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STRONG AND WEAK CONVERGENCE THEOREMS FOR GENERALIZED MIXED EQUILIBRIUM PROBLEM WITH PERTURBATION AND FIXED POINTED PROBLEM OF INFINITELY MANY NONEXPANSIVE MAPPINGS

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Abstract. Very recently, Plubtieng and Kumam [S. Plubtieng, P. Kumam, Weak convergence theorem for monotone mappings and a countable family of nonexpansive mappings, J. Comput. Appl. Math. 224 (2009) 614-621] proposed an iterative algorithm for finding a common solution of a variational inequality problem for an inverse-strongly monotone mapping and a fixed point problem of a countable family of nonexpansive mappings, and obtained a weak convergence theorem. In this paper, based on Plubtieng-Kumam's iterative algorithm we introduce a new iterative algorithm for finding a common solution of a generalized mixed equilibrium problem with perturbation and a fixed point problem of a countable family of nonexpansive mappings in a Hilbert space. We first derive a strong convergence theorem for this new algorithm under appropriate assumptions and then consider a special case of this new algorithm. Moreover, we establish a weak convergence theorem for this special case under some weaker assumptions. Such a weak convergence theorem unifies, improves and extends Plubtieng-Kumam's weak convergence theorem. It is worth pointing out that the proof method of strong convergence theorem is very different from the one of weak convergence theorem.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $S : C \to C$ be a self-mapping on C.

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We denote by $F(S)$ the set of fixed points of S and by P_C the metric projection of H onto C. Moreover, we also denote by $\mathbf{R}$ the set of all real numbers. A mapping S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

If C is a nonempty bounded closed convex subset and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty. A mapping A of C into H is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in C.$$

A is called α-inverse-strongly-monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2, \quad \forall u, v \in C.$$

It is obvious that any α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous.

Very recently, Peng and Yao [1] introduced the following generalized mixed equilibrium problem of finding $\bar{x} \in C$ such that

$$f(\bar{x}, y) + \varphi(y) - \varphi(\bar{x}) + \langle A\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $A : C \to H$ is a nonlinear mapping, $\varphi : C \to \mathbf{R}$ is a function and $f : C \times C \to \mathbf{R}$ is a bifunction. The set of solutions of problem (1.1) is denoted by $GMEP$. Subsequently, this problem was also considered by Yao, Liou and Yao [2], and Ceng and Yao [29]. Inspired by problem (1.1) we introduce and investigate the following generalized mixed equilibrium problem with perturbation: Find $\bar{x} \in C$ such that

$$f(\bar{x}, y) + \varphi(y) - \varphi(\bar{x}) + \langle (A + B)\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C, \tag{1.2}$$

where $A, B : C \to H$ are nonlinear mappings, $\varphi : C \to \mathbf{R}$ is a function and $f : C \times C \to \mathbf{R}$ is a bifunction. The set of solutions of problem (1.2) is denoted by *GMEPP*.

If $B = 0$, then problem (1.2) reduces to problem (1.1).

If $A = B = 0$, then problem (1.2) reduces to the following mixed equilibrium problem of finding $\bar{x} \in C$ such that

$$f(\bar{x}, y) + \varphi(y) - \varphi(\bar{x}) \geq 0, \quad \forall y \in C,$$

which was considered by Ceng and Yao [3]. The set of solutions of this problem is denoted by $MEP(f, \varphi)$.

If $\varphi = 0$ and $B = 0$, then problem (1.2) reduces to the following generalized equilibrium problem of finding $\bar{x} \in C$ such that

$$f(\bar{x}, y) + \langle A\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C, \tag{1.3}$$

which was studied by Takahashi and Takahashi [4].

If $\varphi = 0$ and $A = B = 0$, then problem (1.2) reduces to the following equilibrium problem of finding $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in C.$$

If $f = 0$, $\varphi = 0$ and $B = 0$, then problem (1.2) reduces to the following classical variational inequality problem of finding $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of problem (1.4) is denoted by $VI(C, A)$. The variational inequality problem has been extensively studied in the literature; see [5-15,26] and the references therein. Recently, Nadezhkina and Takahashi [12] and Zeng and Yao [14] proposed some variants of Korpelevich's extragradient method [13] for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem.

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others; see, e.g., [1-4,16-21,29]. In order to solve problem (1.1), Peng and Yao [1] developed a CQ method and Ceng and Yao [29] gave a new variant of Korpelevich's extragradient method [13]. Peng and Yao [1] established some strong convergence results for finding a common element of the set of solutions of problem (1.1), the set of solutions of problem (1.4), and the set of fixed points of a nonexpansive mapping. Moreover, Ceng and Yao [29] derived some strong convergence theorems for finding a common element of the set of solutions of problem (1.1), the set of solutions for a general system of generalized equilibria, and the set of fixed points of a k-strictly pseudocontractive mapping.

On the other hand, Aoyama, Kimura, Takahashi and Toyoda [22] recently introduced an iterative scheme defined by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n x_n, \quad n = 1, 2, ..., \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, C is a nonempty closed convex subset of H and $\{S_n\}$ is a sequence of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. They also proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a common fixed point of nonexpansive mappings T_n, $n = 1, 2, ...$

Very recently, Plubtieng and Kumam [30] proposed the following iteration process for finding a common element of the set of solutions of variational inequality (1.4) and the set of common fixed points of infinitely many nonexpansive mappings $\{S_n\}_{n=1}^{\infty}$ of C into itself and proved the weak convergence of the sequence generated by this iteration process to an element of $\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A)$.

Theorem PK. ([30, Theorem 4]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly-monotone mapping of C into H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C defined by $x_0 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n P_C(x_n - \lambda_n A x_n),$$

for all $n = 0, 1, 2, ...$, where $0 < a < \lambda_n < b < 2\alpha,\ 0 < c < \alpha_n < d < 1$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. Then $\{x_n\}$ converges weakly to $z \in F(S) \cap VI(C, A)$, where $z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.

Throughout this paper, suppose that $\{S_n\}$ is a sequence of nonexpansive self-mappings on a nonempty closed convex subset C of a real Hilbert space H. Motivated and inspired by Aoyama, Kimura, Takahashi and Toyoda [22], Takahashi and Takahashi [4], Plubtieng and Kumam [30] and Ceng and Yao [29] we introduce the following iterative algorithm for finding a common solution of problem (1.2) and the fixed point problem of infinitely many nonexpansive mappings $\{S_n\}$: For fixed $u \in C$ and $x_1 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by

$$\begin{aligned} y_n &= T_{\lambda_n}^{(f,\varphi)}(x_n - \lambda_n(A + B)x_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S_n y_n, \end{aligned} \tag{1.6}$$

for all $n = 1, 2, ...$, where $0 \leq \lambda_n \leq \min\{\alpha, \beta\}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$, $f : C \times C \to \mathbf{R}$ is a bifunction, $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function, and $A, B : C \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. On one hand, following the idea of the proof in Ceng and Yao [29, Theorem 3.1] we derive a strong convergence theorem for algorithm (1.6) under appropriate assumptions. On the other hand, we consider a special case of algorithm (1.6): For fixed $u \in C$ and $x_1 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by

$$\begin{aligned} y_n &= T_{\lambda_n}^{(f,\varphi)}(x_n - \lambda_n(A + B)x_n), \\ x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n S_n y_n, \end{aligned} \tag{1.7}$$

for all $n = 1, 2, \ldots$. Following the idea of the proof in Plubtieng and Kumam [30, Theorem 4] we establish a weak convergence theorem for algorithm (1.7) under some weaker assumptions. Such a weak convergence theorem unifies, improves and extends Plubtieng and Kumam [30, Theorem 4]. It is worth pointing out that the proof method of strong convergence theorem for algorithm (1.6) is very different from the one of weak convergence theorem for algorithm (1.7).

2. Preliminaries and Notations

Let C be a nonempty closed convex subset of a real Hilbert space H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Such a P_C is called the metric projection of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C, i.e.,

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

It is also known that, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{2.1}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|P_C x - y\|^2 \tag{2.2}$$

for all $x \in H$ and $y \in C$. In a real Hilbert space H, it is well known that

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle \tag{2.3}$$

and

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{2.4}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. It is also known that H satisfies the Opial condition [23], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

In the context of the variational inequality problem, it is easy to see that

$$u \in VI(C, A) \ \Leftrightarrow \ u = P_C(u - \lambda A u), \quad \forall \lambda > 0. \tag{2.5}$$

Recall that a mapping $A : C \to H$ is called α-inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that any α-inverse-strongly monotone mapping A is Lipschitz continuous. Meantime, we have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} &\|(I-\lambda A)u-(I-\lambda A)v\|^2 \\ &= \|(u-v)-\lambda(Au-Av)\|^2 \\ &= \|u-v\|^2-2\lambda\langle u-v, Au-Av\rangle+\lambda^2\|Au-Av\|^2 \\ &\leq \|u-v\|^2+\lambda(\lambda-2\alpha)\|Au-Av\|^2. \end{aligned} \tag{2.6}$$

So, if $\lambda \leq 2\alpha$, then $I-\lambda A$ is a nonexpansive mapping of C into H.

The following lemmas will be used for proving the convergence result of this paper in the sequel.

Lemma 2.1. ([22, Lemma 3.2]). *Let C be a nonempty closed subset of a Banach space and let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself. Suppose that $\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z-T_nz\| : z \in C\} < \infty$ Then, for each $y \in C$, $\{T_ny\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by*

$$Ty = \lim_{n\to\infty} T_ny, \quad \forall y \in C.$$

Then $\lim_{n\to\infty}\sup\{\|T_nz-Tz\| : z \in C\} = 0$.

Lemma 2.2. (see [24]). *Demiclosedness principle. Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If T has a fixed point, then $I-T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

Lemma 2.3. (see [26]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of real numbers such that

(*i*) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$, *or equivalently,*

$$\prod_{n=1}^{\infty}(1-\alpha_n) := \lim_{n\to\infty}\prod_{k=1}^{n}(1-\alpha_k) = 0;$$

(*ii*) $\limsup_{n\to\infty}\beta_n \leq 0$, *or*

$(ii)'$ $\sum_{n=1}^{\infty} \alpha_n \beta_n$ *is convergent.*

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. ([3, Lemma 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (H1)-(H4):*

$(H1)$ $f(x, x) = 0, \ \forall x \in C$;

$(H2)$ f *is monotone, i.e.,* $f(x, y) + f(y, x) \leq 0, \ \forall x, y \in C$;

$(H3)$ *for each* $y \in C, \ x \mapsto f(x, y)$ *is weakly upper semicontinuous;*

$(H4)$ *for each* $x \in C, \ y \mapsto f(x, y)$ *is convex and lower semicontinuous.*

Let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $T_r^{(f,\varphi)} : H \to C$ as follows:

$$T_r^{(f,\varphi)}(x) = \{z \in C : f(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\}$$

for all $x \in H$. Assume that either (A1) or (A2) holds:

(1) *for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \backslash D_x$,*

$$f(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(2) *C is a bounded set.*

Then there hold following:

(i) $T_r^{(f,\varphi)}(x) \neq \emptyset$ *for each $x \in H$ and $T_r^{(f,\varphi)}$ is single-valued;*

(ii) $T_r^{(f,\varphi)}$ *is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r^{(f,\varphi)}x - T_r^{(f,\varphi)}y\|^2 \leq \langle T_r^{(f,\varphi)}x - T_r^{(f,\varphi)}y, x - y\rangle;$$

(iii) $F(T_r^{(f,\varphi)}) = MEP(f, \varphi)$;

(iv) $MEP(f, \varphi)$ *is closed and convex.*

Remark 2.1. If $\varphi = 0$, then $T_r^{(f,\varphi)}$ is rewritten as T_r^f.

Lemma 2.5. ([29, Proposition 2.1]). *Let C, H, f, φ and $T_r^{(f,\varphi)}$ be as in Lemma 2.4. Then the following holds:*

$$\|T_s^{(f,\varphi)}x - T_t^{(f,\varphi)}x\|^2 \leq \frac{s - t}{s}\langle T_s^{(f,\varphi)}x - T_t^{(f,\varphi)}x, T_s^{(f,\varphi)}x - x\rangle$$

for all $s, t > 0$ and $x \in H$.

Corollary 2.1. ([4, Lemma 2.3]). *Let C, H, f and T_r^f be as in Remark 2.1. Then the following holds:*

$$\|T_s^f x - T_t^f x\|^2 \leq \frac{s-t}{s}\langle T_s^f x - T_t^f x, T_s^f x - x\rangle$$

for all $s, t > 0$ and $x \in H$.

Lemma 2.6. (see [27]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.7. ([5, Lemma 3.2]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H such that*

$$\|x_{n+1} - y\| \leq \|x_n - y\|,$$

for all $y \in C$ and $n \geq 1$. Then the sequence $\{P_C(x_n)\}$ converges strongly to some point in C.

Lemma 2.8. (see [25, p. 303]). *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.9. ([5, Lemma 3.1]). *Let H be a real Hilbert space. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n = 1, 2, ...$, and let $\{v_n\}$ and let $\{w_n\}$ be sequences of H such that*

$$\limsup_{n\to\infty} \|v_n\| \leq c, \quad \limsup_{n\to\infty} \|w_n\| \leq c,$$

and

$$\lim_{n\to\infty} \|\alpha_n v_n + (1-\beta_n)w_n\| = c,$$

for some $c > 0$. Then,

$$\lim_{n\to\infty} \|v_n - w_n\| = 0.$$

3. Strong and Weak Convergence Theorems

In this section, we prove some strong and weak convergence theorems for a generalized mixed equilibrium problem with perturbation and a countable family of nonexpansive mappings.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \times C \to \mathbf{R}$ be a bifunction which satisfies assumptions (H1)-(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with assumptions (A1) or (A2). Let $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, and $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap GMEPP \neq \emptyset$. For fixed $u \in C$ and $x_1 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by*

$$
(3.1) \qquad \begin{aligned} y_n &= T_{\lambda_n}^{(f,\varphi)}(x_n - \lambda_n(A+B)x_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S_n y_n, \end{aligned}
$$

for all $n = 1, 2, \ldots$, where $0 \leq \lambda_n \leq \min\{\alpha, \beta\}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for all* $n \geq 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$;

(v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \min\{\alpha, \beta\}$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. Then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap GMEPP}u$.

Proof. We divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, take $z \in F(S) \cap GMEPP$ arbitrarily. Since $z = T_{\lambda_n}^{(f,\varphi)}(z - \lambda_n(A + B)z) = Sz$, A and B are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, and $0 \leq \lambda_n \leq \min\{\alpha, \beta\}$, we know that, for any $n \geq 0$,

$$
(3.2) \qquad \begin{aligned} &\|y_n - z\|^2 \\ &= \|T_{\lambda_n}^{(f,\varphi)}(x_n - \lambda_n(A+B)x_n) - T_{\lambda_n}^{(f,\varphi)}(z - \lambda_n(A+B)z)\|^2 \\ &\leq \|(x_n - \lambda_n(A+B)x_n) - (z - \lambda_n(A+B)z)\|^2 \\ &= \|\tfrac{1}{2}[x_n - z - 2\lambda_n(Ax_n - Az)] + \tfrac{1}{2}[x_n - z - 2\lambda_n(Bx_n - Bz)]\|^2 \\ &\leq \tfrac{1}{2}\|x_n - z - 2\lambda_n(Ax_n - Az)\|^2 + \tfrac{1}{2}\|x_n - z - 2\lambda_n(Bx_n - Bz)\|^2 \\ &\leq \tfrac{1}{2}[\|x_n - z\|^2 + 4\lambda_n(\lambda_n - \alpha)\|Ax_n - Az\|^2] \\ &\quad + \tfrac{1}{2}[\|x_n - z\|^2 + 4\lambda_n(\lambda_n - \beta)\|Bx_n - Bz\|^2] \\ &= \|x_n - z\|^2 + 2\lambda_n(\lambda_n - \alpha)\|Ax_n - Az\|^2 + 2\lambda_n(\lambda_n - \beta)\|Bx_n - Bz\|^2 \\ &\leq \|x_n - z\|^2, \end{aligned}
$$

for every $n = 1, 2, \ldots$. Hence, from (3.1) and (3.2) it follows that

$$\begin{aligned} &\|x_{n+1} - z\| \\ &= \|\alpha_n(u - z) + \beta_n(x_n - z) + \gamma_n(y_n - z) + \delta_n(S_n y_n - z)\| \\ &\le \alpha_n\|u - z\| + \beta_n\|x_n - z\| + \gamma_n\|y_n - z\| + \delta_n\|S_n y_n - z\| \\ &\le \alpha_n\|u - z\| + \beta_n\|x_n - z\| + (\gamma_n + \delta_n)\|y_n - z\| \\ &\le \alpha_n\|u - z\| + \beta_n\|x_n - z\| + (\gamma_n + \delta_n)\|x_n - z\| \\ &\le \alpha_n\|u - z\| + (1 - \alpha_n)\|x_n - z\|. \end{aligned} \tag{3.3}$$

By induction, we obtain that for all $n \ge 1$

$$\|x_n - z\| \le \max\{\|u - z\|, \|x_0 - z\|\}.$$

Thus, $\{x_n\}$ is bounded. Consequently, we deduce immediately that $\{Ax_n\}, \{Bx_n\}$, $\{y_n\}$ and $\{S_n y_n\}$ are bounded.

Step 2. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
Indeed, define $x_{n+1} = \beta_n x_n + (1 - \beta_n)w_n$ for all $n \ge 1$. It follows that

$$\begin{aligned} &w_{n+1} - w_n \\ &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}u + \gamma_{n+1}y_{n+1} + \delta_{n+1}S_{n+1}y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n y_n + \delta_n S_n y_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}u}{1-\beta_{n+1}} - \frac{\alpha_n u}{1-\beta_n} + \frac{\gamma_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n y_n}{1-\beta_n} + \frac{\delta_{n+1}S_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n S_n y_n}{1-\beta_n} \\ &= (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})u + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(y_{n+1} - y_n) + (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n})y_n \\ &\quad + \frac{\delta_{n+1}}{1 - \beta_{n+1}}(S_{n+1}y_{n+1} - S_n y_{n+1}) + \frac{\delta_{n+1}}{1 - \beta_{n+1}}(S_n y_{n+1} - S_n y_n) \\ &\quad + (\frac{\delta_{n+1}}{1 - \beta_{n+1}} - \frac{\delta_n}{1 - \beta_n})S_n y_n. \end{aligned} \tag{3.4}$$

Observe that

$$\begin{aligned} &\|(x_{n+1} - \lambda_{n+1}(A + B)x_{n+1}) - (x_n - \lambda_n(A + B)x_n)\| \\ &= \|x_{n+1} - x_n - \lambda_{n+1}((A + B)x_{n+1} - (A + B)x_n) \\ &\quad +(\lambda_n - \lambda_{n+1})(A + B)x_n\| \\ &\le \|x_{n+1} - x_n - \lambda_{n+1}((A + B)x_{n+1} - (A + B)x_n)\| \\ &\quad +|\lambda_{n+1} - \lambda_n|\|(A + B)x_n\| \\ &\le \tfrac{1}{2}[\|x_{n+1} - x_n - 2\lambda_{n+1}(Ax_{n+1} - Ax_n)\| \\ &\quad +\|x_{n+1} - x_n - 2\lambda_{n+1}(Bx_{n+1} - Bx_n)\|] \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&+|\lambda_{n+1}-\lambda_n|\|(A+B)x_n\| \\
&\le \tfrac{1}{2}[\|x_{n+1}-x_n\|+\|x_{n+1}-x_n\|]+|\lambda_{n+1}-\lambda_n|\|(A+B)x_n\| \\
&= \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|(A+B)x_n\|,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\|y_{n+1}-y_n\| \\
&= \|T^{(f,\varphi)}_{\lambda_{n+1}}(x_{n+1}-\lambda_{n+1}(A+B)x_{n+1})-T^{(f,\varphi)}_{\lambda_n}(x_n-\lambda_n(A+B)x_n)\| \\
&\le \|T^{(f,\varphi)}_{\lambda_{n+1}}(x_{n+1}-\lambda_{n+1}(A+B)x_{n+1})-T^{(f,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(A+B)x_n)\| \\
&\quad +\|T^{(f,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(A+B)x_n)-T^{(f,\varphi)}_{\lambda_n}(x_n-\lambda_n(A+B)x_n)\| \\
&\le \|(x_{n+1}-\lambda_{n+1}(A+B)x_{n+1})-(x_n-\lambda_n(A+B)x_n)\| \\
&\quad +\|T^{(f,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(A+B)x_n)-T^{(f,\varphi)}_{\lambda_n}(x_n-\lambda_n(A+B)x_n)\| \\
&\le \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|(A+B)x_n\| \\
&\quad +\|T^{(f,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(A+B)x_n)-T^{(f,\varphi)}_{\lambda_n}(x_n-\lambda_n(A+B)x_n)\|.
\end{aligned}
\tag{3.6}
$$

Note that

$$
\begin{aligned}
\frac{\delta_{n+1}}{1-\beta_{n+1}}-\frac{\delta_n}{1-\beta_n} &= \frac{1-\alpha_{n+1}-\beta_{n+1}-\gamma_{n+1}}{1-\beta_{n+1}}-\frac{1-\alpha_n-\beta_n-\gamma_n}{1-\beta_n} \\
&= -\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)-\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right).
\end{aligned}
$$

So, it follows from (3.4) and (3.6) that

$$
\begin{aligned}
&\|w_{n+1}-w_n\| \\
&\le \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}+\frac{\alpha_n}{1-\beta_n}\right)\|u\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\| \\
&\quad +\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|\|y_n\|+\frac{\delta_{n+1}}{1-\beta_{n+1}}\|S_{n+1}y_{n+1}-S_ny_{n+1}\| \\
&\quad +\frac{\delta_{n+1}}{1-\beta_{n+1}}\|S_ny_{n+1}-S_ny_n\|+\left|\frac{\delta_{n+1}}{1-\beta_{n+1}}-\frac{\delta_n}{1-\beta_n}\right|\|S_ny_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|u\|+\|S_ny_n\|)+\frac{\alpha_n}{1-\beta_n}(\|u\|+\|S_ny_n\|) \\
&\quad +\frac{\gamma_{n+1}+\delta_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\|+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|(\|y_n\|+\|S_ny_n\|) \\
&\quad +\frac{\delta_{n+1}}{1-\beta_{n+1}}\|S_{n+1}y_{n+1}-S_ny_{n+1}\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|u\|+\|S_ny_n\|)+\frac{\alpha_n}{1-\beta_n}(\|u\|+\|S_ny_n\|) \\
&\quad +\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|(A+B)x_n\|
\end{aligned}
\tag{3.7}
$$

$$+\|T^{(f,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(A+B)x_n)-T^{(f,\varphi)}_{\lambda_n}(x_n-\lambda_n(A+B)x_n)\|$$
$$+|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}|(\|y_n\|+\|S_ny_n\|)$$
$$+\frac{\delta_{n+1}}{1-\beta_{n+1}}\|S_{n+1}y_{n+1}-S_ny_{n+1}\|.$$

Note that $0<\liminf_{n\to\infty}\lambda_n\leq\limsup_{n\to\infty}\lambda_n<\min\{\alpha,\beta\}$ and $\lim_{n\to\infty}(\lambda_n-\lambda_{n+1})=0$. Then utilizing Lemma 2.5 we have

$$\lim_{n\to\infty}\|T^{(f,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(A+B)x_n)-T^{(f,\varphi)}_{\lambda_n}(x_n-\lambda_n(A+B)x_n)\|=0. \tag{3.8}$$

Since $\lim_{n\to\infty}\sup\{\|S_{n+1}z-S_nz\|:z\in B\}=0$ for any bounded subset B of C, we get

$$\lim_{n\to\infty}\|S_{n+1}y_{n+1}-S_ny_{n+1}\|=0. \tag{3.9}$$

Consequently, it follows from (3.8), (3.9) and conditions (ii), (iv), (v) that

$$\begin{aligned}&\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)\\ \leq&\limsup_{n\to\infty}\{\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|u\|+\|S_ny_n\|)+\frac{\alpha_n}{1-\beta_n}(\|u\|+\|S_ny_n\|)\\ &+|\lambda_{n+1}-\lambda_n|\|(A+B)x_n\|\\ &+\|T^{(f,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(A+B)x_n)-T^{(f,\varphi)}_{\lambda_n}(x_n-\lambda_n(A+B)x_n)\|\\ &+|\tfrac{\gamma_{n+1}}{1-\beta_{n+1}}-\tfrac{\gamma_n}{1-\beta_n}|(\|y_n\|+\|S_ny_n\|)+\tfrac{\delta_{n+1}}{1-\beta_{n+1}}\|S_{n+1}y_{n+1}-S_ny_{n+1}\|\}\\ =&\,0.\end{aligned}$$

Therefore, by Lemma 2.6 we obtain $\lim_{n\to\infty}\|w_n-x_n\|=0$. This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|w_n-x_n\|=0. \tag{3.10}$$

Step 3. $\lim_{n\to\infty}\|Ax_n-Az\|=\lim_{n\to\infty}\|Bx_n-Bz\|=0.$

Indeed, from (3.1) we get

$$\begin{aligned}&\|x_{n+1}-z\|^2\\ =&\langle\alpha_n(u-z)+\beta_n(x_n-z)+\gamma_n(y_n-z)+\delta_n(S_ny_n-z),x_{n+1}-z\rangle\\ =&\alpha_n\langle u-z,x_{n+1}-z\rangle+\beta_n\langle x_n-z,x_{n+1}-z\rangle\\ &+\langle\gamma_n(y_n-z)+\delta_n(S_ny_n-z),x_{n+1}-z\rangle\\ \leq&\alpha_n\langle u-z,x_{n+1}-z\rangle+\beta_n\|x_n-z\|\|x_{n+1}-z\|\\ &+\|\gamma_n(y_n-z)+\delta_n(S_ny_n-z)\|\|x_{n+1}-z\|\\ \leq&\alpha_n\langle u-z,x_{n+1}-z\rangle+\beta_n\|x_n-z\|\|x_{n+1}-z\|\end{aligned}$$

$$
\begin{aligned}
&+(\gamma_n+\delta_n)\|y_n-z\|\|x_{n+1}-z\| \\
&\le \alpha_n\langle u-z, x_{n+1}-z\rangle + \tfrac{\beta_n}{2}(\|x_n-z\|^2+\|x_{n+1}-z\|^2) \\
&+\tfrac{\gamma_n+\delta_n}{2}(\|y_n-z\|^2+\|x_{n+1}-z\|^2),
\end{aligned}
$$

that is,

$$
\|x_{n+1}-z\|^2 \le \frac{2\alpha_n}{1+\alpha_n}\langle u-z, x_{n+1}-z\rangle + \frac{\beta_n}{1+\alpha_n}\|x_n-z\|^2 + \frac{\gamma_n+\delta_n}{1+\alpha_n}\|y_n-z\|^2. \tag{3.11}
$$

So, in terms of (3.2) and (3.11) we have

$$
\begin{aligned}
&\|x_{n+1}-z\|^2 \\
&\le \frac{2\alpha_n}{1+\alpha_n}\|u-z\|\|x_{n+1}-z\| + \frac{\beta_n}{1+\alpha_n}\|x_n-z\|^2 \\
&+\frac{\gamma_n+\delta_n}{1+\alpha_n}[\|x_n-z\|^2+2\lambda_n(\lambda_n-\alpha)\|Ax_n-Az\|^2+2\lambda_n(\lambda_n-\beta)\|Bx_n-Bz\|^2] \\
&\le \frac{2\alpha_n}{1+\alpha_n}\|u-z\|\|x_{n+1}-z\| + \frac{\beta_n}{1+\alpha_n}\|x_n-z\|^2 + \frac{\gamma_n+\delta_n}{1+\alpha_n}\|x_n-z\|^2 \\
&+\frac{\gamma_n+\delta_n}{1+\alpha_n}[2\lambda_n(\lambda_n-\alpha)\|Ax_n-Az\|^2+2\lambda_n(\lambda_n-\beta)\|Bx_n-Bz\|^2] \\
&= \frac{2\alpha_n}{1+\alpha_n}\|u-z\|\|x_{n+1}-z\| + \frac{1-\alpha_n}{1+\alpha_n}\|x_n-z\|^2 \\
&+\frac{\gamma_n+\delta_n}{1+\alpha_n}[2\lambda_n(\lambda_n-\alpha)\|Ax_n-Az\|^2+2\lambda_n(\lambda_n-\beta)\|Bx_n-Bz\|^2].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&2\lambda_n(\lambda_n-\alpha)\|Ax_n-Az\|^2+2\lambda_n(\lambda_n-\beta)\|Bx_n-Bz\|^2 \\
&\le \frac{2\alpha_n}{\gamma_n+\delta_n}\|u-z\|\|x_{n+1}-z\| + \frac{1-\alpha_n}{\gamma_n+\delta_n}(\|x_n-z\|^2-\|x_{n+1}-z\|^2) \\
&\le \frac{2\alpha_n}{\gamma_n+\delta_n}\|u-z\|\|x_{n+1}-z\| + \frac{1-\alpha_n}{\gamma_n+\delta_n}(\|x_n-z\|+\|x_{n+1}-z\|)\|x_n-x_{n+1}\|.
\end{aligned}
$$

Since $\alpha_n \to 0$, $\|x_n-x_{n+1}\| \to 0$, $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < \min\{\alpha,\beta\}$, and $\liminf_{n\to\infty}(\gamma_n+\delta_n) > 0$, we have

$$
\lim_{n\to\infty}\|Ax_n-Az\| = \lim_{n\to\infty}\|Bx_n-Bz\| = 0. \tag{3.12}
$$

Step 4. $\lim_{n\to\infty}\|S_n y_n - y_n\| = 0$.

Indeed, utilizing the firm nonexpansivity of $T_{\lambda_n}^{(f,\varphi)}$, we conclude from (3.2) that

$$
\begin{aligned}
&\|y_n - z\|^2 \\
&= \|T_{\lambda_n}^{(f,\varphi)}(x_n - \lambda_n(A+B)x_n) - T_{\lambda_n}^{(f,\varphi)}(z - \lambda_n(A+B)z)\|^2 \\
&\le \langle (x_n - \lambda_n(A+B)x_n) - (z - \lambda_n(A+B)z), y_n - z\rangle \\
&= \tfrac{1}{2}[\|(x_n - \lambda_n(A+B)x_n) - (z - \lambda_n(A+B)z)\|^2 + \|y_n - z\|^2 \\
&\quad - \|(x_n - \lambda_n(A+B)x_n) - (z - \lambda_n(A+B)z) - (y_n - z)\|^2] \\
&\le \tfrac{1}{2}[\|x_n - z\|^2 + \|y_n - z\|^2 - \|(x_n - y_n) - \lambda_n((A+B)x_n - (A+B)z)\|^2]
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\|y_n - z\|^2 \\
&\le \|x_n - z\|^2 - \|(x_n - y_n) - \lambda_n((A+B)x_n - (A+B)z)\|^2 \\
&= \|x_n - z\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\langle x_n - y_n, (A+B)x_n - (A+B)z\rangle \\
&\quad - \lambda_n^2\|(A+B)x_n - (A+B)z\|^2 \\
&\le \|x_n - z\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\|x_n - y_n\|\|(A+B)x_n - (A+B)z\|.
\end{aligned} \tag{3.13}
$$

From (3.11) and (3.13), we have

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\le \frac{2\alpha_n}{1+\alpha_n}\langle u - z, x_{n+1} - z\rangle + \frac{\beta_n}{1+\alpha_n}\|x_n - z\|^2 + \frac{\gamma_n + \delta_n}{1+\alpha_n}\|y_n - z\|^2 \\
&\le \frac{2\alpha_n}{1+\alpha_n}\langle u - z, x_{n+1} - z\rangle + \frac{\beta_n}{1+\alpha_n}\|x_n - z\|^2 + \frac{\gamma_n + \delta_n}{1+\alpha_n}[\|x_n - z\|^2 - \|x_n - y_n\|^2 \\
&\quad + 2\lambda_n\|x_n - y_n\|\|(A+B)x_n - (A+B)z\|].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\frac{\gamma_n + \delta_n}{1+\alpha_n}\|x_n - y_n\|^2 \\
&\le \frac{2\alpha_n}{1+\alpha_n}\langle u - z, x_{n+1} - z\rangle + \frac{1-\alpha_n}{1+\alpha_n}\|x_n - z\|^2 \\
&\quad - \|x_{n+1} - z\|^2 + \frac{2\lambda_n(\gamma_n + \delta_n)}{1+\alpha_n}\|x_n - y_n\|\|(A+B)x_n - (A+B)z\| \\
&\le \frac{2\alpha_n}{1+\alpha_n}\|u - z\|\|x_{n+1} - z\| + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_n - x_{n+1}\| \\
&\quad + \frac{2\lambda_n(\gamma_n + \delta_n)}{1+\alpha_n}\|x_n - y_n\|\|(A+B)x_n - (A+B)z\|.
\end{aligned}
$$

Since $\alpha_n \to 0,\ \|x_{n+1} - x_n\| \to 0$ and $\|(A+B)x_n - (A+B)z\| \to 0$, we conclude that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Utilizing (3.1), we deduce from $\liminf_{n\to\infty}\delta_n > 0$ that

$$\begin{aligned}\|S_n y_n - y_n\| &= \frac{1}{\delta_n}\|x_{n+1} - y_n - [\alpha_n(u - y_n) + \beta_n(x_n - y_n)]\| \\ &\le \frac{1}{\delta_n}[\|x_{n+1} - x_n\| + \|x_n - y_n\| + \alpha_n\|u - y_n\| + \beta_n\|x_n - y_n\|] \to 0\end{aligned}$$

as $n \to \infty$. That is,

$$\lim_{n\to\infty}\|S_n y_n - y_n\| = 0. \tag{3.14}$$

Step 5. $\limsup_{n\to\infty}\langle u - \bar{x}, x_n - \bar{x}\rangle \le 0$ where $\bar{x} = P_{F(S)\cap GMEPP}u$.

Indeed, take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle u - \bar{x}, y_n - \bar{x}\rangle = \lim_{i\to\infty}\langle u - \bar{x}, y_{n_i} - \bar{x}\rangle. \tag{3.15}$$

Without loss of generality, we may assume that $y_{n_i} \rightharpoonup w$. Next, let us show that $w \in F(S) \cap GMEPP$.

First, we show that $w \in F(S)$. Indeed, since $\{y_n\}$ is bounded, it follows that

$$\sum_{n=1}^{\infty}\sup\{\|S_n x - S_{n+1}x\| : x \in \{y_n\}\} < \infty.$$

Observe that

$$\begin{aligned}\|S y_n - y_n\| &\le \|S y_n - S_n y_n\| + \|S_n y_n - y_n\| \\ &\le \sup_{x\in\{y_n\}}\|Sx - S_n x\| + \|S_n y_n - y_n\|.\end{aligned}$$

Utilizing Lemma 2.1, from (3.14) we get $\lim_{n\to\infty}\|S y_n - y_n\| = 0$. Since $y_{n_i} \rightharpoonup w$, it follows from the demiclosedness principle for S that $z \in F(S)$.

Now, we show that $w \in GMEPP$. Indeed, from $y_n = T_{\lambda_n}^{(f,\varphi)}(x_n - \lambda_n(A + B)x_n)$, we know that

$$f(y_n, x) + \varphi(x) - \varphi(y_n) + \langle (A+B)x_n, x - y_n\rangle + \frac{1}{\lambda_n}\langle x - y_n, y_n - x_n\rangle \ge 0, \quad \forall x \in C.$$

From (H2) it follows that

$$\varphi(x) - \varphi(y_n) + \langle (A+B)x_n, x - y_n\rangle + \frac{1}{\lambda_n}\langle x - y_n, y_n - x_n\rangle \ge f(x, y_n), \quad \forall x \in C.$$

Replacing n by n_i, we have

$$\varphi(x) - \varphi(y_{n_i}) + \langle (A+B)x_{n_i}, x - y_{n_i}\rangle + \langle x - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \ge f(x, y_{n_i}), \ \forall x \in C. \tag{3.16}$$

Put $y_t = tx + (1-t)w$ for all $t \in (0,1]$ and $x \in C$. Then, we have $y_t \in C$. So, from (3.16) we have

$$\begin{aligned}
&\langle y_t - y_{n_i}, (A+B)y_t\rangle \\
&\geq \langle y_t - y_{n_i}, (A+B)y_t\rangle - \varphi(y_t) + \varphi(y_{n_i}) - \langle y_t - y_{n_i}, (A+B)x_{n_i}\rangle \\
&\quad -\langle y_t - y_{n_i}, \tfrac{y_{n_i}-x_{n_i}}{\lambda_{n_i}}\rangle + f(y_t, y_{n_i}) \\
&= \langle y_t - y_{n_i}, (A+B)y_t - (A+B)y_{n_i}\rangle + \langle y_t - y_{n_i}, (A+B)y_{n_i} - (A+B)x_{n_i}\rangle \\
&\quad -\varphi(y_t) + \varphi(y_{n_i}) - \langle y_t - y_{n_i}, \tfrac{y_{n_i}-x_{n_i}}{\lambda_{n_i}}\rangle + f(y_t, y_{n_i}).
\end{aligned}$$

Since $\|y_{n_i} - x_{n_i}\| \to 0$, we have $\|(A+B)y_{n_i} - (A+B)x_{n_i}\| \to 0$. Further, from the monotonicity of $A+B$, we have $\langle y_t - y_{n_i}, (A+B)y_t - (A+B)y_{n_i}\rangle \geq 0$. So, from (H4), the weakly lower semicontinuity of φ, $\frac{y_{n_i}-x_{n_i}}{\lambda_{n_i}} \to 0$ and $y_{n_i} \rightharpoonup w$, we have

$$\langle y_t - w, (A+B)y_t\rangle \geq -\varphi(y_t) + \varphi(w) + f(y_t, w), \tag{3.17}$$

as $i \to \infty$. From (H1), (H4) and (3.17), we also have

$$\begin{aligned}
0 &= f(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq tf(y_t, x) + (1-t)f(y_t, w) + t\varphi(x) + (1-t)\varphi(w) - \varphi(y_t) \\
&= t[f(y_t, x) + \varphi(x) - \varphi(y_t)] + (1-t)[f(y_t, w) + \varphi(w) - \varphi(y_t)] \\
&\leq t[f(y_t, x) + \varphi(x) - \varphi(y_t)] + (1-t)\langle y_t - w, (A+B)y_t\rangle \\
&\leq t[f(y_t, x) + \varphi(x) - \varphi(y_t)] + (1-t)t\langle x - w, (A+B)y_t\rangle,
\end{aligned}$$

and hence

$$0 \leq f(y_t, x) + \varphi(x) - \varphi(y_t) + (1-t)\langle x - w, (A+B)y_t\rangle.$$

Letting $t \to 0$, we have, for each $x \in C$,

$$0 \leq f(w, x) + \varphi(x) - \varphi(w) + \langle x - w, (A+B)w\rangle,$$

which hence implies that $w \in GMEPP$. Therefore, $w \in F(S) \cap GMEPP$. This together with $\|x_n - y_n\| \to 0$ and the property of metric projection, implies that

$$\limsup_{n\to\infty}\langle u - \bar{x}, x_n - \bar{x}\rangle = \lim_{i\to\infty}\langle u - \bar{x}, x_{n_i} - \bar{x}\rangle = \langle u - \bar{x}, w - \bar{x}\rangle \leq 0.$$

Step 6. $x_n \to \bar{x}$ as $n \to \infty$.

Indeed, from (3.2) and (3.11) we have

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &\leq \tfrac{2\alpha_n}{1+\alpha_n}\langle u-\bar{x}, x_{n+1}-\bar{x}\rangle + \tfrac{\beta_n}{1+\alpha_n}\|x_n-\bar{x}\|^2 + \tfrac{\gamma_n+\delta_n}{1+\alpha_n}\|x_n-\bar{x}\|^2 \\ &= (1-\tfrac{2\alpha_n}{1+\alpha_n})\|x_n-\bar{x}\|^2 + \tfrac{2\alpha_n}{1+\alpha_n}\langle u-\bar{x}, x_{n+1}-\bar{x}\rangle.\end{aligned}$$

It is clear that $\sum_{n=1}^{\infty}\frac{2\alpha_n}{1+\alpha_n}=\infty$. Hence, applying Lemma 2.3 to the last inequality, we immediately obtain that $x_n \to \bar{x}$ as $n \to \infty$. This completes the proof. ■

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \times C \to \mathbf{R}$ be a bifunction which satisfies assumptions (H1)-(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with assumptions (A1) or (A2). Let $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, and $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap GMEPP \neq \emptyset$. For fixed $u \in C$ and $x_1 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by (3.1), where $0 \leq \lambda_n \leq \min\{\alpha, \beta\}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ satisfy the following conditions:*

- (i) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for all* $n \geq 1$*;*
- (ii) $\sum_{n=1}^{\infty} \alpha_n < \infty$*;*
- (iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$*;*
- (iv) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$*;*
- (v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \min\{\alpha, \beta\}$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. Then $\{x_n\}$ converges weakly to an element of $F(S) \cap GMEPP$.

Proof. Repeating the same arguments as those of Steps 1-5 in the proof of Theorem 3.1, we know that the following statements hold:

- (a) $\{x_n\}$ is bounded;
- (b) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;
- (c) $\lim_{n\to\infty} \|Ax_n - Az\| = \lim_{n\to\infty} \|Bx_n - Bz\| = 0$ for each $z \in F(S) \cap GMEPP$;
- (d) $\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|S_n y_n - y_n\| = 0$;
- (e) there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup w \in F(S) \cap GMEPP$.

Now, let us show that $\lim_{n\to\infty} \|x_n - z\|$ exists for each $z \in F(S) \cap GMEPP$. In terms of (3.3) we have

$$\|x_{n+1} - z\| \leq \alpha_n\|u - z\| + (1-\alpha_n)\|x_n - z\| \leq \|x_n - z\| + \alpha_n\|u - z\|. \tag{3.18}$$

Utilizing Lemma 2.8, we deduce from condition (ii) that $\lim_{n\to\infty} \|x_n - z\|$ exists.

Next, let us show that $x_n \rightharpoonup w \in F(S) \cap GMEPP$. Suppose that there exist $\{x_{m_j}\} \subset \{x_n\}$ and $p \neq w$ such that $x_{m_j} \rightharpoonup p$. Then, we have $p \in F(S) \cap GMEPP$. From Opial's condition it follows that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - w\| = \lim_{i\to\infty} \|x_{n_i} - w\| &< \lim_{i\to\infty} \|x_{n_i} - p\| \\ &= \lim_{j\to\infty} \|x_{m_j} - p\| < \lim_{j\to\infty} \|x_{m_j} - w\| \\ &= \lim_{n\to\infty} \|x_n - w\|,\end{aligned}$$

which leads to a contradiction. Hence $x_n \rightharpoonup w \in F(S) \cap GMEPP$. ■

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \times C \to \mathbf{R}$ be a bifunction which satisfies assumptions (H1)-(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with assumptions (A1) or (A2). Let $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, and $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap GMEPP \neq \emptyset$. For $x_1 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{aligned} y_n &= T_{\lambda_n}^{(f,\varphi)}(x_n - \lambda_n(A+B)x_n), \\ x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n S_n y_n, \end{aligned} \tag{3.19}$$

for all $n = 1, 2, \ldots$, where $0 \leq \lambda_n \leq \min\{\alpha, \beta\}$, $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ satisfy the following conditions:

(*i*) $\beta_n + \gamma_n + \delta_n = 1$ *for all* $n \geq 1$;

(*ii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(*iii*) $\liminf_{n\to\infty} \delta_n > 0$;

(*iv*) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \min\{\alpha, \beta\}$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. Then $\{x_n\}$ converges weakly to $w \in F(S) \cap GMEPP$, where $w = \lim_{n\to\infty} P_{F(S)\cap GMEPP} x_n$.

Proof. Take $z \in F(S) \cap GMEPP$ arbitrarily. Then $z = J_{\lambda_n}^{(f,\varphi)}(z - \lambda_n(A+B)z)$. Utilizing (3.2) we get

$$\|y_n - z\|^2 \le \|x_n - z\|^2 + 2\lambda_n(\lambda_n - \alpha)\|Ax_n - Az\|^2 + 2\lambda_n(\lambda_n - \beta)\|Bx_n - Bz\|^2 \le \|x_n - z\|^2$$

for every $n \ge 1$. Hence from condition (i) it follows that

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \|\beta_n(x_n - z) + \gamma_n(y_n - z) + \delta_n(S_n y_n - z)\|^2 \\
&\le \beta_n\|x_n - z\|^2 + (\gamma_n + \delta_n)\|\tfrac{\gamma_n}{\gamma_n+\delta_n}(y_n - z) + \tfrac{\delta_n}{\gamma_n+\delta_n}(S_n y_n - z)\|^2 \\
&\le \beta_n\|x_n - z\|^2 + (\gamma_n + \delta_n)[\tfrac{\gamma_n}{\gamma_n+\delta_n}\|y_n - z\|^2 + \tfrac{\delta_n}{\gamma_n+\delta_n}\|S_n y_n - z\|^2] \\
&\le \beta_n\|x_n - z\|^2 + (\gamma_n + \delta_n)\|y_n - z\|^2 \\
&\le \beta_n\|x_n - z\|^2 + (\gamma_n + \delta_n)[\|x_n - z\|^2 \\
&\quad + 2\lambda_n(\lambda_n - \alpha)\|Ax_n - Az\|^2 + 2\lambda_n(\lambda_n - \beta)\|Bx_n - Bz\|^2] \\
&= \|x_n - z\|^2 + 2(1 - \beta_n)[\lambda_n(\lambda_n - \alpha)\|Ax_n - Az\|^2 + \lambda_n(\lambda_n - \beta)\|Bx_n - Bz\|^2] \\
&\le \|x_n - z\|^2
\end{aligned}$$

for all $n \ge 1$. This implies that

$$\|x_{n+1} - z\| \le \|x_n - z\| \tag{3.20}$$

for all $n \ge 1$. Hence $\lim_{n\to\infty}\|x_n - z\|$ exists. Thus from conditions (ii), (iv) it follows that

$$\|Ax_n - Az\| \to 0 \quad \text{and} \quad \|Bx_n - Bz\| \to 0.$$

Then $\{x_n\}$ and $\{y_n\}$ are bounded. From the firm nonexpansivity of $J_{\lambda_n}^{(f,\varphi)}$, we have

$$\begin{aligned}
&\|y_n - z\|^2 \\
&= \|J_{\lambda_n}^{(f,\varphi)}(x_n - \lambda_n(A+B)x_n) - J_{\lambda_n}^{(f,\varphi)}(z - \lambda_n(A+B)z)\|^2 \\
&\le \langle y_n - z, (x_n - \lambda_n(A+B)x_n) - (z - \lambda_n(A+B)z)\rangle \\
&= \tfrac{1}{2}\{\|y_n - z\|^2 + \|(x_n - \lambda_n(A+B)x_n) - (z - \lambda_n(A+B)z)\|^2 \\
&\quad - \|y_n - z - [(x_n - \lambda_n(A+B)x_n) - (z - \lambda_n(A+B)z)]\|^2\} \\
&\le \tfrac{1}{2}\{\|y_n - z\|^2 + \|x_n - z\|^2 - \|(y_n - x_n) + \lambda_n(A+B)x_n - (A+B)z)\|^2\} \\
&= \tfrac{1}{2}\{\|y_n - z\|^2 + \|x_n - z\|^2 - \|y_n - x_n\|^2 - 2\lambda_n\langle y_n - x_n, (A+B)x_n - (A+B)z\rangle \\
&\quad - \lambda_n^2\|(A+B)x_n - (A+B)z\|^2\}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|y_n - z\|^2 \le\ & \|x_n - z\|^2 - \|y_n - x_n\|^2 - 2\lambda_n \\
& \langle y_n - x_n, (A+B)x_n - (A+B)z\rangle - \lambda_n^2\|(A+B)x_n - (A+B)z\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
&\|x_{n+1}-z\|^2 \le \beta_n\|x_n-z\|^2+(\gamma_n+\delta_n)\|y_n-z\|^2\\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)[\|x_n-z\|^2-\|y_n-x_n\|^2\\
&\quad -2\lambda_n\langle y_n-x_n,(A+B)x_n-(A+B)z\rangle\\
&\quad -\lambda_n^2\|(A+B)x_n-(A+B)z\|^2]\\
&= \|x_n-z\|^2-(1-\beta_n)\|y_n-x_n\|^2-2\lambda_n(1-\beta_n)\langle y_n-x_n,(A+B)x_n-(A+B)z\rangle\\
&\quad -\lambda_n^2(1-\beta_n)\|(A+B)x_n-(A+B)z\|^2\\
&\le \|x_n-z\|^2-(1-\beta_n)\|y_n-x_n\|^2+2\min\{\alpha,\beta\}\|y_n-x_n\|\|(A+B)x_n-(A+B)z\|,
\end{aligned}$$

which implies that

$$(1-\beta_n)\|y_n-x_n\|^2\le\|x_n-z\|^2-\|x_{n+1}-z\|^2+2\min\{\alpha,\beta\}\|y_n-x_n\|\|(A+B)x_n-(A+B)z\|.$$

Since $\|(A+B)x_n-(A+B)z\|\to 0$, $\{x_n\}$ and $\{y_n\}$ are bounded and $\lim_{n\to\infty}\|x_n-z\|$ exists, so it follows from condition (ii) that

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.21}$$

Now, let us show that

$$\lim_{n\to\infty}\|S_ny_n-y_n\|=0. \tag{3.22}$$

Indeed, since

$$\begin{aligned}
&\|\frac{\gamma_n}{1-\beta_n}(y_n-z)+\frac{\delta_n}{1-\beta_n}(S_ny_n-z)\|\\
&\le \frac{\gamma_n}{1-\beta_n}\|y_n-z\|+\frac{\delta_n}{1-\beta_n}\|S_ny_n-z\|\\
&\le \frac{\gamma_n}{1-\beta_n}\|y_n-z\|+\frac{\delta_n}{1-\beta_n}\|y_n-z\|\\
&= \|y_n-z\|\le\|x_n-z\|,
\end{aligned}$$

we get $\limsup_{n\to\infty}\|\frac{\gamma_n}{1-\beta_n}(y_n-z)+\frac{\delta_n}{1-\beta_n}(S_ny_n-z)\|\le c$, where $c=\lim_{n\to\infty}\|x_n-z\|$. Furthermore, we have

$$\begin{aligned}
&\lim_{n\to\infty}\|\beta_n(x_n-z)+(1-\beta_n)[\frac{\gamma_n}{1-\beta_n}(y_n-z)+\frac{\delta_n}{1-\beta_n}(S_ny_n-z)]\|\\
&=\lim_{n\to\infty}\|\beta_n(x_n-z)+\gamma_n(y_n-z)+\delta_n(S_ny_n-z)\|\\
&=\lim_{n\to\infty}\|x_{n+1}-z\|=c.
\end{aligned}$$

Utilizing Lemma 2.9, we have

$$\lim_{n\to\infty}\|\frac{\gamma_n}{1-\beta_n}(y_n-z)+\frac{\delta_n}{1-\beta_n}(S_ny_n-z)-(x_n-z)\|=0.$$

Thai is,

$$\lim_{n\to\infty} \| \frac{\gamma_n}{1-\beta_n}(y_n - x_n) + \frac{\delta_n}{1-\beta_n}(S_n y_n - x_n)\| = 0. \tag{3.23}$$

Since

$$\begin{aligned}
&\frac{\delta_n}{1-\beta_n}\|S_n y_n - x_n\| \\
&= \| \frac{\gamma_n}{1-\beta_n}(y_n - x_n) + \frac{\delta_n}{1-\beta_n}(S_n y_n - x_n) - \frac{\gamma_n}{1-\beta_n}(y_n - x_n)\| \\
&\le \| \frac{\gamma_n}{1-\beta_n}(y_n - x_n) + \frac{\delta_n}{1-\beta_n}(S_n y_n - x_n)\| + \frac{\gamma_n}{1-\beta_n}\|y_n - x_n\|,
\end{aligned}$$

from (3.21), (3.23) and conditions (ii), (iii), it follows that

$$\lim_{n\to\infty} \|S_n y_n - x_n\| = 0. \tag{3.24}$$

This together with (3.21) implies that (3.22) holds. Note that $\{y_n\}$ is bounded. Hence there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup w \in C$. Repeating the argument of Step 5 in the proof of Theorem 3.1, we can obtain $w \in F(S) \cap GMEPP$. Also, repeating the same argument as in the proof of Theorem 3.2, we can derive $x_n \rightharpoonup w \in F(S) \cap GMEPP$.

Finally we prove that $\lim_{n\to\infty} z_n = w$, where $z_n = P_{F(S)\cap GMEPP} x_n$ for each $n \ge 1$. Utilizing (3.20) and Lemma 2.7, we know that there is $z_0 \in F(S) \cap GMEPP$ such that $z_n \to z_0$. From $z_n = P_{F(S)\cap GMEPP} x_n$ and $w \in F(S) \cap GMEPP$, we have

$$\langle x_n - z_n, z_n - w\rangle \ge 0, \quad \forall n \ge 1.$$

It follows from $z_n \to z_0$ and $x_n \rightharpoonup w$ that

$$\langle w - z_0, z_0 - w\rangle \ge 0$$

and hence $z_0 = w$. This completes the proof. ■

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be κ-inverse-strongly monotone, and $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, T) \neq \emptyset$. For $x_1 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S_n P_C(x_n - \lambda_n T x_n), \tag{3.25}$$

for all $n = 1, 2, ...$, where $0 \le \lambda_n \le 2\kappa$ and $\{\beta_n\} \subset [0, 1]$ satisfy the following conditions:

(i) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\kappa$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ *for any bounded subset* B *of* C. *Let* S *be a mapping of* C *into itself defined by* $Sz = \lim_{n\to\infty} S_n z$ *for all* $z \in C$ *and suppose that* $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. *Then* $\{x_n\}$ *converges weakly to* $w \in F(S) \cap VI(C, T)$, *where* $w = \lim_{n\to\infty} P_{F(S)\cap VI(C,T)} x_n$.

Proof. Put $f = 0,\ \varphi = 0,\ A = B = \frac{1}{2}T,\ \gamma_n = 0$ and $\delta_n = 1 - \beta_n$ in Theorem 3.3. In this case, we get $GMEPP = VI(C, T)$. Moreover, it is clear that

$$y_n = T_{\lambda_n}^{(f,\varphi)}(x_n - \lambda_n(A + B)x_n) = P_C(x_n - \lambda_n T x_n)$$

and

$$x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n S_n y_n = \beta_n x_n + (1 - \beta_n) S_n y_n.$$

Note that $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Thus we have

$$\liminf_{n\to\infty} \delta_n = \liminf_{n\to\infty} (1 - \beta_n) > 0.$$

Since $T : C \to H$ is κ-inverse-strongly monotone, we have for all $x, y \in C$

$$\begin{aligned}\langle Ax - Ay, x - y\rangle = \langle Bx - By, x - y\rangle &= \frac{1}{2}\langle Tx - Ty, x - y\rangle \\ &\geq \frac{1}{2}\kappa\|Tx - Ty\|^2 = \frac{1}{2}\kappa\|2 \cdot \frac{Tx - Ty}{2}\|^2 \\ &= 2\kappa\|Ax - Ay\|^2 = 2\kappa\|Bx - By\|^2.\end{aligned}$$

Hence both A and B are 2κ-inverse-strongly monotone. Thus we get $\min\{\alpha, \beta\} = 2\kappa$. This shows that conditions (i)-(iv) in Theorem 3.3 are satisfied. Therefore, by Theorem 3.3 we derive the desired conclusion. This completes the proof. ∎

Remark 3.1. Compared with Theorem 4 of Plubtieng and Kumam [30], Theorem 3.4 coincides essentially with it. Therefore, Theorem 3.3 includes it as a special case. Indeed, Theorem 3.3 unifies, improves and extends it in the following aspects:

(i) the problem of finding an element of $F(S) \cap GMEPP$ is more general than the one of finding an element of $F(S) \cap VI(C, T)$ because the generalized mixed equilibrium problem with perturbation includes the variational inequality problem as a special case.

(ii) the iterative scheme (3.1) is more general than (3.25) because (3.1) reduces to (3.25) by putting $f = 0,\ \varphi = 0,\ A = B = \frac{1}{2}T,\ \gamma_n = 0$ and $\delta_n = 1 - \beta_n$ in Theorem 3.3.

Let $F : C \to C$ be a k-strictly pseudocontractive mapping with $k \in [0, 1)$. For recent convergence result for strictly pseudocontractive mappings, we refer to Zeng, Wong and Yao [28]. Putting $T = I - F$, we know that for all $x, y \in C$

$$\|(I-T)x-(I-T)y\|^2 \le \|x-y\|^2 + k\|Tx-Ty\|^2.$$

Since

$$\|(I-T)x-(I-T)y\|^2 = \|x-y\|^2 + \|Tx-Ty\|^2 - 2\langle x-y, Tx-Ty\rangle,$$

we have for all $x, y \in C$

$$\langle x-y, Tx-Ty\rangle \ge \frac{1-k}{2}\|Tx-Ty\|^2.$$

Consequently, if $F : C \to C$ is a k-strictly pseudocontractive mapping, then the mapping $T = I - F$ is $(1-k)/2$-inverse-strongly monotone.

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to C$ be k-strictly pseudocontractive mapping, and $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, T) \neq \emptyset$, where $T = I - F$. For $x_1 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S_n P_C((1-\lambda_n)x_n + \lambda_n F x_n),$$

for all $n = 1, 2, ...$, where $\lambda_n \subset [0, 1-k]$ and $\{\beta_n\} \subset [0, 1]$ satisfy the following conditions:

(i) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 1-k$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. Then $\{x_n\}$ converges weakly to $w \in F(S) \cap VI(C, T)$, where $w = \lim_{n\to\infty} P_{F(S)\cap VI(C,T)} x_n$.

Putting $S_n = S$ in Theorems 3.1, 3.2 and 3.3, we immediately obtain the following strong and weak convergence results.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \times C \to \mathbf{R}$ be a bifunction which satisfies assumptions (H1)-(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with assumptions (A1) or (A2). Let $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, and S be a nonexpansive mapping from C into* *itself such that $F(S) \cap GMEPP \neq \emptyset$. For fixed $u \in C$ and $x_1 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by*

$$(3.26)\qquad \begin{aligned} y_n &= T_{\lambda_n}^{(f,\varphi)}(x_n - \lambda_n(A+B)x_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \end{aligned}$$

for all $n = 1, 2, ...$, where $0 \leq \lambda_n \leq \min\{\alpha, \beta\}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ satisfy the following conditions:

(*i*) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for all* $n \geq 1$;

(*ii*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(*iv*) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$;

(*v*) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \min\{\alpha, \beta\}$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap GMEPP}u$.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \times C \to \mathbf{R}$ be a bifunction which satisfies assumptions (H1)-(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with assumptions (A1) or (A2). Let $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, and S be a nonexpansive mapping from C into itself such that $F(S) \cap GMEPP \neq \emptyset$. For fixed $u \in C$ and $x_1 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by (3.26), where $0 \leq \lambda_n \leq \min\{\alpha, \beta\}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ satisfy the following conditions:*

(*i*) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for all* $n \geq 1$;

(*ii*) $\sum_{n=1}^{\infty} \alpha_n < \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(*iv*) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$;

(*v*) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \min\{\alpha, \beta\}$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Then $\{x_n\}$ converges weakly to an element $w \in F(S) \cap GMEPP$.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \times C \to \mathbf{R}$ be a bifunction which satisfies assumptions (H1)-(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with assumptions (A1) or (A2). Let $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, and S be a nonexpansive mapping from C into* *itself such that* $F(S) \cap GMEPP \neq \emptyset$*. For* $x_1 \in C$ *arbitrary, let* $\{x_n\} \subset C$ *be a sequence generated by*

$$y_n = T_{\lambda_n}^{(f,\varphi)}(x_n - \lambda_n(A+B)x_n),$$
$$x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n S y_n,$$

for all $n = 1, 2, ...$*, where* $0 \leq \lambda_n \leq \min\{\alpha, \beta\}$*,* $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ *satisfy the following conditions:*

(*i*) $\beta_n + \gamma_n + \delta_n = 1$ *for all* $n \geq 1$*;*

(*ii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(*iii*) $\liminf_{n\to\infty} \delta_n > 0$*;*

(*iv*) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \min\{\alpha, \beta\}$.

Then $\{x_n\}$ *converges weakly to* $w \in F(S) \cap GMEPP$*, where* $w = \lim_{n\to\infty} P_{F(S)\cap GMEPP} x_n$.

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