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QUOTIENTS OF QUANTUM BORNOLOGICAL SPACES

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Abstract. In the note we investigate the main duality properties of quantum (or local operator) spaces involving quantum bornology. Namely, we prove that each finite complete bornology admits precisely one quantization and each complete quantum space is a matrix bornology quotient of a local trace class algebra.

1. Introduction

One of the principal foundations of the quantum functional analysis [9, 11] is the well developed duality theory for operator spaces, which is mainly due to Blecher [1-3], Effros and Ruan [9, 4.2]. This duality theory for operator spaces is based on the following two central results. First one asserts that if $X \subseteq V$ is an operator space inclusions then the restriction $\varphi : V^* \to X^*$, $\varphi(f) = f|_X$, is *an exact matrix quotient mapping*, that is, $\varphi^{(\infty)}(\mathrm{ball}\, M(V^*)) = \mathrm{ball}\, M(X^*)$, where $\varphi^{(\infty)} : M(V^*) \to M(X^*)$ is the canonical extension of φ over all finite matrices, and ball indicates the unit ball of the relevant normed space. The second one asserts that each complete operator space V is a matrix quotient of an L^1-direct sum of finite dimensional trace class algebras up to a matrix isometry [1], [9, 4.2.3]. Thus there is *a matrix quotient mapping* $\varphi : T_J \to V$ in the sense that $\varphi^{(\infty)}(\mathrm{ball}\, M(T_J))$ is dense in $\mathrm{ball}\, M(V)$, where $T_J = \overset{1}{\bigoplus_{i\in J}} T_{n_i}$ and T_{n_i} is the n_i-square trace class matrix algebra. These duality results play key roles in tensor products of operator spaces [2, 9]. The duality theory for general quantum (or local operator) spaces has been partially developed in [4, 6, 7, 10] (see [5] for applications). The quantizations of all polynormed topologies compatible with the given duality (V, W) have been classified in [7] over a local von Neumann algebra.

In the present note we investigate the main duality properties of quantum spaces. First note that the "normed tricks" used in the duality theory can not be applied to the

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general quantum spaces. The main reason is that the properties to be open quotient and bornology quotient should be separated for the general quantum spaces. In fact we need to develop a bornology theory for the quantum spaces independently. By *a matrix* (or *quantum*) *bornology on a linear space* V we mean a classical bornology $\mathfrak{S}$ in the matrix space $M(V)$ such that it contains all absolutely matrix convex hulls $\operatorname{amc}\mathfrak{B}$ for each $\mathfrak{B} \in \mathfrak{S}$. The pair $(V, \mathfrak{S})$ is called *a quantum bornological space.* For instance, if (V, W) is a dual pair and $\mathcal{N}$ is a neighborhood filter base of a certain quantum topology in $M(V)$ compatible with the duality, that is, $(V, \mathcal{N}|V)' = W$, then the family $\mathcal{N}^{\odot}$ of matrix (or operator) polars is a base of matrix bornology in $M(W)$. Conversely, if $\mathfrak{S}$ is a matrix bornology in $M(V)$ of weakly bounded matrix sets then $\mathfrak{S}^{\odot}$ is a neighborhood filter base of a certain quantum topology in $M(W)$. Actually each matrix bornology $\mathfrak{S}$ automatically generates a (vector) bornology $\mathfrak{s}$ on V. We are saying that $\mathfrak{S}$ is *a quantization of* $\mathfrak{s}$. Our first central result asserts that the canonical bornology of a finite dimensional space V admits precisely one quantization. Further, we prove the main duality theorems for quantum space. Namely, let $(T, \mathfrak{T})$ and $(V, \mathfrak{S})$ be quantum bornological space and let $\varphi : (T, \mathfrak{T}) \to (V, \mathfrak{S})$ be a linear mapping. We say that φ is *a matrix quotient mapping* if the weak closures $\mathfrak{S}^{-}$ and $\varphi^{(\infty)}(\mathfrak{T})^{-}$ coincide. If $\mathfrak{S} = \varphi^{(\infty)}(\mathfrak{T})$ then we say that φ is *an exact matrix quotient mapping.* The first duality result asserts that if V is a quantum space with its neighborhood filter base $\mathcal{N}$ of absolutely matrix convex sets and $X \subseteq V$ is a linear subspace, then the restriction mapping $(V', \mathcal{N}^{\odot}) \to \left(X', (M(X) \cap \mathcal{N})^{\odot}\right)$, $f \mapsto f|_X$ is an exact matrix quotient mapping of the quantum bornological spaces. Thus $\left(X', (M(X) \cap \mathcal{N})^{\odot}\right)$ is a bornology quotient of $(V', \mathcal{N}^{\odot})$. Finally, we prove that if V is a complete quantum space and $\mathfrak{S}$ is a matrix bornology in $M(V)$ of $\sigma(V, V')$-bounded matrix sets, then there is a matrix quotient mapping $\varphi : (\mathcal{T}_J, \beta) \to (V, \mathfrak{S})$ of the matrix bornological spaces, where $\mathcal{T}_J = \operatorname{op}\bigoplus_{\kappa} T_{J_\kappa}$ is the quantum inductive sum of the trace class algebras equipped with its strong matrix bornology β. Thus $(V, \mathfrak{S})$ is a bornology quotient of $(\mathcal{T}_J, \beta)$. In the normed case, these assertions are reduced to the main duality results mentioned above.

2. PRELIMINARIES

The linear space of all $m \times n$-matrices $x = [x_{ij}]$ over a linear space V is denoted by $M_{m,n}(V)$, and we set $M_{m,n} = M_{m,n}(\mathbb{C})$, $M_m(V) = M_{m,m}(V)$. Further, $M(V)$ (respectively, M) denotes the linear space of all infinite (respectively, scalar) matrices $[x_{ij}]$, $x_{ij} \in V$, where all but finitely many of x_{ij} are zero. Note that $M_{m,n}(L(E)) = L(E^n, E^m)$ is the space of all linear transformations up to the canonical identification. In particular, if $E = H$ is a Hilbert space then $M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$ is a normed space of all bounded linear operators. So is

$M_{m,n}$ equipped with the operator norm $\|\cdot\|$ between the canonical Hilbert spaces $\mathbb{C}^n$ and $\mathbb{C}^m$. Hence M is a normed space.

Let us introduce the main quantum operations, *the direct sum* and *M-bimodule structure* in the space $M(V)$ of all matrices over V. If $v \in M_{s,t}(V)$ and $w \in M_{m,n}(V)$ then we have their direct sum $v \oplus w \in M_{s+m,t+n}(V)$. If $a \in M_{m,s}$, $v \in M_{s,t}(V)$ and $b \in M_{t,n}$ then we have the matrix product $avb = \left[\sum_{k,l} a_{ik} v_{kl} b_{lj}\right]_{i,j} \in M_{m,n}(V)$. A linear mapping $\varphi : V \to W$ has the canonical linear extensions $\varphi^{(n)} : M_n(V) \to M_n(W)$ (respectively, $\varphi^{(\infty)} : M(V) \to M(W)$) over all matrix spaces defined as $\varphi^{(n)}([x_{ij}]) = [\varphi(x_{ij})]$ (respectively, $\varphi^{(\infty)} | M_n(V) = \varphi^{(n)}$). By a *matrix set* $\mathfrak{B}$ *in* $M(V)$ we mean a collection $\mathfrak{B} = (\mathfrak{b}_n)$ of subsets $\mathfrak{b}_n \subseteq M_n(V)$, $n \in \mathbb{N}$. Each subset $\mathfrak{b} \subseteq V$ determines a matrix set $\mathfrak{b} = (\mathfrak{b}_n)$ with $\mathfrak{b}_1 = \mathfrak{b}$ and $\mathfrak{b}_n = \{0\}$ if $n > 1$. The set of all matrix combinations $\sum_i a_i v_i b_i$ with $a_i, b_i \in M$ and $v_i \in \mathfrak{B}$ is called *the matrix span of a matrix set* $\mathfrak{B}$ and it is denoted by $\operatorname{mspan}(\mathfrak{B})$. A matrix set $\mathfrak{B}$ in $M(V)$ is said to be *absolutely matrix convex* if $\mathfrak{B} \oplus \mathfrak{B} \subseteq \mathfrak{B}$ and $a\mathfrak{B}b \subseteq \mathfrak{B}$ for all $a, b \in \operatorname{ball} M$. One can easily verify that an absolutely matrix convex set $\mathfrak{B}$ turns out to be absolutely convex set, that is, so are all $\mathfrak{b}_n$, $n \in \mathbb{N}$. Using the canonical identification $M(V \times Y) \to M(V) \times M(Y)$, $[(v_{ij}, y_{ij})] \mapsto ([v_{ij}], [y_{ij}])$ for linear spaces V and Y, one can easily prove that the direct product $\mathfrak{A} \times \mathfrak{B}$ of absolutely matrix convex sets $\mathfrak{A} \subseteq M(V)$ and $\mathfrak{B} \subseteq M(Y)$ is an absolutely matrix convex set in $M(V \times Y)$. The intersection of all absolutely matrix convex sets containing a matrix set $\mathfrak{A}$ is called *the absolutely matrix convex hull of* $\mathfrak{A}$ and it is denoted by $\operatorname{amc} \mathfrak{A}$. The known lemma by B. E. Johnson [10, Lemma 3.2] asserts that if $\mathfrak{M} = (\mathfrak{m}_n) = \operatorname{amc} \mathfrak{A}$ then $\mathfrak{m}_n$ consists of those $\sum_s a_s v_s b_s$, $a_s \in M_{n,k_s}$, $v_s \in \mathfrak{b}_{k_s}$, $b_s \in M_{k_s,n}$, such that $\sum_s a_s a_s^* \leq 1$ and $\sum_s b_s^* b_s \leq 1$.

Lemma 2.1. *If* $\mathfrak{b} \subseteq V$ *and* $\mathfrak{M} = \operatorname{amc} \mathfrak{b}$ *then* $\mathfrak{m}_1 = \operatorname{abc} \mathfrak{b}$ *is the absolutely convex hull of* $\mathfrak{b}$.

Proof. Since $\mathfrak{m}_1$ is absolutely convex set containing $\mathfrak{b}$, it follows that $\operatorname{abc} \mathfrak{b} \subseteq \mathfrak{m}_1$. Conversely, take $x \in \mathfrak{m}_1$. Using Johnson's lemma, we derive that $x = \sum_i \alpha_i v_i \beta_i$ for some $\alpha_i, \beta_i \in \mathbb{C}$, $v_i \in \mathfrak{b}$ such that $\sum_i |\alpha_i|^2 \leq 1$ and $\sum_i |\beta_i|^2 \leq 1$. But $\sum_i |\alpha_i \beta_i| \leq \left(\sum_i |\alpha_i|^2\right)^{1/2} \left(\sum_i |\beta_i|^2\right)^{1/2} \leq 1$. Hence $x \in \operatorname{abc} \mathfrak{b}$. ■

Lemma 2.2. *Let* $\mathfrak{B} = (\mathfrak{b}_n)$ *be an absolutely matrix convex set in* $M(V)$. *Then* $\operatorname{mspan}(\mathfrak{B}) = M(V_{\mathfrak{B}})$, *where* $V_{\mathfrak{B}} = \operatorname{span}(\mathfrak{b}_1)$.

Proof. By its very definition, $\operatorname{mspan}(\mathfrak{B})$ consists of finite sums $\sum_i a_i v_i b_i$ with $a_i, b_i \in M$ and $v_i \in \mathfrak{B}$. Actually, $\sum_i a_i v_i b_i = avb^*$ with $a = [\cdots a_i \cdots]$, $b = [\cdots b_i \cdots] \in M$ and $v = \oplus_i v_i \in \mathfrak{B}$. But $avb^* = \|a\| \|b\| \left(\|a\|^{-1} av \|b\|^{-1} b^*\right) \in \|a\| \|b\| \mathfrak{B}$. If $w = [w_{ij}] \in \mathfrak{b}_n$ then $w_{ij} = \epsilon_i w \epsilon_j^* \in \epsilon_i \mathfrak{b}_n \epsilon_j^* \subseteq \mathfrak{b}_1$ for all i, j, where $\epsilon_i = [\cdots 1 \cdots]$ is the canonical row matrix, that is, $w \in M(\mathfrak{b}_1)$. Hence $avb^* \in M(V_{\mathfrak{B}})$. Thus $\mathrm{mspan}(\mathfrak{B}) \subseteq M(V_{\mathfrak{B}})$.

Conversely, if $u = [u_{ij}] \in M_n(V_{\mathfrak{B}})$ then $u_{ij} = \sum_{k=1}^{p} \alpha_{ij}^k v_{ij}^k$ with $v_{ij}^k \in \mathfrak{b}_1$, $\alpha_{ij}^k \in \mathbb{C}$. It can be assumed that $u_{ij} = \alpha_{ij} v_{ij}$ with $v_{ij} \in \mathfrak{b}_1$ and $\alpha_{ij} \in \mathbb{C}$ for all i, j. Put $\alpha = [\alpha_{ij}] \in M_n$ and $v = [v_{ij}] \in M_n(\mathfrak{b}_1)$. First note that $n^{-1}v \in \mathfrak{b}_n$. Indeed, $n^{-1}v = \sum_{i,j} n^{-1/2}\epsilon_i^* v_{ij} n^{-1/2}\epsilon_j \in \mathfrak{b}_n$ (Johnson lemma), for $\sum_{i,j} n^{-1}\epsilon_i^*\epsilon_i = I_n$. Further, $u = E(\alpha \otimes v)E^*$ with $E = \epsilon_1 \oplus \cdots \oplus \epsilon_n$. It follows that $u = nE(\alpha \otimes I_n)\left(n^{-1}v\right)^{\oplus n} E^* \in nE(\alpha \otimes I_n)\mathfrak{B}E^* \subseteq \mathrm{mspan}(\mathfrak{B})$, that is, $M(V_{\mathfrak{B}}) \subseteq \mathrm{mspan}(\mathfrak{B})$. ∎

If $p_{\mathfrak{B}}$ is the Minkowski functional of $\mathfrak{B}$ then $p_{\mathfrak{B}}$ is a matrix seminorm on $M(V_{\mathfrak{B}})$ thanks to Lemma 2.2. Note that $p_{\mathfrak{B}}(v) \leq \sum \|a_i\| \, p(v_i) \, \|b_i\| < \infty$ whenever $v = \sum_i a_i v_i b_i \in M(V_{\mathfrak{B}})$. Hence $(V_{\mathfrak{B}}, p_{\mathfrak{B}})$ is a matrix seminormed space. We say that $\mathfrak{B}$ is *a matrix norming* if $p_{\mathfrak{B}}$ is a matrix norm on $M(V_{\mathfrak{B}})$, that is, $(V_{\mathfrak{B}}, p_{\mathfrak{B}})$ is an operator space. If $V_{\mathfrak{B}}$ is a complete operator space then we say that $\mathfrak{B}$ is *matrix completant.* These notions play the central roles to develop the matrix bornology technique (see [12], [13] for the classical case).

Let $\mathfrak{B}$ be a matrix set in $M(V)$. The matrix set

$$\mathcal{HS}(\mathfrak{B}) = \{avb : a, b \in M, v \in \mathfrak{B}, \|a\|_2 \|b\|_2 \leq 1\}$$

is called *the Hilbert-Schmidt boundary of* $\mathfrak{B}$ [6], where $\|x\|_2 = \mathrm{tr}(x^*x)^{1/2}$ is the Hilbert -Schmidt norm of a matrix $x \in M$. If $\mathfrak{B}$ is absolutely matrix convex then $\mathcal{HS}(\mathfrak{B}) \subseteq \mathfrak{B}$. Actually, $n^{-1}\mathfrak{b}_n \subseteq \mathfrak{m}_n \subseteq \mathfrak{b}_n$ for all n, whenever $\mathfrak{M} = (\mathfrak{m}_n) = \mathcal{HS}(\mathfrak{B})$ (see [6]). Moreover, if $X \subseteq V$ is a linear subspace and $\mathfrak{B}$ is an absolutely matrix convex set in $M(V)$ then

$$\mathcal{HS}(M(X) \cap \mathfrak{B}) = M(X) \cap \mathcal{HS}(\mathfrak{B}). \tag{2.1}$$

If $\varphi : X \to V$ is a linear mapping and $\mathfrak{B} \subseteq M(X)$ is a matrix set then $\varphi^{(\infty)}(\mathcal{HS}(\mathfrak{B})) = \mathcal{HS}\left(\varphi^{(\infty)}(\mathfrak{B})\right)$.

Now let $\mathfrak{F} = \{\mathfrak{I}\}$ be a family of absorbent absolutely matrix convex sets in $M(V)$ such that $\cap\mathfrak{F} = \{0\}$. Consider the family $\{t\mathfrak{U}\}$, where t runs over all positive real numbers and $\mathfrak{U}$ runs over all finite intersections of the matrix sets from $\mathfrak{F}$. This family defines a Hausdorff polynormed (or locally convex) topology in $M(V)$ whose neighborhood filter base of the origin is $\mathfrak{F}$. This topology is called *a quantum topology in* $M(V)$. The linear space V with a quantum topology in $M(V)$ is called *a quantum space.* Note that the quantum topology inherits a polynormed topology in V, and the topology in $M_n(V)$ is just the direct product topology of V^{n^2} (see [10]). Each polynormed topology in V admits a *quantization*, that is, it is a trace of a certain quantum topology in $M(V)$. All these quantizations of the polynormed space V rate between the $\min$ and $\max$ quantizations [15, 2.2.2]. A matrix set $\mathfrak{A} \subseteq M(V)$ is said to be *a matrix bounded set* if it is absorbed by each $\mathfrak{J}$ from $\mathfrak{F}$. Finally, a linear mapping $\varphi : V \to Y$ of quantum spaces is said to be *a matrix continuous* if the mapping $\varphi^{(\infty)} : M(V) \to M(Y)$ is continuous in the usual sense.

Let V and W be linear spaces. These spaces are said to be in duality if there is a pairing $\langle \cdot, \cdot \rangle : V \times W \to \mathbb{C}$ such that $\{\langle v, \cdot \rangle : v \in V\}$ and $\{\langle \cdot, w \rangle : w \in W\}$ are separating families of functionals on W and V, respectively. We briefly say that (V, W) is a dual pair. The spaces V and W are equipped with the relevant weak $\sigma(V, W)$ and weak * $\sigma(W, V)$ topologies, respectively. The given pairing between V and W determines the *matrix pairing*

$$\langle\langle \cdot, \cdot \rangle\rangle : M_m(V) \times M_n(W) \to M_{mn}, \quad \langle\langle v, w \rangle\rangle = [\langle v_{ij}, w_{st} \rangle],$$

where $v = [v_{ij}] \in M_m(V)$, $w = [w_{st}] \in M_n(W)$. Each $M_m(V)$ (respectively, $M_n(W)$) is a polynormed space equipped with the direct product $(V, \sigma(V, W))^{n^2}$ (respectively, $(W, \sigma(W, V))^{n^2}$) topology. One can easily verify (see [7] for the details) that these polynormed topologies in $M_n(V)$ and $M_n(W)$ are just the weak $\sigma(M_n(V), M_n(W))$ and weak* $\sigma(M_n(W), M_n(V))$ topologies determined by the scalar pairing $\langle \cdot, \cdot \rangle : M_n(V) \times M_n(W) \to \mathbb{C}$, $\langle v, w \rangle = \sum_{i,j} \langle v_{ij}, w_{ij} \rangle$. Given a matrix set $\mathfrak{B}$ in $M(V)$ let us introduce the *matrix* (or *operator*) *polar* $\mathfrak{B}^{\odot}$ *in* $M(W)$ to be a matrix set $(\mathfrak{b}_n^{\odot})$ defined as $\mathfrak{b}_n^{\odot} = \{w \in M_n(W) : \|\langle\langle v, w \rangle\rangle\| \leq 1, v \in \mathfrak{b}_s, s \in \mathbb{N}\}$. We briefly write $\mathfrak{B}^{\odot} = \{w \in M(W) : \sup \|\langle\langle \mathfrak{B}, w \rangle\rangle\| \leq 1\}$. It can be proved (see [10]) that $\mathfrak{b}_1^{\odot}$ coincides with the classical absolute polar of $\mathfrak{b}_1$ in W, that is, $\mathfrak{b}_1^{\odot} = \mathfrak{b}_1^{\circ} = \{w \in W : \sup |\langle \mathfrak{b}_1, w \rangle| \leq 1\}$. Similarly, one can define the absolute matrix polar $\mathfrak{M}^{\odot} \subseteq M(V)$ for a matrix set $\mathfrak{M} = (\mathfrak{m}_n)$ in $M(W)$. A matrix set $\mathfrak{B}$ in $M(V)$ is said to be *weakly closed* if each $\mathfrak{b}_n$ is $\sigma(M_n(V), M_n(W))$-closed in $M_n(V)$. Note that $\mathfrak{B}^{\odot}$ is an absolutely matrix convex and weakly closed set in $M(W)$. The following quantum version of the bipolar theorem was proved in [10] by Effros and Webster.

Theorem 2.1. *Let (V, W) be a dual pair and let $\mathfrak{B}$ be a matrix set in $M(V)$. Then $\mathfrak{B}^{\odot\odot} = (\operatorname{amc} \mathfrak{B})^{-}$, where $(\operatorname{amc} \mathfrak{B})^{-}$ is the weak closure of $\operatorname{amc} \mathfrak{B}$.*

In order to compare the classical and matrix polars of a matrix set we introduce the *classical polar* $\mathfrak{B}^{\circ} \subseteq M(W)$ of a matrix set $\mathfrak{B} \subseteq M(V)$ as the matrix set $(\mathfrak{b}_n^{\circ})$ of the polars $\mathfrak{b}_n^{\circ} = \{w \in M_n(W) : \sup |\langle \mathfrak{b}_n, w \rangle| \leq 1\}$ with respect to the scalar pairing. The following assertions were proved in [6].

Theorem 2.2. *If $\mathfrak{B} \subseteq M(V)$ is a matrix set then $\mathfrak{B}^{\odot} = \mathcal{HS}(\mathfrak{B})^{\circ}$. If $\mathfrak{B}$ is absolutely matrix convex then $\mathcal{HS}(\mathfrak{B}^{\odot})^{-} = \mathfrak{B}^{\circ}$, where $\mathcal{HS}(\mathfrak{B}^{\odot})^{-}$ indicates the weak* closure of $\mathcal{HS}(\mathfrak{B}^{\odot})$.*

The family of matrix polars $\mathfrak{B}_w = \{w\}^{\odot}$, $w \in M(W)$, determines the quantum topology $\mathfrak{s}(V, W)$ in $M(V)$ called *the weak quantum topology*. For each $w \in$

$M(W)$ we put $p_w(v) = \|\langle\langle v, w\rangle\rangle\|$, $v \in M(V)$. One can easily verify that the family $\{p_w : w \in M(W)\}$ of matrix seminorms defines $\mathfrak{s}(V, W)$.

Theorem 2.3. *The weak topology $\sigma(V, W)$ admits precisely one quantization $\mathfrak{s}(V, W)$.*

Similar result for a nuclear quantum space was proved in [10].

3. Matrix Bornology

The bornology in the quantum space theory can independently be developed using the properties of matrix bounded sets in quantum spaces.

3.1. The quantizations of a bornology

Let V be a linear space. Recall [12] that a family $\mathfrak{f}$ of subsets in V is said to be *a (vector) bornology on* V if $\cup\mathfrak{f} = V$, and the family $\mathfrak{f}$ is closed with respect to the taking subsets, finite sums and absolutely convex hulls. In particular, $\mathfrak{f}$ is closed with respect to the finite unions, scalar multipliers and it contains all singletons. Indeed, take $A, B \in \mathfrak{f}$. It can be assumed that both are absolutely convex sets. Then $A \cup B \subseteq A + B \in \mathfrak{f}$, hence $A \cup B \in \mathfrak{f}$. Since $\cup\mathfrak{f} = V$, it follows that each $v \in V$ belongs to a certain $A \in \mathfrak{f}$, therefore $\{v\} \in \mathfrak{f}$. Finally, for each $\lambda \in \mathbb{C} \setminus \{0\}$ and an absolutely convex set $A \in \mathfrak{f}$, we have $|\lambda| \leq n$ for some $n \in \mathbb{N}$, and $\lambda A \subseteq |\lambda| \left(\lambda |\lambda|^{-1} A\right) \subseteq |\lambda| A \subseteq nA \subseteq A + \cdots + A \in \mathfrak{f}$. A subfamily $\mathfrak{s} \subseteq \mathfrak{f}$ of absolutely convex sets is called *a base of bornology* if each set from $\mathfrak{f}$ is contained in a certain set from $\mathfrak{s}$. The pair $(V, \mathfrak{f})$ is called *a bornological space* and the sets from the bornology $\mathfrak{f}$ are called *bounded sets.* We assume that $\{0\}$ is the only bounded linear subspace in V, that is, $\mathfrak{f}$ is *a separated bornology.* A (base of) bornology $\mathfrak{f}$ is said to be *finite-complete* if $\operatorname{span}(A)$ has the finite dimension for each $A \in \mathfrak{f}$. In particular, it turns out to be a finite dimensional Banach space with respect to the Minkowski functional of A.

Remark 3.1. Let $\mathfrak{s}$ be a family of absolutely convex sets in V. Then $\mathfrak{s}$ is a base of a bornology iff $\cup\mathfrak{s} = V$ and for each couple $A_0, B_0 \in \mathfrak{s}$ there exists $C_0 \in \mathfrak{s}$ such that $A_0 + B_0 \subseteq C_0$. Indeed, only possible choice for the bornology $\mathfrak{f}$ is the collection of those subsets which are contained in an element of $\mathfrak{s}$. So, for each $A \in \mathfrak{f}$ there is $A_0 \in \mathfrak{s}$ such that $A \subseteq A_0$. If $A, B \in \mathfrak{f}$ then $A + B \subseteq A_0 + B_0$ for some $A_0, B_0 \in \mathfrak{s}$. But $A_0 + B_0 \subseteq C_0$ for a certain $C_0 \in \mathfrak{s}$. Hence $A + B \in \mathfrak{f}$. The rest is clear.

Assume $\mathfrak{f}$ and $\mathfrak{l}$ are bornologies on the space V. We say that $\mathfrak{f}$ is *finer than* $\mathfrak{l}$ and we write $\mathfrak{l} \preceq \mathfrak{f}$ if $\mathfrak{l} \subseteq \mathfrak{f}$ as the families of sets. Thus for each $A \in \mathfrak{l}$ there corresponds $B \in \mathfrak{f}$ such that $A \subseteq B$.

Definition 3.1. Let V be a linear space. A family $\mathfrak{S}$ of matrix subsets in the matrix space $M(V)$ is said to be a matrix bornology on V if the following conditions hold:

(i) $M(V) = \cup\mathfrak{S}$;

(ii) if $\mathfrak{B} \in \mathfrak{S}$ and $\mathfrak{M} \subseteq \mathfrak{B}$ then $\mathfrak{M} \in \mathfrak{S}$;

(iii) if $\mathfrak{B}, \mathfrak{M} \in \mathfrak{S}$ then $\mathfrak{B} + \mathfrak{M} \in \mathfrak{S}$;

(iv) if $\mathfrak{B} \in \mathfrak{S}$ then $\operatorname{amc}\mathfrak{B} \in \mathfrak{S}$.

The sets from $\mathfrak{S}$ are called *matrix bounded sets.* Again we assume that $\{0\}$ is the only matrix bounded subspace in $M(V)$. Actually, $\mathfrak{S}$ is a bornology on $M(V)$ in the classical sense, for $\operatorname{abc}\mathfrak{B} \subseteq \operatorname{amc}\mathfrak{B}$. In particular, $\mathfrak{S}$ is closed with respect to the finite unions and scalar multiples, and it contains all singletons. Note that $\mathfrak{S}$ is a matrix bornology on V iff it is a bornology on $M(V)$ which contains $\operatorname{amc}\mathfrak{B}$ for each $\mathfrak{B} \in \mathfrak{S}$. We are saying that $(V, \mathfrak{S})$ is *a quantum bornological space.* A subfamily $\mathcal{S} \subseteq \mathfrak{S}$ of absolutely matrix convex sets is called *a base of matrix bornology* if each set from $\mathfrak{S}$ is contained in a certain set from $\mathcal{S}$. Using Remark 3.1, we derive that $\mathcal{S}$ is a base of a matrix bornology on V iff $\cup\mathcal{S} = M(V)$ and for each couple $\mathfrak{A}_0, \mathfrak{B}_0 \in \mathcal{S}$ there exist $\mathfrak{C}_0 \in \mathcal{S}$ such that $\mathfrak{A}_0 + \mathfrak{B}_0 \subseteq \mathfrak{C}_0$.

Let $\mathcal{S}$ be a family of absolutely matrix convex sets in $M(V)$. Since each $\mathfrak{B} \in \mathcal{S}$ is a matrix set $\mathfrak{B} = (\mathfrak{b}_n)$, we put $\mathcal{S}|V = \{\mathfrak{b}_1 : \mathfrak{B} \in \mathcal{S}\}$, which consists of absolutely convex sets.

Lemma 3.1. *If $\mathfrak{S}$ is a (base of) matrix bornology on V then $\mathfrak{S}|V$ is a (base of) bornology on V.*

Proof. Obviously, $V = \cup(\mathfrak{S}|V)$. Moreover, $\mathfrak{b}_1 + \mathfrak{m}_1 \in \mathfrak{S}|V$ whenever $\mathfrak{B}$, $\mathfrak{M} \in \mathfrak{S}$. Indeed, it can be assumed that both $\mathfrak{B}$ and $\mathfrak{M}$ are absolutely matrix convex sets. In particular, they are absolutely convex sets. But $\mathfrak{b}_1 + \mathfrak{m}_1 \subseteq \mathfrak{B} + \mathfrak{M}$ as the matrix sets, hence $\mathfrak{b}_1 + \mathfrak{m}_1 \in \mathfrak{S}$, that is, $\mathfrak{b}_1 + \mathfrak{m}_1 \in \mathfrak{S}|V$. Finally, take $\mathfrak{B} \in \mathfrak{S}$ with $\mathfrak{M} = \operatorname{amc}\mathfrak{B}$. Then $\operatorname{abc}\mathfrak{b}_1 \subseteq \mathfrak{m}_1$, for $\mathfrak{m}_1$ is absolutely convex. But $\operatorname{abc}\mathfrak{b}_1 \subseteq \mathfrak{M}$ as the matrix sets in $M(V)$, hence $\operatorname{abc}\mathfrak{b}_1 \in \mathfrak{S}$ (see Definition 3.1), or $\operatorname{abc}\mathfrak{b}_1 \in \mathfrak{S}|V$. The rest is clear. ∎

Based on Lemma 3.1, we say that a (base of) matrix bornology $\mathcal{S}$ is *a quantization of a* (*base of*) *bornology* $\mathfrak{s}$ if $\mathcal{S}|V = \mathfrak{s}$. Note that if $\mathcal{S}$ is a base of matrix bornology on V then $(V_{\mathfrak{B}}, p_{\mathfrak{B}})$ is an operator space for each $\mathfrak{B} \in \mathcal{S}$, where $V_{\mathfrak{B}} = \operatorname{span}(\mathfrak{b}_1)$ (see Lemma 2.2). Thus each $\mathfrak{B} \in \mathcal{S}$ is matrix norming. If $V_{\mathfrak{B}}$ has the finite dimension then we are saying that $\mathfrak{B}$ is *matrix finite-completant.* In particular, each base of a matrix bornology on a finite dimensional space consists of matrix finite-completant sets.

Now let (V, W) be a dual pair with the duality $\langle\cdot,\cdot\rangle$, and let $\mathcal{S}$ be a base of matrix bornology on V. The family of the weak closures $\mathcal{S}^{-} = \{\mathfrak{A}^{-} : \mathfrak{A} \in \mathcal{S}\}$ turns out to be a matrix bornology base on V. Indeed, $\mathcal{S}^{-}$ consists of absolutely matrix convex sets. Moreover, for each $\mathfrak{A}, \mathfrak{B} \in \mathcal{S}$ there is $\mathfrak{C} \in \mathcal{S}$ such that $\mathfrak{A} + \mathfrak{B} \subseteq \mathfrak{C}$. It follows that $\mathfrak{A}^{-} + \mathfrak{B}^{-} \subseteq (\mathfrak{A} + \mathfrak{B})^{-} \subseteq \mathfrak{C}^{-}$. Hence $\mathcal{S}^{-}$ is a bornology base thanks to Remark 3.1. If $\mathfrak{s}$ is a base of bornology on V, then we put $\mathfrak{s}^{\min} = \{\mathfrak{a}^{\odot\odot} : \mathfrak{a} \in \mathfrak{s}\}$ and $\mathfrak{s}^{\max} = \left\{(\mathfrak{a}^{\circ})^{\odot} : \mathfrak{a} \in \mathfrak{s}\right\}$, which consist of absolutely matrix convex, weakly closed matrix sets in $M(V)$.

Proposition 3.1. *Let (V, W) be a dual pair and let $\mathfrak{s}$ be a base of bornology on V with its weak closure $\mathfrak{s}^{-}$. Then $\mathfrak{s}^{\min}$ and $\mathfrak{s}^{\max}$ are base of matrix bornologies on V, and $\mathfrak{s}^{\min}|V = \mathfrak{s}^{-} = \mathfrak{s}^{\max}|V$. Moreover, if $\mathcal{S}$ is a quantization of $\mathfrak{s}^{-}$ then*

$$\mathfrak{s}^{\min} \preceq \mathcal{S}^{-} \preceq \mathfrak{s}^{\max}.$$

Proof. First let us prove that $\mathfrak{s}^{\min}$ is a matrix bornology base on V. Take $\mathfrak{a}, \mathfrak{b} \in \mathfrak{s}$. Then $\mathfrak{a} + \mathfrak{b} \subseteq \mathfrak{c}$ for some $\mathfrak{c} \in \mathfrak{s}$ (see Remark 3.1). Using the Bipolar Theorem 2.1, we derive that $\mathfrak{a}^{\odot\odot} + \mathfrak{b}^{\odot\odot} = (\operatorname{amc} \mathfrak{a})^{-} + (\operatorname{amc} \mathfrak{b})^{-} \subseteq (\operatorname{amc} \mathfrak{a} + \operatorname{amc} \mathfrak{b})^{-}$. But $\operatorname{amc} \mathfrak{a} + \operatorname{amc} \mathfrak{b} \subseteq 2 \operatorname{amc} \mathfrak{c}$. Indeed, take $x \in \operatorname{amc} \mathfrak{a}$, $y \in \operatorname{amc} \mathfrak{b}$ and assume that $x, y \in M_k(V)$. By Johnson's lemma (see Section), $x = avb$ and $y = a'v'b'$ with $a, a', b, b' \in \operatorname{ball} M$ and $v \in \mathfrak{a}^{\oplus n}$ and $v' \in \mathfrak{b}^{\oplus m}$. Then $x + y = c(v \oplus v') d^{*}$ with $c = \left[\begin{array}{cc} a & a' \end{array}\right]$ and $d = \left[\begin{array}{cc} b & b' \end{array}\right]$. Note that $v \oplus v' \in \mathfrak{c}^{\oplus(n+m)}$, $cc^{*} = aa^{*} + a'a'^{*} \leq 2$ and $dd^{*} = bb^{*} + b'b'^{*} \leq 2$. Hence $x + y \in 2 \operatorname{amc} \mathfrak{c}$. Thus $\mathfrak{a}^{\odot\odot} + \mathfrak{b}^{\odot\odot} \subseteq (\operatorname{amc}(2\mathfrak{c}))^{-} \subseteq (2\mathfrak{c})^{\odot\odot}$. But $2\mathfrak{c} \subseteq \mathfrak{c} + \mathfrak{c} \subseteq \mathfrak{d}$ for some $\mathfrak{d} \in \mathfrak{s}$ (see Remark 3.1). It remains to prove that $\cup \mathfrak{s}^{\min} = M(V)$. Take a matrix $v \in M_n(V)$. Then $v_{ij} \in \mathfrak{a}_{ij}$ for some $\mathfrak{a}_{ij} \in \mathfrak{s}$. But $\cup \mathfrak{a}_{ij} \subseteq \sum_{i,j} \mathfrak{a}_{ij} \subseteq \mathfrak{a}$ for some $\mathfrak{a} \in \mathfrak{s}$. Then $n^{-1} v = \sum_{i,j} n^{-1/2} \epsilon_i^{*} v_{ij} n^{-1/2} \epsilon_j \in \operatorname{amc}(\mathfrak{a})$, for $\sum_{i,j} n^{-1} \epsilon_i^{*} \epsilon_i = I_n$. Therefore $v \in (n\mathfrak{a})^{\odot\odot}$. But again $n\mathfrak{a} \subseteq \mathfrak{b}$ for some $\mathfrak{b} \in \mathfrak{s}$. Hence $v \in \mathfrak{b}^{\odot\odot}$. Thus $\mathfrak{s}^{\min}$ is a matrix bornology base on V.

Now let us prove that $\mathfrak{s}^{\max}$ is a matrix bornology base on V. Take $x \in (\mathfrak{a}^{\circ})^{\odot}$, $y \in (\mathfrak{b}^{\circ})^{\odot}$ with $\mathfrak{a}, \mathfrak{b} \in \mathfrak{s}$. Then $\left\|\left\langle\left\langle 2^{-1}(x + y), w\right\rangle\right\rangle\right\| \leq 2^{-1} \|\langle\langle x, w\rangle\rangle\| + 2^{-1} \|\langle\langle y, w\rangle\rangle\| \leq 1$ for all $w \in \mathfrak{a}^{\circ} \cap \mathfrak{b}^{\circ}$. But $\mathfrak{a} + \mathfrak{b} \subseteq \mathfrak{c}$ for a certain $\mathfrak{c} \in \mathfrak{s}$. It follows that $\mathfrak{c}^{\circ} \subseteq (\mathfrak{a} + \mathfrak{b})^{\circ} \subseteq \mathfrak{a}^{\circ} \cap \mathfrak{b}^{\circ}$ and $(\mathfrak{a}^{\circ} \cap \mathfrak{b}^{\circ})^{\odot} \subseteq (\mathfrak{c}^{\circ})^{\odot}$. In particular, $2^{-1}(x + y) \in (\mathfrak{c}^{\circ})^{\odot}$ or $x + y \in ((2\mathfrak{c})^{\circ})^{\odot}$. Thus $(\mathfrak{a}^{\circ})^{\odot} + (\mathfrak{b}^{\circ})^{\odot} \subseteq ((2\mathfrak{c})^{\circ})^{\odot}$. But $2\mathfrak{c} \subseteq \mathfrak{c} + \mathfrak{c} \subseteq \mathfrak{d}$ for some $\mathfrak{d} \in \mathfrak{s}$. Hence $(\mathfrak{a}^{\circ})^{\odot} + (\mathfrak{b}^{\circ})^{\odot} \subseteq (\mathfrak{d}^{\circ})^{\odot}$. It remains to prove that $\cup \mathfrak{s}^{\max} = M(V)$. Take $v \in M_n(V)$. As above we can assume that all $v_{ij} \in \mathfrak{a}$ for a certain $\mathfrak{a} \in \mathfrak{s}$. If $y \in \mathfrak{a}^{\circ}$ then $\left\|\left\langle\left\langle n^{-2} v, y\right\rangle\right\rangle\right\| = n^{-2} \|[\langle v_{ij}, y\rangle]\| \leq n^{-2} \sum_{i,j} |\langle v_{ij}, y\rangle| \leq 1$, that is, $v \in \left(\left(n^2 \mathfrak{a}\right)^{\circ}\right)^{\odot}$. But $n^2 \mathfrak{a} \subseteq \mathfrak{b}$ for some $\mathfrak{b} \in \mathfrak{s}$. Hence $v \in (\mathfrak{b}^{\circ})^{\odot}$, that is, $\mathfrak{s}^{\max}$ is a base of matrix bornology on V.

Further, fix $\mathfrak{a} \in \mathfrak{s}$. Put $\mathfrak{M} = \mathfrak{a}^{\odot\odot}$. On the grounds of the Bipolar Theorem 2.1, $\mathfrak{M}$ is the weak closure of the matrix set $\mathfrak{B} = \operatorname{amc} \mathfrak{a}$. But $\mathfrak{b}_1 = \operatorname{abc} \mathfrak{a} = \mathfrak{a}$ thanks to Lemma 2.1. Therefore $\mathfrak{m}_1 = \mathfrak{b}_1^{-} = \mathfrak{a}^{-}$. In particular, $\mathfrak{s}^{\min}|V = \mathfrak{s}^{-}$. Similarly, if $\mathfrak{M} = (\mathfrak{a}^{\circ})^{\odot}$ then $\mathfrak{m}_1 = (\mathfrak{a}^{\circ})^{\circ}$ (see Section). But $(\mathfrak{a}^{\circ})^{\circ} = \mathfrak{a}^{-}$ by the classical Bipolar Theorem. Hence $\mathfrak{s}^{-} = \mathfrak{s}^{\max}|V$.

Finally, let $\mathcal{S}$ be a base of matrix bornology on V such that $\mathcal{S}|V = \mathfrak{s}^{-}$. Take $\mathfrak{a} \in \mathfrak{s}$. If $\mathfrak{b} = \mathfrak{a}^{-}$ then $\mathfrak{b} = \mathfrak{b}_1 \in \mathcal{S}|V$ for a certain $\mathfrak{B} \in\mathcal{S}$. Since $\mathfrak{a} \subseteq \mathfrak{B}$ as the matrix sets then $\mathfrak{a}^{\odot\odot} \subseteq \mathfrak{B}^{\odot\odot} = \mathfrak{B}^{-}$ thanks to Theorem 2.1. Hence $\mathfrak{s}^{\min} \preceq \mathcal{S}^{-}$. Further, if $\mathfrak{A} = (\mathfrak{a}_n) \in \mathcal{S}$ then $\mathfrak{a}_1 = \mathfrak{b}^{-}$ for some $\mathfrak{b} \in \mathfrak{s}$ and $\mathfrak{a}_1^{\odot} = \mathfrak{a}_1^{\circ} = \mathfrak{b}^{\circ} \subseteq \mathfrak{A}^{\odot}$ as the matrix sets. It follows that $\mathfrak{A}^{-} = \mathfrak{A}^{\odot\odot} \subseteq (\mathfrak{b}^{\circ})^{\odot}$, that is, $\mathcal{S}^{-} \preceq \mathfrak{s}^{\max}$. ■

3.2. The dual matrix bornology

Now we investigate the matrix bornologies obtained as matrix polars of the neighborhoods from a quantum topology.

Lemma 3.2. *Let (V, W) be a dual pair and let $\mathcal{N}$ be a neighborhood filter base of a certain quantum topology in $M(V)$ compatible with the duality. Then $\mathcal{N}^{\odot} = \{\mathfrak{U}^{\odot} : \mathfrak{U} \in\mathcal{N}\}$ is a base of matrix bornology in $M(W)$ which consists of weak* compact matrix sets.*

Proof. Take $w \in M(W)$. Then $\{w\}^{\odot}$ is a neighborhood of the origin in $M(V)$ with respect to the weak quantum topology $\mathfrak{s}(V, W)$ (see Section). Note that

$$\mathfrak{s}(V, W) = \max \sigma(V, W) = \min \sigma(V, W) \subseteq \min(\mathcal{N}|V) \subseteq \mathcal{N}$$

thanks to Theorem 2.3. In particular, $\{w\}^{\odot} \supseteq \mathfrak{U}$ for a certain $\mathfrak{U} \in\mathcal{N}$. Then $\{w\} \subseteq \{w\}^{\odot\odot} \subseteq \mathfrak{U}^{\odot}$, that is, $\cup\mathcal{N}^{\odot} = M(W)$. Further, if $\mathfrak{U} \in\mathcal{N}$ and $\mathfrak{B} =\mathcal{HS}(\mathfrak{U})$ then $\mathfrak{u}_n^{\odot} = \mathfrak{b}_n^{\circ}$ by virtue of Theorem 2.2. Moreover, $\mathfrak{b}_n$ is a neighborhood in $M_n(V)$, for $n^{-1}\mathfrak{u}_n \subseteq \mathfrak{b}_n$. By Alaoglu-Bourbaki Theorem, $\mathfrak{b}_n^{\circ}$ is the weak $\sigma(M_n(W), M_n(V))$-compact set. Hence $\mathfrak{U}^{\odot}$ is a weak* compact matrix set. It remains to prove that $\mathcal{N}^{\odot}$ is a base of matrix bornology. Take $\mathfrak{U}, \mathfrak{V} \in\mathcal{N}$. Since $\mathcal{N}$ is a neighborhood filter base of a quantum topology, it follows that $\mathfrak{W} \subseteq 2^{-1}\mathfrak{U} \cap \mathfrak{V}$ for a certain $\mathfrak{W} \in\mathcal{N}$. Then $\mathfrak{U}^{\odot} + \mathfrak{V}^{\odot} \subseteq \left(2^{-1}\mathfrak{U} \cap \mathfrak{V}\right)^{\odot} \subseteq \mathfrak{W}^{\odot}$. Hence $\mathcal{N}$ is a base of matrix bornology. ■

The assertion from Lemma 3.2 can be reversed by the following way.

Lemma 3.3. *Let (V, W) be a dual pair and let $\mathcal{S}$ be a base of matrix bornology in $M(V)$ which consists of weakly matrix bounded sets. Then $\mathcal{S}^{\odot} = \{\mathfrak{B}^{\odot} : \mathfrak{B} \in\mathcal{S}\}$ is a neighborhood filter base of a quantum topology in $M(W)$.*

Proof. Since each $\mathfrak{B} \in \mathcal{S}$ is weakly matrix bounded, it follows that $\mathfrak{B}^{\odot}$ is absorbent absolutely matrix convex set in $M(W)$. Take $\mathfrak{A}, \mathfrak{B} \in \mathcal{S}$. Then $\mathfrak{A} \cup \mathfrak{B} \subseteq \mathfrak{C}$ for a certain $\mathfrak{C} \in\mathcal{S}$. It follows that $\mathfrak{C}^{\odot} \subseteq (\mathfrak{A} \cup \mathfrak{B})^{\odot} = \mathfrak{A}^{\odot} \cap \mathfrak{B}^{\odot}$, that is, $\mathcal{S}^{\odot}$ is a filter base. Furthermore, $\cap\mathcal{S}^{\odot} = (\cup\mathcal{S})^{\odot} = (M(V))^{\odot} = \{0\}$. Hence $\mathcal{S}^{\odot}$ defines a (Hausdorff) quantum topology in $M(W)$. ■

Now we prove the main result on a matrix bornology.

Theorem 3.1. *Let $\mathfrak{s}$ be a base of bornology on V. If $\mathfrak{s}$ is finite-complete then the closures of all quantizations of $\mathfrak{s}^{-}$ coincide, that is, if $\mathcal{S}|V = \mathfrak{s}^{-} = \mathcal{T}|V$ for* *some bases $\mathcal{S}$ and $\mathcal{T}$ of matrix bornologies on V then $\mathcal{S}^- = \mathcal{T}^-$, where all closures are taken with respect to the finest locally convex topology.*

Proof. Let W be the subspace of those linear functionals $w : V \to \mathbb{C}$ such that $\sup |w(\mathfrak{b})| < \infty$ for all $\mathfrak{b} \in \mathfrak{s}$. Thus W is the bornology dual of $(V, \mathfrak{s})$ or the space of all $\mathfrak{s}$-bounded linear functionals on V. Since $V = \varinjlim \{\operatorname{span}(\mathfrak{b}), \mathfrak{b} \in \mathfrak{s}\}$ is the inductive limit of the finite-dimensional spaces, it follows that the relevant inductive topology in V is just the finest polynormed topology. But W is the space of all continuous linear functionals on V. Hence $W = V^*$ is just the algebraic dual space of V.

Obviously, all sets from $\mathfrak{s}$ are weakly bounded. Furthermore, the polar sets $\mathfrak{s}^\circ$ in W define the weak* topology $\sigma(W, V)$. Indeed, take a finite subset $F \subseteq V$. Then each $v \in F$ belongs to a certain $\mathfrak{b}_v \in \mathfrak{s}$. It follows that $F \subseteq \cup_{v \in F} \mathfrak{b}_v \subseteq \sum_{v \in F} \mathfrak{b}_v \subseteq \mathfrak{b}$ for some $\mathfrak{b} \in \mathfrak{s}$ (see Remark 3.1). In particular, $\mathfrak{b}^\circ \subseteq F^\circ$, that is, $\mathfrak{s}^\circ$ is finer than $\sigma(W, V)$. Conversely, fix $\mathfrak{b} \in \mathfrak{s}$ whose Minkowski functional is denoted by p. Thus $(\operatorname{span}(\mathfrak{b}), p)$ is a finite dimensional normed space. Take a basis $e = (e_1, \ldots, e_n)$ in $\operatorname{span}(\mathfrak{b})$, and let l_e be the ℓ_1-norm with respect to e. Then $cl_e \leq p$ for some positive c. But $cl_e = l_x$, where $x = c^{-1}e$. Then $\mathfrak{b} \subseteq \operatorname{ball} p \subseteq \operatorname{ball} l_x = \operatorname{abc} x$, which in turn implies that $x^\circ = (\operatorname{abc} x)^\circ \subseteq \mathfrak{b}^\circ$. Hence $\mathfrak{s}^\circ$ is coarser than $\sigma(W, V)$. Thus $\mathfrak{s}^\circ = \sigma(W, V)$.

Now take a base of matrix bornology $\mathcal{S}$ on V such that $\mathcal{S}|V = \mathfrak{s}^-$. Note that $\mathfrak{s}^-$ is just reduced to the weak(-$\sigma(V, W)$) closure of $\mathfrak{s}$ thanks to Mazur's theorem. By Proposition 3.1, $\mathcal{S}^- \preceq \mathfrak{s}^{\max}$. Let us prove that $\mathfrak{s}^{\max}$ consists of weakly matrix bounded sets. Take $\mathfrak{b} \in \mathfrak{s}$. Since $\mathfrak{b}$ is weakly bounded, it follows that $\mathfrak{b}^\circ$ is absorbent in W. Fix $w \in M_n(W)$. Then all $m^{-1} w_{ij} \in \mathfrak{b}^\circ$ for a certain $m \in \mathbb{N}$. Then $(nm)^{-1} w = \sum_{i,j} n^{-1/2} \epsilon_i^* \left(m^{-1} w_{ij}\right) n^{-1/2} \epsilon_j \in \operatorname{amc} \mathfrak{b}^\circ$. Thus $\operatorname{amc} \mathfrak{b}^\circ$ is absorbent in $M(W)$. Since $\operatorname{amc} \mathfrak{b}^\circ \subseteq (\mathfrak{b}^\circ)^{\odot\odot}$, it follows that $(\mathfrak{b}^\circ)^{\odot\odot}$ is absorbent in $M(W)$. Hence $(\mathfrak{b}^\circ)^\odot$ is a weakly matrix bounded set in $M(V)$, that is, $\mathfrak{s}^{\max}$ consists of weakly matrix bounded sets. In particular, $\mathcal{S}$ consists of weakly matrix bounded sets in $M(V)$. By Lemma 3.3, $\mathcal{S}^\odot$ defines a quantum topology in $M(W)$. Further, take $\mathfrak{B} \in \mathcal{S}$. Then $\mathfrak{b}_1 = \mathfrak{a}^-$ for some $\mathfrak{a} \in \mathfrak{s}$. But $\mathfrak{b}_1^\odot = \mathfrak{b}_1^\circ = \mathfrak{a}^\circ$. Hence $\mathcal{S}^\odot|W$ defines the polar topology $\mathfrak{s}^\circ$ in W, which in turn is reduced to the weak* topology $\sigma(W, V)$. But $\sigma(W, V)$ admits precisely one quantization $\mathfrak{s}(W, V)$ thanks to Theorem 2.3. Therefore $\mathcal{S}^\odot = \mathfrak{s}(W, V)$.

Finally, if $\mathcal{S}|V = \mathfrak{s}^- = \mathcal{T}|V$ for some bases $\mathcal{S}$ and $\mathcal{T}$ of matrix bornologies on V then $\mathcal{S}^\odot = \mathfrak{s}(W, V) = \mathcal{T}^\odot$. Using the Bipolar Theorem 2.1, we derive that $\mathcal{S}^- = \mathcal{S}^{\odot\odot} = \mathcal{T}^{\odot\odot} = \mathcal{T}^-$, that is, $\mathcal{S}^- = \mathcal{T}^-$. ∎

It seems similar argument can be applied to a nuclear quantum bornology.

3.3. The finite dimensional matrix bornology

Now let V be a finite dimensional linear space. It has a canonical finite-complete bornology $\mathfrak{f}$ of bounded sets, which is unique up to an isomorphism [12]. The family $\mathfrak{s}$ of all closed (with respect to the canonical normed topology in V) absolutely convex bounded sets is a base of this bornology. Let $\mathfrak{S}$ be a matrix bornology in $M(V)$ such that $\mathfrak{S}|V = \mathfrak{f}$, that is, $\mathfrak{S}$ is a quantization of $\mathfrak{f}$. Consider the family $\mathcal{S}$ of those absolutely matrix convex sets $\mathfrak{B}$ from $\mathfrak{S}$ whose first members $\mathfrak{b}_1$ are closed.

Lemma 3.4. *The family $\mathcal{S}$ is a base of the matrix bornology $\mathfrak{S}$. Moreover, $\mathcal{S}|V = \mathfrak{s}$.*

Proof. First let us prove that $\cup\mathcal{S} = M(V)$. Take an absolutely matrix convex set $\mathfrak{B} \in \mathfrak{S}$. Then the closure $\mathfrak{b}_1^-$ is a bounded set in V, that is, $\mathfrak{b}_1^- \in \mathfrak{f}$. Since $\mathfrak{S}|V = \mathfrak{f}$, it follows that $\mathfrak{b}_1^- = \mathfrak{m}_1$ for a certain $\mathfrak{M} \in \mathfrak{S}$. But $\mathfrak{m}_1 \subseteq \mathfrak{M}$ as the matrix sets. Hence $\mathfrak{b}_1^- \in \mathfrak{S}$, and $\mathfrak{B} \cup \mathfrak{b}_1^- \in \mathfrak{S}$. Consider $\mathfrak{A} = \operatorname{amc}\left(\mathfrak{B} \cup \mathfrak{b}_1^-\right)$ which belongs to $\mathfrak{S}$ too (see Definition 3.1). Let us prove that $\mathfrak{a}_1 = \mathfrak{b}_1^-$. Take $v \in \mathfrak{a}_1$. Then $v = a(v_1 \oplus \cdots \oplus v_s)\, b$ with $v_i \in \mathfrak{B} \cup \mathfrak{b}_1^-$, $a, b \in \operatorname{ball} M$. If $v_i \in \mathfrak{b}_1^-$ for some i, then $v_i = \lim_n v_{in}$ for a certain sequence $v_{in} \in \mathfrak{b}_1$. Put $v_n = a(v_{1n} \oplus \cdots \oplus v_{sn})\, b$, where $v_{in} = v_i$ if $v_i \in \mathfrak{B}\backslash\mathfrak{b}_1^-$. Note that $v_n \in \operatorname{amc}\mathfrak{B} = \mathfrak{B}$ for all n. Actually $v_n \in \mathfrak{b}_1$. Moreover, $v = \lim_n v_n \in \mathfrak{b}_1^-$. Therefore $\mathfrak{a}_1 = \mathfrak{b}_1^-$. Thus $\mathfrak{A} \in\mathcal{S}$ and $\mathfrak{B} \subseteq \mathfrak{A}$. In particular, $\cup\mathcal{S} = M(V)$.

It remains to observe that if $\mathfrak{A}, \mathfrak{B} \in\mathcal{S}$ then $\mathfrak{A} + \mathfrak{B} \in\mathcal{S}$, for $\mathfrak{A} + \mathfrak{B}$ is absolutely matrix convex and $\mathfrak{a}_1 + \mathfrak{b}_1$ is closed being a sum of two compact sets in a finite dimensional space. Thus $\mathcal{S}$ is a base of $\mathfrak{S}$.

Finally, let us prove that $\mathcal{S}|V = \mathfrak{s}$. Evidently, $\mathcal{S}|V \preceq \mathfrak{s}$. Take $\mathfrak{a} \in \mathfrak{s}$. Then $\mathfrak{a} = \mathfrak{a}_1$ for a certain $\mathfrak{A} \in \mathfrak{S}$. Since $\mathfrak{a} \subseteq \mathfrak{A}$ as the matrix sets, it follows that $\mathfrak{a} \in \mathfrak{S}$. Put $\mathfrak{B} = \operatorname{amc}\mathfrak{a} \in \mathfrak{S}$. But $\mathfrak{b}_1 = \operatorname{abc}\mathfrak{a} = \mathfrak{a}$ by Lemma 2.1. Hence $\mathfrak{B} \in \mathcal{S}$ and $\mathfrak{b}_1 = \mathfrak{a}$, that is, $\mathfrak{a} \in\mathcal{S}|V$. ■

Theorem 3.2. *The canonical bornology $\mathfrak{f}$ on a finite dimensional space V admits precisely one quantization, that is, if $\mathfrak{S}$ and $\mathfrak{T}$ are matrix bornologies on V such that $\mathfrak{S}|V = \mathfrak{f} = \mathfrak{T}|V$, then $\mathfrak{S} = \mathfrak{T}$.*

Proof. By Lemma 3.4, $\mathcal{S}$ is a base of the matrix bornology $\mathfrak{S}$ such that $\mathcal{S}|V = \mathfrak{s}$. Similarly, if $\mathfrak{T}$ is another quantization of $\mathfrak{f}$ then is has the base $\mathcal{T}$ of absolutely matrix convex sets with their closed first members. Moreover, $\mathcal{T}|V = \mathfrak{s}$ by Lemma 3.4. Using Theorem 3.1, we derive that $\mathcal{S}^- = \mathcal{T}^-$. Hence $\mathfrak{S}^- = \mathfrak{T}^-$. It remains to prove that $\mathfrak{S} = \mathfrak{T}$.

First we consider the one dimensional case, that is, $\dim(V) = 1$. Take a bounded set $\mathfrak{b} \in \mathfrak{f}$. Then $\mathfrak{b} \subseteq \mathfrak{a} = \operatorname{abc}\{v\}$ for a certain $v \in V\backslash\{0\}$. But $\mathfrak{a}$ is a compact set and $\mathfrak{a} \in \mathfrak{s}$. Consider the matrix set $\mathfrak{A} = \operatorname{amc}(\mathfrak{a}) \subseteq M(V)$. Let us prove that $\mathfrak{A}$ is closed, that is each $\mathfrak{a}_n$ is closed in the finite dimensional space $M_n(V)$. Take $w \in \mathfrak{a}_n$. Then $w = \sum_{i=1}^m a_i w_i b_i$ with $a_i \in M_{n,1}$, $w_i \in \mathfrak{a}$, $b_i \in M_{1,n}$, and

$\sum_i a_i a_i^*, \sum_i b_i^* b_i \leq 1$. But $w_i = \lambda_i v$ with $|\lambda_i| \leq 1$. Put $\theta_i = \lambda_i |\lambda_i|^{-1/2}$ (if $\lambda_i = 0$ then $\theta_i = 0$) and $\eta_i = |\lambda_i|^{1/2}$. It follows that $w = \sum_{i=1}^m (\theta_i a_i) v (\eta_i b_i) = av^{\oplus m} b^* = cv^{\oplus n}$, where $a = \begin{bmatrix} \theta_1 a_1 & \cdots & \theta_m a_m \end{bmatrix} \in M_{n,m}$, $b = \begin{bmatrix} \eta_1 b_1^* & \cdots & \eta_m b_m^* \end{bmatrix} \in M_{m,n}$, and $c = ab^* \in M_n$. Note that

$$aa^* = \sum_{i=1}^m |\theta_i|^2 a_i a_i^* = \sum_{i=1}^m |\lambda_i| a_i a_i^* \leq 1, \quad bb^* = \sum_{i=1}^m |\eta_i|^2 b_i^* b_i = \sum_{i=1}^m |\lambda_i| b_i^* b_i \leq 1,$$

that is, $a, b \in \operatorname{ball} M$. In particular, $c \in \operatorname{ball} M_n$. Conversely, if $w = cv^{\oplus n}$ for some $c \in \operatorname{ball} M_n$, then $w = u|c| v^{\oplus n} = uv^{\oplus n} |c|$, where $c = u|c|$ is the polar decomposition of c. But $u, |c| \in \operatorname{ball} M_n$. Thus $\mathfrak{a}_n = \{cv^{\oplus n} : c \in \operatorname{ball} M_n\}$, which in turn implies that $\mathfrak{a}_n$ is a compact set in M_n. Moreover, using the Bipolar Theorem 2.1, we conclude that $\mathfrak{b}^{\odot\odot} = (\operatorname{amc} \mathfrak{b})^- \subseteq \mathfrak{A} = \operatorname{amc} \mathfrak{a}$. But $\mathfrak{a} = \mathfrak{c}_1$ for a certain $\mathfrak{C} \in \mathcal{S}$ thanks to Lemma 3.4. Since $\mathfrak{C}$ is absolutely matrix convex, it follows that $\operatorname{amc}(\mathfrak{a}) \subseteq \mathfrak{C}$, that is, $\mathfrak{b}^{\odot\odot} \subseteq \mathfrak{C}$. Hence $\mathfrak{s}^{\min} \preceq \mathcal{S}$. Using Proposition 3.1 and Theorem 3.1, we derive that $\mathcal{S} \preceq \mathcal{S}^- = (\mathfrak{s}^{\min})^- = \mathfrak{s}^{\min} \preceq \mathcal{S}$, that is, $\mathfrak{s}^{\min} = \mathcal{S} = \mathcal{S}^-$. In particular, $\mathcal{S} = \mathcal{S}^- = \mathcal{T}^- = \mathcal{T}$, which in turn implies that $\mathfrak{S} = \mathfrak{T}$. Thus the assertion has been proved for the case $\dim(V) = 1$.

In the general case, we fix a decomposition $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_q$, where $q = \dim(V)$. For brevity, we assume that $q = 2$. If $\mathfrak{S}_k$ is the unique matrix bornology on $\mathbb{C}e_k$ then $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2$ is a matrix bornology on V with the base $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$. Note that $\mathcal{S}$ is closed and $\mathcal{S}|V = \mathfrak{s}$. Therefore $\mathcal{T}^- = \mathcal{S}$ for any matrix bornology $\mathfrak{T}$ on V with its basis $\mathcal{T}$ such that $\mathfrak{f} = \mathfrak{T}|V$. In particular, $\mathcal{T} \preceq \mathcal{S}$. Conversely, take $\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2 \in \mathcal{S}$. Since $\mathcal{S}_k = \mathfrak{s}_k^{\min}$, $k = 1, 2$, it follows that $\mathfrak{B}_k \subseteq \operatorname{amc}(\mathfrak{a}_k)$ for some $\mathfrak{a}_k = \operatorname{abc}\{v_k\} \in \mathfrak{s}_k$. But $\operatorname{amc}(\mathfrak{a}_1) \times \operatorname{amc}(\mathfrak{a}_2) \subseteq \operatorname{amc}(\mathfrak{c})$, where $\mathfrak{c} = 2\mathfrak{a}_1 \times 2\mathfrak{a}_2$ is a compact set in V. Indeed, if $x = \left(a_1 v_1^{\oplus n_1} b_1, a_2 v_2^{\oplus n_2} b_2\right)$ with $a_k, b_k \in \operatorname{ball} M$, then $x = 2^{-1/2} a_1 2 v_1^{\oplus n_1} 2^{-1/2} b_1 + 2^{-1/2} a_2 2 v_2^{\oplus n_2} 2^{-1/2} b_2$ in $M(V)$. Note that $2^{-1} a_1 a_1^* + 2^{-1} a_2 a_2^* \leq 2^{-1} \|a_1\|^2 I + 2^{-1} \|a_2\|^2 I \leq I$. Similarly, $2^{-1} b_1^* b_1 + 2^{-1} b_2^* b_2 \leq I$. Hence $x \in \operatorname{amc} \mathfrak{c}$. But $\mathfrak{c} = \mathfrak{c}_1$ for some $\mathfrak{C} \in \mathcal{T}$. It follows that $\mathfrak{B} \subseteq \operatorname{amc} \mathfrak{c} \subseteq \mathfrak{C}$, that is, $\mathcal{S} \preceq \mathcal{T}$. ■

Let $(T, \mathfrak{T})$ and $(V, \mathfrak{S})$ be quantum bornological spaces and let $\varphi : (T, \mathfrak{T}) \to (V, \mathfrak{S})$ be a surjective linear mapping. The range $\varphi^{(\infty)}(\mathfrak{T}) = \left\{\varphi^{(\infty)}(\mathfrak{L}) : \mathfrak{L} \in \mathfrak{T}\right\}$ of the matrix bornology $\mathfrak{T}$ in $M(V)$ turns out to be a matrix bornology. We say that φ is *a matrix quotient mapping* if $\mathfrak{S}^- = \varphi^{(\infty)}(\mathfrak{T})^-$. If $\mathfrak{S} = \varphi^{(\infty)}(\mathfrak{T})$ then we say that φ is *an exact matrix quotient mapping.*

Corollary 3.1. *Let V be a finite dimensional linear space with its canonical bornology $\mathfrak{f}$, $(T, \mathfrak{t})$ a bornological space and let $\varphi : (T, \mathfrak{t}) \to (V, \mathfrak{f})$ be a bornological quotient mapping, that is, $\varphi(\mathfrak{t}) = \mathfrak{f}$. Then φ is an exact matrix quotient mapping for any quantizations of the original bornologies.*

Proof. Assume $\mathfrak{T}$ is a matrix bornology on T such that $\mathfrak{T}|T = \mathfrak{t}$. By Theorem 3.2, $\mathfrak{f}$ admits precisely one quantization $\mathfrak{S}$. Then $\varphi^{(\infty)}(\mathfrak{T})$ is a matrix bornology on V such that $\varphi^{(\infty)}(\mathfrak{T})|V = \varphi(\mathfrak{T}|T) = \varphi(\mathfrak{t}) = \mathfrak{f} = \mathfrak{S}|V$. Using Theorem 3.2 again, we derive that $\varphi^{(\infty)}(\mathfrak{T}) = \mathfrak{S}$. ■

In the infinite dimensional case we use the duality (see Theorem 4.2 below).

4. The Duality Theorems

In this section we prove the main duality results for quantum spaces.

As above we fix a dual pair (V, W) of linear spaces. Consider a quantum topology $\mathcal{N}$ in $M(V)$ which is compatible with the duality. We identify $\mathcal{N}$ with its neighborhood filter base of weakly closed absolutely matrix convex sets in $M(V)$. By Lemma 3.2, $\mathcal{N}^{\odot}$ is a base of matrix bornology in $M(W)$. Let $X \subseteq V$ be a linear subspace, which is a quantum subspace in V with its neighborhood filter base $M(X) \cap \mathcal{N} = \{M(X) \cap \mathfrak{U} : \mathfrak{U} \in \mathcal{N}\}$. If $Y = W/X^{\perp}$ then the spaces X and Y are in the canonical duality associated with the dual pair (V, W). Since $M(X) \cap \mathcal{N}$ defines a quantum topology in $M(X)$, it follows that $(M(X) \cap \mathcal{N})^{\odot}$ is a base of matrix bornology in $M(Y)$ (see Lemma 3.2). So, we have the quantum bornological spaces $(W, \mathcal{N}^{\odot})$ and $\left(Y, (M(X) \cap \mathcal{N})^{\odot}\right)$, and the quotient mapping $\varphi : W \to Y$.

Theorem 4.1. *The mapping $\varphi : (W, \mathcal{N}^{\odot}) \to \left(Y, (M(X) \cap \mathcal{N})^{\odot}\right)$ is an exact matrix quotient mapping of the quantum bornological spaces, that is,*

$$\varphi^{(\infty)}\left(\mathcal{N}^{\odot}\right) = (M(X) \cap \mathcal{N})^{\odot}.$$

Proof. Fix $n \in \mathbb{N}$ and consider the polynormed space $M_n(V)$ equipped with the weak topology, and the linear mapping $M_n(W) \to M_n(V)'$, $f \mapsto F$, $F(v) = \langle v, f \rangle$. If $F \in M_n(V)'$ is a continuous linear functional then all functionals $f_{ij} : V \to \mathbb{C}$, $f_{ij}(v) = F(\epsilon_i^* v \epsilon_j)$, are weakly continuous. Indeed,

$$\begin{aligned} |f_{ij}(v)| &= |F(\epsilon_i^* v \epsilon_j)| \leq p_w(\epsilon_i^* v \epsilon_j) = \|\langle\langle \epsilon_i^* v \epsilon_j, w \rangle\rangle\| = \|\epsilon_i^* \otimes 1 \langle\langle v, w \rangle\rangle \epsilon_j \otimes 1\| \\ &\leq \|\langle\langle v, w \rangle\rangle\| \leq \sum_{s,t} |\langle v, w_{st} \rangle| = \sum_{s,t} p_{w_{st}}(v) \end{aligned}$$

for a certain $w \in M(W)$ (see Section). Hence $f_{ij} \in W$ for all i, j. In particular, $f = [f_{ij}] \in M_n(W)$ and $\langle v, f \rangle = \sum_{i,j} \langle v_{ij}, f_{ij} \rangle = \sum_{i,j} F(\epsilon_i^* v_{ij} \epsilon_j) = F(v)$. Thus $M_n(W) = M_n(V)'$ up to the canonical identification. Further, if $i_n : M_n(X) \to M_n(V)$ is the inclusion mapping then $(i_n)' = \varphi^{(n)} : M_n(W) \to M_n(Y)$ is the dual mapping. Indeed, $(i_n)'(F)(x) = F(i_n(x)) = F(x) = \langle x, f \rangle = \left\langle x, \varphi^{(n)}(f) \right\rangle$ for all $x \in M_n(X)$.

Take $\mathfrak{U} \in \mathcal{N}$ and consider its Hilbert-Schmidt boundary $\mathfrak{B} = \mathcal{HS}(\mathfrak{U})$, which is an absolutely convex set. Note that $\mathfrak{b}_n^{\circ}$ is $\sigma(M_n(W), M_n(V))$-compact thanks to Alaoglu-Bourbaki theorem (see to the proof of Lemma 3.2). Since $\varphi^{(n)}(\mathfrak{b}_n^{\circ}) =$

$(i_n)'(\mathfrak{b}_n^\circ)$ and $(i_n)'$ is weak* continuous, it follows that $\varphi^{(n)}(\mathfrak{b}_n^\circ)$ is weak* compact too. But $\varphi^{(n)}(\mathfrak{b}_n^\circ)^\bullet = i_n^{-1}(\mathfrak{b}_n^{\circ\circ})$ (this is a well known equality, see for instance [8, (8.6.2)]), that is, $\varphi^{(n)}(\mathfrak{b}_n^\circ)^\bullet = \mathfrak{b}_n^- \cap M_n(X)$, where $\varphi^{(n)}(\mathfrak{b}_n^\circ)^\bullet$ is the polar of $\varphi^{(n)}(\mathfrak{b}_n^\circ)$ with respect to the dual pair $(M_n(X), M_n(Y))$, and $\mathfrak{b}_n^-$ is the weak closure of $\mathfrak{b}_n$. But $\mathfrak{b}_n^- \cap M_n(X) = (\mathfrak{b}_n \cap M_n(X))^-$ is the weak closure of $\mathfrak{b}_n \cap M_n(X)$ in the subspace $M_n(X)$ with respect to the induced topology $\sigma(M_n(V), M_n(W)) | M_n(X)$. Since $\sigma(M_n(V), M_n(W)) | M_n(X) = \sigma(M_n(X), M_n(Y))$ [14, Section 4.4], it follows that $\mathfrak{b}_n^- \cap M_n(X)$ is the weak $\sigma(M_n(X), M_n(Y))$-closure of $\mathfrak{b}_n \cap M_n(X)$. Then

$$(\mathfrak{b}_n \cap M_n(X))^\bullet = \left(\mathfrak{b}_n^- \cap M_n(X)\right)^\bullet = \varphi^{(n)}(\mathfrak{b}_n^\circ)^{\bullet\bullet} = \varphi^{(n)}(\mathfrak{b}_n^\circ),$$

for $\varphi^{(n)}(\mathfrak{b}_n^\circ)$ is weak* compact. Thus $\varphi^{(\infty)}(\mathcal{HS}(\mathfrak{U})^\circ) = (\mathcal{HS}(\mathfrak{U}) \cap M(X))^\bullet$. Using (2.1) and Theorem 2.2 twice, we derive that

$$\begin{aligned}\varphi^{(\infty)}\left(\mathfrak{U}^\odot\right) &= \varphi^{(\infty)}(\mathcal{HS}(\mathfrak{U})^\circ) = (\mathcal{HS}(\mathfrak{U}) \cap M(X))^\bullet = \mathcal{HS}(M(X) \cap \mathfrak{U})^\bullet \\ &= (M(X) \cap \mathfrak{U})^\odot,\end{aligned}$$

that is, $\varphi^{(\infty)}(\mathcal{N}^\odot) = (M(X) \cap \mathcal{N})^\odot$. ■

Corollary 4.1. *Let V be a quantum space with its neighborhood filter base $\mathcal{N}$ of absolutely matrix convex sets and let $X \subseteq V$ be a linear subspace. Then the restriction mapping $(V', \mathcal{N}^\odot) \to \left(X', (M(X) \cap \mathcal{N})^\odot\right)$, $f \mapsto f|_X$, is an exact matrix quotient mapping.*

Proof. It suffices to apply Theorem 4.1 to the dual pair (V, V'). ■

Corollary 4.2. *Let V be an operator space, $X \subseteq V$ a linear subspace and let $\varphi : V^* \to X^*$, $\varphi(f) = f|_X$, be the restriction mapping. Then*

$$\varphi^{(\infty)}(\operatorname{ball} M(V^*)) = \operatorname{ball} M(X^*).$$

In particular, $V^*/X^\perp = X^*$ up to the matrix isometry.

Proof. It suffices to put $\mathcal{N} = \{s \operatorname{ball} M(V) : s > 0\}$ in Corollary 4.1. ■

Now let again (V, W) be a dual pair and let $\mathfrak{S}$ be a matrix bornology in $M(V)$ of weakly bounded matrix sets. By Lemma 3.3, the space $M(W)$ can be equipped with the quantum topology $\mathfrak{S}^\odot$. The relevant quantum space is denoted by $M(W_\mathfrak{S})$.

Theorem 4.2. *Let (T, D) and (V, W) be dual pairs with their matrix bornologies $\mathfrak{T}$ and $\mathfrak{S}$ of weakly bounded matrix sets, respectively, and let $\varphi : T \to V$ be a weakly continuous linear mapping. If the dual mapping $\varphi' : W \to D$ implements a matrix isomorphism $(\varphi')^{(\infty)} : M(W_\mathfrak{S}) \to M(D_\mathfrak{T})$ onto its range which is a weak* homeomorphism too, then $\varphi : (T, \mathfrak{T}) \to (V, \mathfrak{S})$ is a matrix quotient mapping of the matrix bornological spaces.*

Proof. We denote $\varphi^{'}$ by Φ. Since $\Phi : W \to D$ is a weak* homeomorphism onto its range, it follows that φ is a mapping onto [8, 8.6.4]. Further, take $\mathfrak{B} \in \mathfrak{S}$. Since $\Phi^{(\infty)} : M(W_{\mathfrak{S}}) \to M(D_{\mathfrak{T}})$ is a topological isomorphism onto its range, it follows that $\Phi^{(\infty)}(\mathfrak{B}^{\odot}) \supseteq \Phi^{(\infty)}(M(W)) \cap \mathfrak{L}^{\odot}$ or $\mathfrak{B}^{\odot} \supseteq \left(\Phi^{(\infty)}\right)^{-1}(\mathfrak{L}^{\odot})$ for a certain $\mathfrak{L} \in \mathfrak{T}$. But $\varphi^{(\infty)}(\mathfrak{L})^{\odot} = \left(\Phi^{(\infty)}\right)^{-1}(\mathfrak{L}^{\odot})$. Indeed, (see [8, (8.6.2)])

$$\varphi^{(\infty)}(\mathfrak{L})^{\odot} = \mathcal{HS}\left(\varphi^{(\infty)}(\mathfrak{L})\right)^{\circ} = \varphi^{(\infty)}(\mathcal{HS}(\mathfrak{L}))^{\circ} = \left(\Phi^{(\infty)}\right)^{-1}(\mathcal{HS}(\mathfrak{L})^{\circ})$$
$$= \left(\Phi^{(\infty)}\right)^{-1}\left(\mathfrak{L}^{\odot}\right).$$

Thus $\mathfrak{B}^{\odot} \supseteq \varphi^{(\infty)}(\mathfrak{L})^{\odot}$. Using the Bipolar Theorem 2.1, we derive that $\mathfrak{B} \subseteq \varphi^{(\infty)}(\mathfrak{L})^{\odot\odot} = \varphi^{(\infty)}(\mathfrak{L})^{-}$, that is, $\mathfrak{B} \in \varphi^{(\infty)}(\mathfrak{T})^{-}$. Hence $\mathfrak{S}^{-} \preceq \varphi^{(\infty)}(\mathfrak{T})^{-}$.

Conversely, take $\mathfrak{L} \in \mathfrak{T}$. Since $\Phi^{(\infty)} : M(W_{\mathfrak{S}}) \to M(D_{\mathfrak{T}})$ is continuous, it follows that $\varphi^{(\infty)}(\mathfrak{L})^{\odot} = \left(\Phi^{(\infty)}\right)^{-1}(\mathfrak{L}^{\odot}) \supseteq \mathfrak{B}^{\odot}$ for a certain $\mathfrak{B} \in \mathfrak{S}$. Then $\varphi^{(\infty)}(\mathfrak{L}) \subseteq \mathfrak{B}^{-}$, that is, $\varphi^{(\infty)}(\mathfrak{L})^{-} \in \mathfrak{S}^{-}$. Whence $\mathfrak{S}^{-} = \varphi^{(\infty)}(\mathfrak{T})^{-}$, that is, φ is a matrix quotient mapping of the relevant quantum bornological spaces. ∎

Now fix a family $J = \{J_\kappa\}$ of sets with a mapping $n : \vee J \to \mathbb{N}$ over its disjoint union. For each κ we have the operator space (von Neumann algebra) $M_\kappa = \bigoplus_{w \in J_\kappa}^{\infty} M_{n(w)}$ (-direct sum of the full matrix algebras). The family J associates the quantum space (local von Neumann algebra) $\mathfrak{D}_J = \mathrm{op}\prod_\kappa M_\kappa$, which is the quantum (or operator) direct product of von Neumann algebras. The quantum space $\mathfrak{D}_J$ has a realization as unbounded operators [4]. Note that the predual of the local von Neumann algebra $\mathfrak{D}_J$ (in the strong dual sense) is the space $\mathcal{T}_J = \mathrm{op}\bigoplus_\kappa \bigoplus_{w \in J_\kappa}^{1} \mathcal{T}_{n(w)}$ of all trace class matrices in $\mathfrak{D}_J$, that is, $\mathfrak{D}_J = (\mathcal{T}_J)'_\beta$ [7]. In particular, we have the strong matrix bornology β on $\mathcal{T}_J$ of all matrix bounded sets in $M(\mathcal{T}_J)$.

Corollary 4.3. *Let V be a complete quantum space and let $\mathfrak{S}$ be a matrix bornology in $M(V)$ of $\sigma(V, V')$-bounded matrix sets. Then there is a matrix quotient mapping $\varphi : (\mathcal{T}_J, \beta) \to (V, \mathfrak{S})$ of the matrix bornological spaces for some family J.*

Proof. Using the dual realization theorem for quantum spaces [7], we conclude that there is a topological matrix isomorphism $\Phi : V'_{\mathfrak{S}} \to \mathfrak{D}_J$ onto its range, which is a weak* homeomorphism too. Then $\Phi = \varphi'$ for the uniquely defined weakly continuous linear mapping $\varphi : \mathcal{T}_J \to V$. Since Φ is the weak* homeomorphism onto its range, it follows that φ is onto [8, 8.6.4]. By Theorem 4.2, $\varphi : (\mathcal{T}_J, \beta) \to (V, \mathfrak{S})$ is a matrix quotient mapping of the matrix bornological spaces. ∎

In the normed case Corollary 4.3 is reduced to the fact that each complete operator space is a matrix quotient of an L^1-direct sum of finite dimensional trace class algebras up to a matrix isometry.

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