\bar{u}| + |Q\bar{u}| + \|\tilde{u}\|_\infty)^\alpha (|Q\bar{u}| + \|\tilde{u}\|_\infty) + M_2 (|Q\bar{u}| + \|\tilde{u}\|_\infty) \\ \leq & (1+\varepsilon) M_1 |P\bar{u}|^\alpha (|Q\bar{u}| + \|\tilde{u}\|_\infty) + B(\varepsilon) M_1 (|Q\bar{u}| + \|\tilde{u}\|_\infty)^{\alpha+1} \\ & + M_2 (|Q\bar{u}| + \|\tilde{u}\|_\infty) \\ \leq & (1+\varepsilon) M_1 |P\bar{u}|^\alpha (M_3 + \|\tilde{u}\|_\infty) + B(\varepsilon) M_1 (M_3 + \|\tilde{u}\|_\infty)^{\alpha+1} \\ & + M_2 (M_3 + \|\tilde{u}\|_\infty) \\ \leq & (1+\varepsilon) M_1 |P\bar{u}|^\alpha \|\tilde{u}\|_\infty + (1+\varepsilon) M_1 M_3 |P\bar{u}|^\alpha + 2^\alpha B(\varepsilon) M_1 \|\tilde{u}\|_\infty^{\alpha+1} \\ & + M_2 \|\tilde{u}\|_\infty + 2^\alpha B(\varepsilon) M_1 M_3^{\alpha+1} + M_2 M_3 \\ \leq & \frac{1}{pa_1} \left(\frac{q+1}{T} \right)^{p/q^2} \|\tilde{u}\|_\infty^p + \frac{[(1+\varepsilon)M_1]^q a_1^{q/p}}{q} \left(\frac{T}{q+1} \right)^{1/q} |P\bar{u}|^{q\alpha} \end{aligned} \tag{3.7}$$

$$+(1+\varepsilon)M_1M_3|P\bar{u}|^\alpha + 2^\alpha B(\varepsilon)M_1\|\tilde{u}\|_\infty^{\alpha+1}$$
$$+M_2\|\tilde{u}\|_\infty + 2^\alpha B(\varepsilon)M_1M_3^{\alpha+1} + M_2M_3$$
$$\le \frac{1}{pa_1}\left(\frac{T}{q+1}\right)^{1/q}\|\dot{u}\|_{L^p}^p + \frac{[(1+\varepsilon)M_1]^q a_1^{q/p}}{q}\left(\frac{T}{q+1}\right)^{1/q}|P\bar{u}|^{q\alpha}$$
$$+(1+\varepsilon)M_1M_3|P\bar{u}|^\alpha$$
$$+2^\alpha B(\varepsilon)M_1\left(\frac{T}{q+1}\right)^{(\alpha+1)/q}\|\dot{u}\|_{L^p}^{\alpha+1}$$
$$+\left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}\|_{L^p} + 2^\alpha B(\varepsilon)M_1M_3^{\alpha+1} + M_2M_3$$

for all $u \in W_T^{1,p}$. It follows from the boundedness of $\{\varphi(u_n)\}$, (1.3), (3.5), (3.6) and (3.7) that

$$C_4 \le \varphi(u_n)$$
$$= \frac{1}{p}\int_0^T |\dot{u}_n(t)|^p dt - \int_0^T [F(t,u_n(t)) - F(t,P\bar{u}_n)]dt$$
$$-\int_0^T F(t,P\bar{u}_n)dt + \int_0^T (e(t),\tilde{u}_n(t))dt$$
$$\le \left[\frac{1}{p} + \frac{1}{pa_1}\left(\frac{T}{q+1}\right)^{1/q}\right]\|\dot{u}_n\|_{L^p}^p + \frac{[(1+\varepsilon)M_1]^q a_1^{q/p}}{q}\left(\frac{T}{q+1}\right)^{1/q}$$
$$|P\bar{u}_n|^{q\alpha} + (1+\varepsilon)M_1M_3|P\bar{u}|^\alpha$$
$$+2^\alpha B(\varepsilon)M_1\left(\frac{T}{q+1}\right)^{(\alpha+1)/q}\|\dot{u}_n\|_{L^p}^{\alpha+1} + \left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}_n\|_{L^p}$$
$$+2^\alpha B(\varepsilon)M_1M_3^{\alpha+1} + M_2M_3$$
$$-\int_0^T F(t,P\bar{u}_n)dt + \left(\frac{T}{q+1}\right)^{1/q}M_4\|\dot{u}_n\|_{L^p}$$
$$\le \left[\frac{1}{p} + \frac{1}{pa_1}\left(\frac{T}{q+1}\right)^{1/q}\right]\{[(1+\varepsilon)a_1M_1]^q|P\bar{u}_n|^{q\alpha} + C_2\}$$
$$+\frac{[(1+\varepsilon)M_1]^q a_1^{q/p}}{q}\left(\frac{T}{q+1}\right)^{1/q}|P\bar{u}_n|^{q\alpha}$$
$$+(1+\varepsilon)M_1M_3|P\bar{u}_n|^\alpha + 2^\alpha B(\varepsilon)M_1\left(\frac{T}{q+1}\right)^{(\alpha+1)/q}$$

$$\begin{aligned}
&\left\{[(1+\varepsilon)a_1M_1]^{q/p}|P\bar{u}_n|^{q\alpha/p}+C_3\right\}^{\alpha+1}\\
&+\left(\frac{T}{q+1}\right)^{1/q}(M_2+M_4)\left\{[(1+\varepsilon)a_1M_1]^{q/p}|P\bar{u}_n|^{q\alpha/p}+C_3\right\}\\
&+2^{\alpha}B(\varepsilon)M_1M_3^{\alpha+1}+M_2M_3-\int_0^T F(t,P\bar{u}_n)dt\\
\le\ &\left[\frac{1}{p}+\frac{1}{a_1}\left(\frac{T}{q+1}\right)^{1/q}\right][(1+\varepsilon)a_1M_1]^{q}|P\bar{u}_n|^{q\alpha}\\
&+C_5|P\bar{u}_n|^{q\alpha(\alpha+1)/p}+C_6|P\bar{u}_n|^{q\alpha/p}\\
&+(1+\varepsilon)M_1M_3|P\bar{u}_n|^{\alpha}+C_7-\int_0^T F(t,P\bar{u}_n)dt\\
=\ &-|P\bar{u}_n|^{q\alpha}\left\{|P\bar{u}_n|^{-q\alpha}\int_0^T F(t,P\bar{u}_n)dt-\left[\frac{1}{p}+\frac{1}{a_1}\left(\frac{T}{q+1}\right)^{1/q}\right][(1+\varepsilon)a_1M_1]^{q}\right\}\\
&+C_5|P\bar{u}_n|^{q\alpha(\alpha+1)/p}+C_6|P\bar{u}_n|^{q\alpha/p}+(1+\varepsilon)M_1M_3|P\bar{u}_n|^{\alpha}+C_7,
\end{aligned}$$

where C_5, C_6 and C_7 are positive constants. Then it follows from (3.1), $q\alpha(\alpha+1)/p < q\alpha$, and $q\alpha/p < q\alpha$ that $\{|P\bar{u}_n|\}$ is bounded. Furthermore, $\{\tilde{u}_n\}$ is bounded by (2.10) and (3.6), which implies that $\{u_n\}$ is bounded in $W_T^{1,p}$. Since $W_T^{1,p}$ is a reflexive Banach space, both boundedness and weak compactness are equivalent, going if necessary to a subsequence, we can assume that

$$u_n \rightharpoonup u \ \text{ weakly in } W_T^{1,p}. \tag{3.8}$$

Furthermore, by Proposition 1.2 in [5], we have

$$u_n \to u \ \text{ strongly in } \ C([0,T],\mathbf{R}^N). \tag{3.9}$$

By (2.12), we have

$$\begin{aligned}
&\langle\varphi'(u_n),u_n-u\rangle\\
=\ &\int_0^T (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t),\dot{u}_n(t)-\dot{u}(t))dt\\
&-\int_0^T(\nabla F(t,u_n(t)),u_n(t)-u(t))dt+\int_0^T(e(t),u_n(t)-u(t))dt.
\end{aligned}\tag{3.10}$$

Since $\{\|u_n\|\}$ is bounded and $\varphi'(u_n)\to 0$, we have

$$|\langle\varphi'(u_n),u_n-u\rangle|\le\|\varphi'(u_n)\|\ \|u_n-u\|\to 0 \quad \text{as } \ n\to\infty. \tag{3.11}$$

By assumption (A) and (3.9), one has

$$\int_0^T \big(\nabla F(t,u_n(t)),u_n(t)-u(t)\big)dt\to 0 \quad \text{as } n\to\infty, \tag{3.12}$$

and

$$\int_0^T (e(t), u_n(t) - u(t))dt \to 0 \quad \text{as} \quad n \to \infty. \tag{3.13}$$

Hence, it follows from (3.10), (3.11), (3.12) and (3.13) that

$$\int_0^T (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t))dt \to 0 \quad \text{as} \quad n \to \infty. \tag{3.14}$$

On the other hand, it is easy to derive from (3.9) and the boundedness of $\{u_n\}$ that

$$\int_0^T (|u_n(t)|^{p-2}u_n(t), u_n(t) - u(t))dt \to 0 \quad \text{as} \quad n \to \infty. \tag{3.15}$$

Set

$$\psi(u) = \frac{1}{p}\left(\int_0^T |u(t)|^p dt + \int_0^T |\dot{u}(t)|^p dt\right).$$

Then we have

$$\begin{aligned} \langle \psi'(u_n), u_n - u\rangle &= \int_0^T (|u_n(t)|^{p-2}u_n(t), u_n(t) - u(t))dt \\ &\quad + \int_0^T (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t))dt, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \langle \psi'(u), u_n - u\rangle &= \int_0^T (|u(t)|^{p-2}u(t), u_n(t) - u(t))dt \\ &\quad + \int_0^T (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{u}_n(t) - \dot{u}(t))dt. \end{aligned} \tag{3.17}$$

From (3.14), (3.15) and (3.16), we obtain

$$\langle \psi'(u_n), u_n - u\rangle \to 0 \quad \text{as} \quad n \to \infty. \tag{3.18}$$

On the other hand, it follows from (3.8) that

$$\langle \psi'(u), u_n - u\rangle \to 0 \quad \text{as} \quad n \to \infty. \tag{3.19}$$

By (3.16), (3.17) and by using the Hölder's inequality, we get By (3.16), (3.17) and by using the Hölder's inequality, we get

$$
\begin{aligned}
&\langle \psi'(u_n) - \psi'(u), u_n - u\rangle \\
=\ & \int_0^T (|u_n(t)|^{p-2}u_n(t), u_n(t)-u(t))dt + \int_0^T (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t)-\dot{u}(t))dt \\
& - \int_0^T (|u(t)|^{p-2}u(t), u_n(t)-u(t))dt - \int_0^T (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{u}_n(t)-\dot{u}(t))dt \\
=\ & \|u_n\|^p + \|u\|^p - \int_0^T (|u_n(t)|^{p-2}u_n(t), u(t))dt - \int_0^T (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}(t))dt \\
& - \int_0^T (|u(t)|^{p-2}u(t), u_n(t))dt - \int_0^T (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{u}_n(t))dt \\
\geq\ & \|u_n\|^p + \|u\|^p - \left(\|u_n\|_{L^p}^{p-1}\|u\|_{L^p} + \|\dot{u}_n\|_{L^p}^{p-1}\|\dot{u}\|_{L^p}\right) \\
& - \left(\|u\|_{L^p}^{p-1}\|u_n\|_{L^p} + \|\dot{u}\|_{L^p}^{p-1}\|\dot{u}_n\|_{L^p}\right) \\
\geq\ & \|u_n\|^p + \|u\|^p - \left(\|u\|_{L^p}^p + \|\dot{u}\|_{L^p}^p\right)^{1/p}\left(\|u_n\|_{L^p}^p + \|\dot{u}_n\|_{L^p}^p\right)^{1/q} \\
& - \left(\|u_n\|_{L^p}^p + \|\dot{u}_n\|_{L^p}^p\right)^{1/p}\left(\|u\|_{L^p}^p + \|\dot{u}\|_{L^p}^p\right)^{1/q} \\
=\ & \|u_n\|^p + \|u\|^p - \|u\|\ \|u_n\|^{p-1} - \|u_n\|\ \|u\|^{p-1} \\
=\ & \left(\|u_n\|^{p-1} - \|u\|^{p-1}\right)\left(\|u_n\| - \|u\|\right).
\end{aligned} \tag{3.20}
$$

It follows that

$$0 \leq \left(\|u_n\|^{p-1} - \|u\|^{p-1}\right)\left(\|u_n\| - \|u\|\right) \leq \langle \psi'(u_n) - \psi'(u), u_n - u\rangle, \tag{3.21}$$

which, together with (3.18) and (3.19) yields $\|u_n\| \to \|u\|$ (see [19]). By the uniform convexity of $W_T^{1,p}$ and (3.8), it follows from the Kadec-Klee property that $u_n \to u$ strongly in $W_T^{1,p}$. Thus we have verified that Ψ satisfies (PS) condition.

Step 2. In order to use Lemma 2.3, we only need to verify the following conditions:

(i) $\inf\{\Psi((z,v)) \mid (z,v) \in Z \times T^r\} > -\infty$;

(ii) $\Psi((y,v)) \to -\infty$ uniformly for $(y,v) \in Y \times T^r$ as $|y| \to \infty$.

For $(z,v) \in Z \times T^r$, set $u = u(t) = v + z(t)$. Then $v = Q\bar{u},\ z(t) = \tilde{u}(t)$. It follows from (1.8) and (2.9) that

$$
\begin{aligned}
&\left|\int_0^T [F(t,u(t)) - F(t,0)]dt\right| \\
=\ & \left|\int_0^T \int_0^1 (\nabla F(t, s(Q\bar{u} + \tilde{u}(t))), Q\bar{u} + \tilde{u}(t))dsdt\right| \\
\leq\ & \int_0^T \int_0^1 f(t)|s(Q\bar{u} + \tilde{u}(t))|^\alpha |Q\bar{u} + \tilde{u}(t)|dsdt + \int_0^T g(t)|Q\bar{u} + \tilde{u}(t)|dt \\
\leq\ & M_1\left(|Q\bar{u}| + \|\tilde{u}\|_\infty\right)^\alpha \left(|Q\bar{u}| + \|\tilde{u}\|_\infty\right) + M_2\left(|Q\bar{u}| + \|\tilde{u}\|_\infty\right) \\
\leq\ & 2^\alpha M_1\left(|Q\bar{u}|^{\alpha+1} + \|\tilde{u}\|_\infty^{\alpha+1}\right) + M_2\left(|Q\bar{u}| + \|\tilde{u}\|_\infty\right) \\
\leq\ & 2^\alpha M_1\left[M_3^{\alpha+1} + \left(\frac{T}{q+1}\right)^{(\alpha+1)/q}\|\dot{u}\|_{L^p}^{\alpha+1}\right] + M_2\left[M_3 + \left(\frac{T}{q+1}\right)^{1/q}\|\dot{u}\|_{L^p}\right].
\end{aligned} \tag{3.22}
$$

Hence, by (1.3), (2.11) and (3.22), we have

$$\begin{aligned}\Psi((z,v)) &= \varphi(u)\\ &= \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt - \int_0^T F(t,0)dt - \int_0^T [F(t,\tilde{u}(t)+Q\bar{u}) - F(t,0)]\,dt\\ &\quad + \int_0^T (e(t),\tilde{u}(t)+Q\bar{u})dt\\ &\geq \frac{1}{p}\|\dot{u}\|_{L^p}^p - 2^\alpha M_1\left[M_3^{\alpha+1} + \left(\frac{T}{q+1}\right)^{(\alpha+1)/q}\|\dot{u}\|_{L^p}^{\alpha+1}\right]\\ &\quad - M_2\left[M_3 + \left(\frac{T}{q+1}\right)^{1/q}\|\dot{u}\|_{L^p}\right]\\ &\quad - \int_0^T F(t,0)dt - M_4\left(\frac{T}{q+1}\right)^{1/q}\|\dot{u}\|_{L^p}.\end{aligned}$$

This shows that condition (i) holds.

For any $(y,v)\in Y\times T^r$, it follows from (1.3), (1.8) and (2.11) that

$$\begin{aligned}\Psi((y,v)) &= \varphi(y+v)\\ &= -\int_0^T F(t,y+v)dt\\ &= -\int_0^T F(t,y)dt - \int_0^T\int_0^1 (\nabla F(t,y+sv),v)dsdt\\ &\leq -\int_0^T F(t,y)dt + |v|\int_0^T\int_0^1 f(t)|y+sv|^\alpha dsdt + |v|\int_0^T g(t)dt\\ &\leq -\int_0^T F(t,y)dt + C_8|y|^\alpha + C_9\\ &= -|y|^{q\alpha}\left(|y|^{-q\alpha}\int_0^T F(t,y)dt\right) + C_8|y|^\alpha + C_9\end{aligned}$$

for some positive constants C_8 and C_9. Hence, the inequality above, $\alpha < q\alpha$ and (1.9) imply condition (ii) holds. It follows from Lemma 2.3 that Ψ has at least $r+1$ critical points. Hence φ has at least $r+1$ geometrically distinct critical points. Therefore, problem (1.7) has at least $r+1$ geometrically distinct solutions in $W_T^{1,p}$.

Proof of Theorem 1.2 By (1.10), we can choose constants $\varepsilon > 0$ and $a_2 > (T/(q+1))^{1/q}$ such that

$$\limsup_{|x|\to\infty} |x|^{-q\alpha}\int_0^T F(t,x)dt < -\frac{[(1+\varepsilon)M_1]^q a_2^{q/p}}{q}\left(\frac{T}{q+1}\right)^{1/q} \tag{3.23}$$

as $x\in 0\times\mathbf{R}^{N-r}$. It follows from (3.7), (1.3) and (2.9) that

$$
\begin{aligned}
&\varphi(u) = \varphi(\hat{u}) \\
&= \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt - \int_0^T F(t,\hat{u}(t))dt + \int_0^T (e(t),\hat{u}(t))dt \\
&= \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt - \int_0^T F(t,P\bar{u})dt + \int_0^T (e(t),\tilde{u}(t))dt \\
&\quad - \int_0^T\int_0^1 (\nabla F(t,P\bar{u}+s(Q\bar{u}+\tilde{u}(t))), Q\bar{u}+\tilde{u}(t))dsdt \\
&\geq \left[\frac{1}{p} - \frac{1}{pa_2}\left(\frac{T}{q+1}\right)^{1/q}\right]\|\dot{u}\|_{L^p}^p - 2^\alpha B(\varepsilon)M_1\left(\frac{T}{q+1}\right)^{(\alpha+1)/q}\|\dot{u}\|_{L^p}^{\alpha+1} \\
&\quad - \left(\frac{T}{q+1}\right)^{1/q}(M_2+M_4)\|\dot{u}\|_{L^p} - M_2M_3 \\
&\quad -2^\alpha B(\varepsilon)M_1M_3^{\alpha+1} - (1+\varepsilon)M_1M_3|P\bar{u}|^\alpha \\
&\quad -|P\bar{u}|^{q\alpha}\left[|P\bar{u}|^{-q\alpha}\int_0^T F(t,P\bar{u})dt + \frac{[(1+\varepsilon)M_1]^q a_2^{q/p}}{q}\left(\frac{T}{q+1}\right)^{1/q}\right]
\end{aligned}
\tag{3.24}
$$

for all $u \in W_T^{1,p}$. It follows from (3.23), (3.24) and $a_2 > (T/(q+1))^{1/q}$ that φ is bounded from below. Let G be a discrete subgroup of $W_T^{1,p}$ defined by (2.14) and let $\pi : W_T^{1,p} \to W_T^{1,p}/G$ be the canonical surjection. By (1.2) and (1.3), it is easy to verify that φ is G-invariant. In what follows, we show that the functional φ satisfies the (PS)_G-condition, that is, for every sequence $\{u_n\}$ in $W_T^{1,p}$ such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$, the sequence $\{\pi(u_n)\}$ has a convergent subsequence. In fact, the boundedness of $\varphi(u_n)$, (3.23), (3.24) and the fact that $a_2 > (T/(q+1))^{1/q}$ imply that $(P\bar{u}_n)$ and $\|\dot{u}_n\|_{L^p}$ are bounded. Furthermore, by (2.10), we know that $(\tilde{u}_n)$ is also bounded. Hence $\{\hat{u}_n\}$ is bounded. Similar to the proof of Theorem 1.1, we can know that $\{\hat{u}_n\}$ has a convergent subsequence. So $\{\pi(u_n)\}$ also has a convergent subsequence since $\pi(u_n) = \pi(\hat{u}_n)$. Thus, by Lemma 2.4, we know that φ has $r+1$ critical orbits. Hence, as in [5, Theorem 4.13], problem (1.7) has at least $r+1$ geometrically distinct solutions in $W_T^{1,p}$. The proof is complete.

Proof of Theorem 1.3. Similar to the proof of Theorem 1.1, we divide our proof into two steps.

Step 3. We prove Ψ defined by (2.15) satisfies (PS)- condition.

First we assume that $\{(y_n+z_n, v_n)\} \subset X \times T^r$ is a (PS)- sequence for Ψ, that is $\{\Psi((y_n+z_n,v_n))\}$ is bounded and $\Psi'((y_n+z_n,v_n)) \to 0$, where $y_n \in Y,\ z_n = z_n(t) \in Z,\ v_n \in T^r$ for $n = 1,2,\ldots$. Set

$$u_n = y_n + v_n + z_n(t), \quad n = 1,2,\ldots.$$

Then it is easy to see that

$$y_n = P\bar{u}_n, \quad v_n = Q\bar{u}_n, \quad z_n(t) = \tilde{u}_n(t), \quad n = 1,2,\ldots.$$

It follows from (2.15) and (2.16) that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$.

By (1.17), we can choose an $a_3 > T/12$ such that

$$\liminf_{|x|\to\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt > a_3 M_1^2 + \frac{TM_1^2}{24} \tag{3.25}$$

as $x \in 0 \times \mathbf{R}^{N-r}$. It follows from (1.4), (1.16), $\alpha \in (0,1)$ and Young's inequality that for all $u \in W_T^{1,2}$

$$\begin{aligned}
&\left|\int_0^T (\nabla F(t, P\bar{u} + Q\bar{u} + \tilde{u}(t)), \tilde{u}(t))dt\right| \\
\le\ & \int_0^T f(t)|P\bar{u} + Q\bar{u} + \tilde{u}(t)|^\alpha |\tilde{u}(t)|dt + \int_0^T g(t)|\tilde{u}(t)|dt \\
\le\ & \int_0^T f(t)(|P\bar{u}|^\alpha + |Q\bar{u}|^\alpha + |\tilde{u}(t)|^\alpha)|\tilde{u}(t)|dt + \int_0^T g(t)|\tilde{u}(t)|dt \\
\le\ & |P\bar{u}|^\alpha \|\tilde{u}\|_\infty \int_0^T f(t)dt + |Q\bar{u}|^\alpha \|\tilde{u}\|_\infty \int_0^T f(t)dt \\
& + \|\tilde{u}\|_\infty^{\alpha+1} \int_0^T f(t)dt + \|\tilde{u}\|_\infty \int_0^T g(t)dt \\
=\ & M_1 |P\bar{u}|^\alpha \|\tilde{u}\|_\infty + M_1 |Q\bar{u}|^\alpha \|\tilde{u}\|_\infty + M_1 \|\tilde{u}\|_\infty^{\alpha+1} + M_2 \|\tilde{u}\|_\infty \\
\le\ & \frac{1}{2a_3}\|\tilde{u}\|_\infty^2 + \frac{a_3 M_1^2}{2}|P\bar{u}|^{2\alpha} + M_1 M_3^\alpha \|\tilde{u}\|_\infty + M_1 \|\tilde{u}\|_\infty^{\alpha+1} + M_2 \|\tilde{u}\|_\infty \\
\le\ & \frac{T}{24a_3}\|\dot{u}\|_{L^2}^2 + \frac{a_3 M_1^2}{2}|P\bar{u}|^{2\alpha} + \sqrt{\frac{T}{12}}(M_2 + M_1 M_3^\alpha)\|\dot{u}\|_{L^2} \\
& + \left(\frac{T}{12}\right)^{(\alpha+1)/2} M_1 \|\dot{u}\|_{L^2}^{\alpha+1}.
\end{aligned} \tag{3.26}$$

Hence we have

$$\begin{aligned}
\|\tilde{u}_n\| \ \ge\ & |\langle \varphi'(u_n), \tilde{u}_n\rangle| \\
=\ & \left|\int_0^T |\dot{u}_n(t)|^2 dt - \int_0^T (\nabla F(t, \hat{u}_n(t)), \tilde{u}_n(t))dt + \int_0^T (e(t), \tilde{u}_n(t))dt\right| \\
\ge\ & \left(1 - \frac{T}{24a_3}\right)\|\dot{u}_n\|_{L^2}^2 - \frac{a_3 M_1^2}{2}|P\bar{u}_n|^{2\alpha} \\
& - \sqrt{\frac{T}{12}}(M_2 + M_4 + M_1 M_3^\alpha)\|\dot{u}_n\|_{L^2} - \left(\frac{T}{12}\right)^{(\alpha+1)/2} M_1 \|\dot{u}_n\|_{L^2}^{\alpha+1}
\end{aligned} \tag{3.27}$$

for large n by the fact that $\varphi'(u_n) \to 0$. It follows from (1.15) and (3.27) that

$$\begin{aligned}
&\left(\frac{T^2}{4\pi^2} + 1\right)^{1/2} \|\dot{u}_n\|_{L^2} \ge \|\tilde{u}_n\| \\
\ge\ & \left(1 - \frac{T}{24a_3}\right)\|\dot{u}_n\|_{L^2}^2 - \frac{a_3 M_1^2}{2}|P\bar{u}_n|^{2\alpha} \\
& - \sqrt{\frac{T}{12}}(M_2 + M_4 + M_1 M_3^\alpha)\|\dot{u}_n\|_{L^2} - \left(\frac{T}{12}\right)^{(\alpha+1)/2} M_1 \|\dot{u}_n\|_{L^2}^{\alpha+1}
\end{aligned}$$

for large n, which implies that

$$\frac{a_3 M_1^2}{2}|P\bar{u}_n|^{2\alpha} \geq \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 + D_1, \tag{3.28}$$

where

$$\begin{aligned} D_1 &= \min_{s\in[0,+\infty)}\left\{\frac{12a_3 - T}{24a_3}s^2 - \left[\sqrt{\frac{T}{12}}(M_2 + M_4 + M_1 M_3^\alpha)\right.\right. \\ &\quad \left.\left. + \left(\frac{T^2}{4\pi^2}+1\right)^{1/2}\right]s - \left(\frac{T}{12}\right)^{\frac{\alpha+1}{2}} M_1 s^{\alpha+1}\right\}. \end{aligned}$$

The fact $a_3 > T/12$ implies that $-\infty < D_1 < 0$. By (3.28), we have

$$\|\dot{u}_n\|_{L^2}^2 \leq a_3 M_1^2 |P\bar{u}_n|^{2\alpha} + D_2 \tag{3.29}$$

and

$$\|\dot{u}_n\|_{L^2} \leq \sqrt{a_3} M_1 |P\bar{u}_n|^\alpha + D_3 \tag{3.30}$$

for all integers $n \geq 1$, where $D_2, D_3 > 0$. It follows from (1.4) and (1.16) that for all $u \in W_T^{1,2}$

$$\begin{aligned} &\left|\int_0^T [F(t, P\bar{u} + Q\bar{u} + \tilde{u}(t)) - F(t, P\bar{u})]dt\right| \\ &= \left|\int_0^T \int_0^1 (\nabla F(t, P\bar{u} + s(Q\bar{u} + \tilde{u}(t))), Q\bar{u} + \tilde{u}(t))dsdt\right| \\ &\leq |P\bar{u}|^\alpha \int_0^T f(t)|Q\bar{u} + \tilde{u}(t)|dt + \int_0^T f(t)|Q\bar{u} + \tilde{u}(t)|^{\alpha+1}dt \\ &\quad + \int_0^T g(t)|Q\bar{u} + \tilde{u}(t)|dt \\ &\leq M_1(|Q\bar{u}| + \|\tilde{u}\|_\infty)|P\bar{u}|^\alpha + 2^\alpha M_1\left(|Q\bar{u}|^{\alpha+1} + \|\tilde{u}\|_\infty^{\alpha+1}\right) + M_2(|Q\bar{u}| + \|\tilde{u}\|_\infty) \\ &\leq M_1 M_3 |P\bar{u}|^\alpha + \frac{1}{2a_3}\|\tilde{u}\|_\infty^2 + \frac{a_3 M_1^2}{2}|P\bar{u}|^{2\alpha} \\ &\quad + 2^\alpha M_1 M_3^{\alpha+1} + 2^\alpha M_1 \|\tilde{u}\|_\infty^{\alpha+1} + M_2 M_3 + M_2\|\tilde{u}\|_\infty \\ &\leq \frac{T}{24a_3}\|\dot{u}\|_{L^2}^2 + \frac{a_3 M_1^2}{2}|P\bar{u}|^{2\alpha} + M_1 M_3 |P\bar{u}|^\alpha + 2^\alpha M_1 M_3^{\alpha+1} + M_2 M_3 \\ &\quad + 2^\alpha M_1 \left(\frac{T}{12}\right)^{(\alpha+1)/2}\|\dot{u}\|_{L^2}^{\alpha+1} + \sqrt{\frac{T}{12}} M_2 \|\dot{u}\|_{L^2}. \end{aligned} \tag{3.31}$$

It follows from the boundedness of $\{\varphi(u_n)\}$, (3.31), (3.29) and (3.30) that

$$
\begin{aligned}
&D_4 \le \varphi(u_n)\\
=\ & \frac{1}{2}\int_0^T |\dot u_n(t)|^2 dt - \int_0^T [F(t,u_n(t)) - F(t,P\bar u_n)]dt\\
&-\int_0^T F(t,P\bar u_n)dt + \int_0^T (e(t),u_n(t))dt\\
\le\ & \left(\frac{1}{2}+\frac{T}{24a_3}\right)\|\dot u_n\|_{L^2}^2 + 2^\alpha M_1\left(\frac{T}{12}\right)^{(\alpha+1)/2}\|\dot u_n\|_{L^2}^{\alpha+1} + \sqrt{\frac{T}{12}}M_2\|\dot u_n\|_{L^2} + 2^\alpha M_1 M_3^{\alpha+1}\\
&+M_2M_3 + \frac{a_3M_1^2}{2}|P\bar u_n|^{2\alpha} + M_1M_3|P\bar u_n|^\alpha - \int_0^T F(t,P\bar u_n)dt + \sqrt{\frac{T}{12}}M_4\|\dot u_n\|_{L^2}\\
\le\ & \left(\frac{1}{2}+\frac{T}{24a_3}\right)\left(a_3M_1^2|P\bar u_n|^{2\alpha}+D_2\right) + 2^\alpha M_1\left(\frac{T}{12}\right)^{(\alpha+1)/2}\left(\sqrt{a_3}M_1|P\bar u_n|^\alpha + D_3\right)^{\alpha+1}\\
&+\sqrt{\frac{T}{12}}(M_2+M_4)\left(\sqrt{a_3}M_1|P\bar u_n|^\alpha + D_3\right) + 2^\alpha M_1M_3^{\alpha+1} + M_2M_3 + \frac{a_3M_1^2}{2}|P\bar u_n|^{2\alpha}\\
&+M_1M_3|P\bar u_n|^\alpha - \int_0^T F(t,P\bar u_n)dt\\
=\ & \left(a_3M_1^2 + \frac{TM_1^2}{24}\right)|P\bar u_n|^{2\alpha} + D_5|P\bar u_n|^{\alpha(\alpha+1)} + D_6|P\bar u_n|^\alpha + D_7 - \int_0^T F(t,P\bar u_n)dt\\
=\ & -|P\bar u_n|^{2\alpha}\left\{|P\bar u_n|^{-2\alpha}\int_0^T F(t,P\bar u_n)dt - \left(a_3M_1^2 + \frac{TM_1^2}{24}\right)\right\}\\
&+D_5|P\bar u_n|^{\alpha(\alpha+1)} + D_6|P\bar u_n|^\alpha + D_7,
\end{aligned}
$$

where $D_5,\ D_6$ and D_7 are positive constants. Then it follows from (3.25) and $\alpha(\alpha+1) < 2\alpha$ that $\{|P\bar u_n|\}$ is bounded. Furthermore, $\{\tilde u_n\}$ is bounded by (1.16) and (3.30), which implies that $\{u_n\}$ is bounded in $W_T^{1,2}$. Arguing then as in Proposition 4.1 in [5], we conclude that the (PS) condition is satisfied. The other proofs are similar to those in Theorem 1.1 with $p = 2$. The proof is complete.

Proof of Theorem 1.4. By (1.18), there exists an $a_4 > T/12$ such that

$$
\limsup_{|x|\to\infty} |x|^{-2\alpha}\int_0^T F(t,x)dt < -\frac{a_4M_1^2}{2} \tag{3.32}
$$

as $x \in 0\times \mathbf{R}^{N-r}$. It follows from (1.3) and (3.31) that for all $u \in W_T^{1,2}$

$$
\begin{aligned}
\varphi(u) &= \varphi(\hat u)\\
&= \frac{1}{2}\int_0^T |\dot u(t)|^2 dt - \int_0^T F(t,P\bar u)dt + \int_0^T (e(t),\tilde u(t))dt\\
&\quad -\int_0^T\int_0^1 (\nabla F(t,P\bar u + s(Q\bar u + \tilde u(t))), Q\bar u + \tilde u(t))dsdt
\end{aligned}
$$

$$\geq \left(\frac{1}{2}-\frac{T}{24a_4}\right)\|\dot u\|^2_{L^2}-\sqrt{\frac{T}{12}}M_4\|\dot u\|_{L^2}-2^\alpha M_1\left(\frac{T}{12}\right)^{(\alpha+1)/2}\|\dot u\|^{\alpha+1}_{L^2}$$
$$-\sqrt{\frac{T}{12}}M_2\|\dot u\|_{L^2}-M_2M_3-2^\alpha M_1M_3^{\alpha+1}-M_1M_3|P\bar u|^\alpha$$
$$-|P\bar u|^{2\alpha}\left(|P\bar u|^{-2\alpha}\int_0^T F(t,P\bar u)dt+\frac{a_4M_1^2}{2}\right).$$

It follows from $a_4 > T/12$ and (3.32) that φ is bounded from below. Arguing then as in Theorem 1.2 with $p = 2$. The proof is complete.

4. Examples

In this section, some examples will be given to illustrate our results.

Example 4.1. Consider the following system:

$$\begin{cases} (|u'(t)|^2u'(t))'+\nabla F(t,u(t))=e(t), \quad \text{a.e. } t\in[0,T], \\ u(0)-u(T)=\dot u(0)-\dot u(T)=0, \end{cases} \tag{4.1}$$

where $T > 0$, $p = 4$ and $q = 4/3$, $e \in L^1(0,T;\mathbf{R}^N)$ satisfies (1.3). Let

$$F(t,x) = (0.5T-t)\left(\sin^2 x_1+\cdots+\sin^2 x_r+\frac{1}{2}\sum_{j=r+1}^N x_j^2\right)^{7/4}$$
$$+(8T^3-t)\left(\sin^2 x_1+\cdots+\sin^2 x_r+\frac{1}{2}\sum_{j=r+1}^N x_j^2\right)^{5/3},$$

where $x = (x_1, x_2, \cdots, x_N)^T \in \mathbf{R}^N$. Obviously, F satisfies (1.2) with $T_i = \pi, i = 1, 2, \cdots, r$. Let

$$y=\sin^2 x_1+\cdots+\sin^2 x_r+\frac{1}{2}\sum_{j=r+1}^N x_j^2$$

and let $z = (\sin 2x_1, \cdots, \sin 2x_r, x_{r+1}, \cdots, x_N)^T$. Then

$$\nabla F(t,x)=\frac{7}{4}(0.5T-t)y^{3/4}z+\frac{5}{3}(8T^3-t)y^{2/3}z.$$

By Young's inequality, we have

$$\begin{aligned}|\nabla F(t,x)| &\leq \frac{7}{4}|0.5T-t|y^{3/4}|z|+\frac{5}{3}|8T^3-t|y^{2/3}|z| \\ &\leq \frac{7}{4}|0.5T-t||x|^{3/2}\left(r+|x|^2\right)^{1/2}+\frac{5}{3}|8T^3-t||x|^{4/3}\left(r+|x|^2\right)^{1/2} \\ &\leq \frac{7}{4}|0.5T-t||x|^{5/2}+\frac{5}{3}|8T^3-t||x|^{7/3}\end{aligned}$$

$$
\begin{aligned}
&+\frac{7}{4}\sqrt{r}|0.5T-t||x|^{3/2}+\frac{5}{3}\sqrt{r}|8T^3-t||x|^{4/3}\\
\le\ & \frac{7}{4}|0.5T-t||x|^{5/2}+\frac{40}{3}T^3|x|^{7/3}+\frac{7}{8}\sqrt{r}T|x|^{3/2}+\frac{40}{3}\sqrt{r}T^3|x|^{4/3}\\
\le\ & \frac{7}{4}\left(|0.5T-t|+\varepsilon\right)|x|^{5/2}+A_1(\varepsilon),
\end{aligned}
$$

where $0<\varepsilon<1,\ A_1(\varepsilon)>1$ is a function of ε. Thus F satisfies (1.8) with $\alpha=5/2$ and

$$f(t)=\frac{7}{4}(|0.5T-t|+\varepsilon),\quad g(t)=A_1(\varepsilon).$$

Then

$$\int_0^T f(t)dt=\frac{7T^2}{16}+\frac{7T\varepsilon}{4},$$

and so

$$\frac{(p+1)T}{p(q+1)}\left(\int_0^T f(t)dt\right)^q=\frac{15T}{28}\left(\frac{7T^2}{16}+\frac{7T\varepsilon}{4}\right)^{4/3}.$$

On the other hand, as $x\in 0\times\mathbf{R}^{N-r}$, we have $|x|=\left(\sum_{j=r+1}^N x_j^2\right)^{1/2}$, and

$$
\begin{aligned}
F(t,x)\ &=\ (0.5T-t)\left(\frac{1}{2}\sum_{j=r+1}^N x_j^2\right)^{7/4}+(8T^3-t)\left(\frac{1}{2}\sum_{j=r+1}^N x_j^2\right)^{5/3}\\
&=\ \frac{0.5T-t}{2^{7/4}}|x|^{7/2}+\frac{8T^3-t}{2^{5/3}}|x|^{10/3}.
\end{aligned}
$$

Then

$$\liminf_{|x|\to\infty}|x|^{-q\alpha}\int_0^T F(t,x)dt=\liminf_{|x|\to\infty}|x|^{-10/3}\int_0^T F(t,x)dt=\frac{16T^4-T^2}{2^{8/3}}.$$

Thus, if $T>0.265$, we can choose $\varepsilon>0$ sufficient small such that

$$
\begin{aligned}
\liminf_{|x|\to\infty}|x|^{-10/3}\int_0^T F(t,x)dt\ &=\ \frac{16T^4-T^2}{2^{8/3}}\\
&>\ \frac{15T}{28}\left(\frac{7T^2}{16}+\frac{7T\varepsilon}{4}\right)^{4/3}\\
&=\ \frac{(p+1)T}{p(q+1)}\left(\int_0^T f(t)dt\right)^q.
\end{aligned}
$$

Hence, when $T>0.265$, F satisfies all conditions of Theorem 1.1. Therefore, system (4.1) has at least $r+1$ geometrically distinct solutions in $W_T^{1,4}$.

Example 4.2. Consider the following system:

$$\text{(4.2)} \qquad \begin{cases} (|u'(t)|^2 u'(t))' + \nabla F(t, u(t)) = e(t), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

where $T > 0$, $p = 4$, $q = 4/3$ and $e \in L^1(0, T; \mathbf{R}^N)$ satisfies (1.3). Let

$$\begin{aligned} F(t, x) \;=\; & (0.5T - t) \left(\sin^2 x_1 + \cdots + \sin^2 x_r + \frac{1}{2} \sum_{j=r+1}^{N} x_j^2 \right)^{7/4} \\ & - 2T^3 \left(\sin^2 x_1 + \cdots + \sin^2 x_r + \frac{1}{2} \sum_{j=r+1}^{N} x_j^2 \right)^{5/3}. \end{aligned}$$

where $x = (x_1, x_2, \cdots, x_N)^T \in \mathbf{R}^N$. Similar to the argument in Example 4.1, it is easy to show that F satisfies (1.8) with $\alpha = 5/2$ and

$$f(t) = \frac{7}{4}(|0.5T - t| + \varepsilon), \quad g(t) = A_2(\varepsilon),$$

where $0 < \varepsilon < 1, \;\; A_2(\varepsilon) > 1$ is a function of ε. Then

$$\int_0^T f(t)dt = \frac{7T^2}{16} + \frac{7T\varepsilon}{4},$$

and so

$$-\frac{T}{q(q+1)} \left(\int_0^T f(t)dt \right)^q = -\frac{9T}{28} \left(\frac{7T^2}{16} + \frac{7T\varepsilon}{4} \right)^{4/3}.$$

On the other hand, as $x \in 0 \times \mathbf{R}^{N-r}$, we have $|x| = \left(\sum_{j=r+1}^{N} x_j^2 \right)^{1/2}$, and

$$\begin{aligned} F(t, x) \;&=\; (0.5T - t) \left(\frac{1}{2} \sum_{j=r+1}^{N} x_j^2 \right)^{7/4} - 2T^3 \left(\frac{1}{2} \sum_{j=r+1}^{N} x_j^2 \right)^{5/3} \\ &=\; \frac{0.5T - t}{2^{7/4}} |x|^{7/2} - \frac{T^3}{2^{2/3}} |x|^{10/3}. \end{aligned}$$

Then

$$\limsup_{|x| \to \infty} |x|^{-q\alpha} \int_0^T F(t, x)dt = \limsup_{|x| \to \infty} |x|^{-10/3} \int_0^T F(t, x)dt = -\frac{T^4}{2^{2/3}}.$$

Then, if $T > 5103/1048576 \approx 0.00487$, we can choose $\varepsilon > 0$ sufficient small such that

$$\begin{aligned} \limsup_{|x| \to \infty} |x|^{-10/3} \int_0^T F(t, x)dt \;&=\; -\frac{T^4}{2^{2/3}} \\ &<\; -\frac{9}{28} T \left(\frac{7T^2}{16} + \frac{7T\varepsilon}{4} \right)^{4/3} \\ &=\; -\frac{T}{q(q+1)} \left(\int_0^T f(t)dt \right)^q. \end{aligned}$$

Hence, if $T > 5103/1048576 \approx 0.00487$, F satisfies all conditions of Theorem 1.2. Therefore, system (4.2) has at least $r+1$ geometrically distinct solutions in $W_T^{1,4}$.

Example 4.3. Consider second-order Hamiltonian system:

$$\begin{cases} \ddot{u}(t) + \nabla F(t, u(t)) = e(t), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{4.3}$$

where $T > 0$, $e \in L^1(0, T; \mathbf{R}^N)$ satisfies (1.3), and F is defined by (1.6). Obviously, F satisfies (1.2) with $T_i = \pi, i = 1, \cdots, r$. Let

$$y = r + 1 + \sin^2 x_1 + \cdots + \sin^2 x_r + \frac{1}{2}\sum_{j=r+1}^{N} x_j^2$$

and let $z = (\sin 2x_1, \cdots, \sin 2x_r, x_{r+1}, \cdots, x_N)^T$. Then

$$\nabla F(t, x) = \frac{7}{8}(0.5T - t)y^{-1/8}z + \frac{3}{64}T^4 y^{-1/4}z.$$

Thus

$$\begin{aligned} |\nabla F(t, x)| &\le \frac{7}{8}|0.5T - t|y^{-1/8}|z| + \frac{3T^4}{64}y^{-1/4}|z| \\ &\le \frac{7}{8}|0.5T - t|\, y^{-1/8}(2y)^{1/2} + \frac{3}{64}T^4\, y^{-1/4}(2y)^{1/2} \\ &\le \frac{7\sqrt{2}}{8}|0.5T - t|\left(r + 1 + |x|^2\right)^{3/8} + \frac{3\sqrt{2}}{64}T^4\left(r + 1 + |x|^2\right)^{1/4} \\ &\le \frac{7\sqrt{2}}{8}|0.5T - t||x|^{3/4} + \frac{7\sqrt{2}}{8}|0.5T - t|\,(r+1)^{3/8} \\ &\quad + \frac{3\sqrt{2}}{64}T^4|x|^{1/2} + \frac{3\sqrt{2}}{64}T^4\,(r+1)^{1/4} \\ &\le \frac{7\sqrt{2}}{8}(|0.5T - t| + \varepsilon)|x|^{3/4} + A_3(\varepsilon), \end{aligned}$$

where $0 < \varepsilon < 1$, $A_3(\varepsilon) > 1$ is a function of ε. Thus F satisfies (1.4) with $\alpha = 3/4$ and

$$f(t) = \frac{7\sqrt{2}}{8}(|0.5T - t| + \varepsilon), \quad g(t) = A_3(\varepsilon).$$

Then

$$\int_0^T f(t)dt = \frac{7\sqrt{2}T^2}{32} + \frac{7\sqrt{2}\,\varepsilon T}{8}.$$

As $x \in 0 \times \mathbf{R}^{N-r}$, then $|x| = \left(\sum_{j=r+1}^N x_j^2\right)^{1/2}$ and

$$\begin{aligned} F(t, x) &= (0.5T - t)\left(r + 1 + \frac{1}{2}\sum_{j=r+1}^{N} x_j^2\right)^{7/8} + \frac{T^4}{16}\left(r + 1 + \frac{1}{2}\sum_{j=r+1}^{N} x_j^2\right)^{3/4} \\ &= (0.5T - t)\left(r + 1 + \frac{1}{2}|x|^2\right)^{7/8} + \frac{T^4}{16}\left(r + 1 + \frac{1}{2}|x|^2\right)^{3/4}. \end{aligned}$$

Thus if $T > 0$, we can choose $\varepsilon > 0$ sufficient small such that

$$\begin{aligned}\liminf_{|x|\to\infty} |x|^{-3/2} \int_0^T F(t,x)dt &= 2^{-19/4}T^5 \\ &> \frac{T}{8}\left(\frac{7\sqrt{2}T^2}{32} + \frac{7\sqrt{2}\,\varepsilon T}{8}\right)^2 \\ &= \frac{T}{8}\left(\int_0^T f(t)dt\right)^2.\end{aligned}$$

Hence, F satisfies all conditions of Theorem 1.3. Therefore, system (4.3) has at least $r+1$ geometrically distinct solutions in $W_T^{1,2}$.

Example 4.4. Consider second-order Hamiltonian system:

$$\text{(4.4)} \qquad \begin{cases} \ddot{u}(t) + \nabla F(t,u(t)) = e(t), & \text{a.e. } t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

where $T > 0$, $e \in L^1(0,T;\mathbf{R}^N)$ satisfies (1.3), and

$$\begin{aligned}F(t,x) = &\ (0.5T - t)\left(r + 1 + \sin^2 x_1 + \cdots + \sin^2 x_r + \frac{1}{2}\sum_{j=r+1}^N x_j^2\right)^{7/8} \\ &-\frac{T^4}{24}\left(r + 1 + \sin^2 x_1 + \cdots + \sin^2 x_r + \frac{1}{2}\sum_{j=r+1}^N x_j^2\right)^{3/4},\end{aligned}$$

where $x = (x_1, x_2, \cdots, x_N)^T \in \mathbf{R}^N$. Obviously, F satisfies (1.2) with $T_i = \pi, i = 1, \cdots, r$. Similar to the argument in Example 4.3, it is easy to show that F satisfies (1.4) with $\alpha = 3/4$ and

$$f(t) = \frac{7\sqrt{2}}{8}(|0.5T - t| + \varepsilon), \quad g(t) = A_4(\varepsilon),$$

where $0 < \varepsilon < 1,\ A_4(\varepsilon) > 1$ is a function of ε. Then

$$\int_0^T f(t)dt = \frac{7\sqrt{2}}{32}T^2 + \frac{7\sqrt{2}}{8}\varepsilon T.$$

As $x \in 0 \times \mathbf{R}^{N-r}$, then $|x| = \left(\sum_{j=r+1}^N x_j^2\right)^{1/2}$ and

$$\begin{aligned}F(t,x) &= (0.5T - t)\left(r + 1 + \frac{1}{2}\sum_{j=r+1}^N x_j^2\right)^{7/8} - \frac{T^4}{24}\left(r + 1 + \frac{1}{2}\sum_{j=r+1}^N x_j^2\right)^{3/4} \\ &= (0.5T - t)\left(r + 1 + \frac{1}{2}|x|^2\right)^{7/8} - \frac{T^4}{24}\left(r + 1 + \frac{1}{2}|x|^2\right)^{3/4}.\end{aligned}$$

Thus if $T > 0$, we can choose $\varepsilon > 0$ sufficient small such that

$$\begin{aligned}\limsup_{|x|\to\infty} |x|^{-3/2} \int_0^T F(t,x)dt &= -\frac{1}{2^{3/4}\cdot 24}T^5 \\ &< -\frac{T}{24}\left(\frac{7\sqrt{2}}{32}T^2 + \frac{7\sqrt{2}}{8}\varepsilon T\right)^2 \\ &= -\frac{T}{24}\left(\int_0^T f(t)dt\right)^2.\end{aligned}$$

Hence, F satisfies all conditions of Theorem 1.4. Therefore, system (4.4) has at least $r+1$ geometrically distinct solutions in $W_T^{1,2}$.

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Xingyong Zhang and Xianhua Tang
School of Mathematical Sciences and Computing Technology
Central South University
Changsha, Hunan 410083
P. R. China
E-mail: xhtang2003@gmail.com