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THE $L(2,1)$-$\mathcal{F}$-LABELING PROBLEM OF GRAPHS

Gerard J. Chang[1] and Changhong Lu[2]

Abstract. In order to unify various concepts of distance-two labelings, we consider a general setting of distance-two labelings as follows. Given a graph H, an $L(2,1)$-H-labeling of a graph G is a mapping f from $V(G)$ to $V(H)$ such that $d_H(f(u), f(v)) \geq 2$ if $d_G(u,v) = 1$ and $d_H(f(u), f(v)) \geq 1$ if $d_G(u,v) = 2$. Suppose $\mathcal{F}$ is a family of graphs. The $L(2,1)$-$\mathcal{F}$-labeling problem is to determine the $L(2,1)$-$\mathcal{F}$-labeling number $\lambda_{\mathcal{F}}(G)$ of a graph G which is the smallest number $|E(H)|$ such that G has an $L(2,1)$-H-labeling for some $H \in \mathcal{F}$. Notice that the $L(2,1)$-$\mathcal{F}$-labeling is the $L(2,1)$-labeling (respectively, the circular distance-two labeling) if $\mathcal{F}$ is the family of all paths (respectively, cycles). The purpose of this paper is to study the $L(2,1)$-$\mathcal{F}$-labeling problem.

1. INTRODUCTION

The problem of vertex labeling with a condition at distance two, studied by Griggs and Yeh [12], is a variation of the T-coloring problem introduced by Hale [13]. Suppose there is a number of transmitters. The problem is to assign a channel to each transmitter such that interference is avoided. In order to avoid interference, any two "close" transmitters must receive different channels, and any two "very close" transmitters must receive channels that are at least two channels apart.

More precisely, we can construct the *interference graph* using the transmitters as the vertices and pairs of "very close" transmitters as the edges. Two transmitters are defined to be "close" if the corresponding vertices are of distance two. An $L(2,1)$-*labeling* of a graph G is a mapping f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u) - f(v)| \geq 2$ if $d_G(u,v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d_G(u,v) = 2$. A k-$L(2,1)$-*labeling* is an $L(2,1)$-labeling

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such that no label is greater than k. The $L(2,1)$-*labeling number* $\lambda(G)$ of G is the smallest number k such that G has a k-$L(2,1)$-labeling. $L(2,1)$-labelings have been extensively studied in the literature, see [3, 7, 8, 9, 11, 14, 15, 17, 18, 19].

There are several variations of the $L(2,1)$-labeling problem, see [1, 2, 3, 4, 10, 16, 17, 21]). Among them, the $L'(2,1)$-labeling problem was introduced by Chang and Kuo [3]. The definitions of an $L'(2,1)$-labeling f, a k-$L'(2,1)$-labeling f, and the $L'(2,1)$-labeling number $\lambda'(G)$ are the same as those of an $L(2,1)$-labeling f, a k-$L(2,1)$-labeling f, and the $L(2,1)$-labeling number $\lambda(G)$, respectively, except that the mapping f is required to be one to one.

Another variation of the $L(2,1)$-labeling, called the k-circular-labeling was introduced by Liu [16], see also [14, 20]. For a positive integer k, a k-*circular-labeling* of a graph G is a function f from $V(G)$ to $\{0,1,2,\ldots,k-1\}$ such that$|f(u)-f(v)|_k \geq 2$ if $d_G(x,y)=1$ and $|f(u)-f(v)|_k \geq 1$ if $d_G(x,y)=2$, where $|x|_k = \min\{|x|, k-|x|\}$ is the circular difference modulo k. The σ-*number* $\sigma(G)$ is the minimum k of a k-circular-labeling of G. Similarly, we can also define $\sigma'(G)$.

In order to unify these concepts, we introduce a more general setting called the $L(2,1)$-$\mathcal{F}$-labeling as follows. Given a graph H, an $L(2,1)$-H-labeling of a graph G is a mapping f from $V(G)$ to $V(H)$ such that $d_H(f(u),f(v)) \geq 2$ if $d_G(u,v)=1$ and $d_H(f(u),f(v)) \geq 1$ if $d_G(u,v)=2$. Notice that a graph may have no $L(2,1)$-H-labeling. For a graph G and a family $\mathcal{F}$ of graphs, let

$$\Omega(\mathcal{F},G) = \{H \in \mathcal{F} : G \text{ has an } L(2,1)\text{-}H\text{-labeling}\}.$$

The $L(2,1)$-$\mathcal{F}$-*labeling number* $\lambda_{\mathcal{F}}(G)$ of a graph G is the smallest number $|E(H)|$ for $H \in \Omega(\mathcal{F},G)$ provided $\Omega(\mathcal{F},G) \neq \emptyset$. The definitions of an $L'(2,1)$-H-labeling f, $\Omega'(\mathcal{F},G)$ and $\lambda'_{\mathcal{F}}(G)$ are the same as those of an $L(2,1)$-H-labeling f, $\Omega(\mathcal{F},G)$ and $\lambda_{\mathcal{F}}^{(G)}$, respectively, except that the mapping f is required to be one to one.

The concept of $L(2,1)$-H-labeling was first introduced by Fiala and Kratochv´ıl [5] using the name $H_{(2,1)}$-labeling, which is closely related to graph homomorphism as follows. Given graphs G and H, a mapping $f : V(G) \to V(H)$ is a *homomorphism* if $f(u)f(v)$ is an edge in H for every edge uv in G. A homomorphism f is *locally injective* if for any vertex u in G, the neighborhood $N_G(u)$ is mapped injectively into $N_H(f(u))$. Fiala and Kratochvíl [5] observed that an $H_{(2,1)}$-labeling of G is precisely a locally injective homomorphsim from G to $\overline{H}$, the complement of H. Computational complexity on locally injective homomorphisms are studied in [5, 6].

In this paper we study the $L(2,1)$-$\mathcal{F}$-labeling problem. Some general results are applicable to the families of paths, cycles, stars and trees, which are denoted by $\mathcal{P}$, $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{T}$, respectively. Notice that $\lambda_{\mathcal{P}}(G) = \lambda(G)$, $\lambda'_{\mathcal{P}}(G) = \lambda'(G)$, $\lambda_{\mathcal{C}}(G) = \sigma(G)$ and $\lambda'_{\mathcal{C}}(G) = \sigma'(G)$.

2. General Propositions

In this section, we establish some general propositions for the $L(2,1)$-$\mathcal{F}$-labeling problem.

Proposition 1. *If $\mathcal{F}$ and $\mathcal{G}$ are two families of graphs with $\mathcal{F} \subseteq \mathcal{G}$ and G is a graph, then $\lambda_{\mathcal{G}}(G) \le \lambda_{\mathcal{F}}(G) \le \lambda'_{\mathcal{F}}(G)$ and $\lambda'_{\mathcal{G}}(G) \le \lambda'_{\mathcal{F}}(G)$.*

Proof. The first two inequalities follows from $\Omega'(\mathcal{F}, G) \subseteq \Omega(\mathcal{F}, G) \subseteq \Omega(\mathcal{G}, G)$ and the third inequality follows from $\Omega'(\mathcal{F}, G) \subseteq \Omega'(\mathcal{G}, G)$. ∎

Proposition 2. *If $\mathcal{F}$ is a family of connected graphs and G is a graph of maximum degree Δ, then $\lambda_{\mathcal{F}}(G) \ge \Delta + 1$. Furthermore, if $\lambda_{\mathcal{F}}(G) = \Delta + 1$, then there is a tree $H \in \Omega(\mathcal{F}, G)$ with $\Delta + 2$ vertices such that $f(x)$ is a leaf in H for any $L(2,1)$-H-labeling f and any vertex x of degree Δ in G.*

Proof. Suppose $H \in \Omega(\mathcal{F}, G)$ and $\lambda_{\mathcal{F}}(G) = |E(H)|$. Choose an $L(2,1)$-H-labeling f of G. Let $N_G(x) = \{y_1, y_2, \ldots, y_\Delta\}$. By definition, $f(x), f(y_1), f(y_2), \ldots, f(y_\Delta)$ are distinct and $N_H(f(x)) \cap \{f(y_1), f(y_2), \ldots, f(y_\Delta)\} = \emptyset$. As H is connected, it has at least $\Delta + 1 + |N_H(f(x))| \ge \Delta + 2$ vertices and hence at least $\Delta + 1$ edges. This gives $\lambda_{\mathcal{F}}(G) \ge \Delta + 1$.

If $\lambda_{\mathcal{F}}(G) = \Delta + 1$, then H has exactly $\Delta + 2$ vertices and $\Delta + 1$ edges, implying that H is a tree. Also, $|N_H(f(x))| = 1$ and so $f(x)$ is a leaf. ∎

An $L(2,1)$-*preserving* from a graph H to a graph I is a mapping g from $V(H)$ to $V(I)$ such that $d_H(x,y) \ge i$ implies $d_I(g(x), g(y)) \ge i$ for $i = 1, 2$.

Proposition 3. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of graphs. If there is a constant $c(\mathcal{F}, \mathcal{G})$ such that for any $H \in \mathcal{F}$ there is some $I \in \mathcal{G}$ with $|E(I)| \le |E(H)| + c(\mathcal{F}, \mathcal{G})$ and there is an $L(2,1)$-preserving g from H to I, then $\lambda_{\mathcal{G}}(G) \le \lambda_{\mathcal{F}}(G) + c(\mathcal{F}, \mathcal{G})$ for any graph G.*

Proof. Suppose $H \in \Omega(\mathcal{F}, G)$ and $\lambda_{\mathcal{F}}(G) = |E(H)|$. Let f be an $L(2,1)$-H-labeling of G. By definition, $g \circ f$ is an $L(2,1)$-I-labeling of G. Thus, $I \in \Omega(\mathcal{G}, G)$ and so $\lambda_{\mathcal{G}}(G) \le |E(I)| \le |E(H)| + c(\mathcal{F}, \mathcal{G}) = \lambda_{\mathcal{F}}(G) + c(\mathcal{F}, \mathcal{G})$. ∎

Theorem 4. ([14]). *For any graph G, we have $\lambda_{\mathcal{P}}(G) + 1 \le \lambda_{\mathcal{C}}(G) \le \lambda_{\mathcal{P}}(G) + 2$.*

Proof. For any $C_n \in \mathcal{C}$, we can choose $P_n \in \mathcal{P}$ such that there is a natural $L(2,1)$-preserving from C_n to P_n with $|E(P_n)| = |E(C_n)| - 1$. By Proposition 3, $\lambda_{\mathcal{P}}(G) \le \lambda_{\mathcal{C}}(G) - 1$ which gives the first inequality.

On the other hand, for any $P_n \in \mathcal{P}$, we can choose $C_{n+1} \in \mathcal{C}$ such that there is an $L(2,1)$-preserving from P_n to C_{n+1} with $|E(C_{n+1})| = |E(P_n)| + 2$. By Proposition 3, $\lambda_{\mathcal{C}}(G) \le \lambda_{\mathcal{P}}(G) + 2$ as desired. ∎

Theorem 5. *Suppose $\mathcal{F}$ is a family of trees. For any graph G, we have $\lambda_{\mathcal{S}}(G) \le \lambda_{\mathcal{F}}(G)+1$. In particular, $\lambda_{\mathcal{S}}(G) \le \lambda_{\mathcal{T}}(G)+1$.*

Proof. For any $H \in \mathcal{F}$ of n vertices, we can choose $K_{1,n} \in \mathcal{S}$ such that there is an $L(2,1)$-preserving from H to $K_{1,n}$ by mapping the vertices of H to the leaves of $K_{1,n}$ bijectively. As $|E(K_{1,n})| = |E(H)|+1$, by Proposition 3, $\lambda_{\mathcal{S}}(G) \le \lambda_{\mathcal{F}}(G)+1$ as desired. ∎

Proposition 6. *If $\mathcal{F}$ is a family of graphs which has a graph H of maximum degree b and $\Delta+b+1$ vertices, then $\lambda_{\mathcal{F}}(G) \le |E(H)|$ for any tree G of maximum degree Δ.*

Proof. Order the vertices of G into $v_1, v_2, \ldots, v_n$ such that for each $i>1$ there is exactly one $j<i$ with $v_iv_j \in E(G)$. We label the vertices of G one by one from v_1 to v_n. After $v_1, v_2, \ldots, v_{i-1}$ having been labeled, it is possible to choose a vertex of H to label v_i so that the distance constraints hold. This is always possible because the only labeled neighbor of v_i forbids at most $b+1$ labels and the labeled distance-two neighbors of v_i forbid at most $\Delta-1$ labels, and in total at most $\Delta+b$ labels forbidden while H has $\Delta+b+1$ vertices. ∎

Theorem 7. ([12]). *If G is a tree of maximum degree Δ, then $\lambda_{\mathcal{P}}(G) = \Delta+1$ or $\Delta+2$.*

Proof. The result follows from Proposition 2 and Proposition 6 with $b=2$. ∎

Proposition 8. *If $\mathcal{F}$ is a family of r-regular graphs in which there is a graph of $\lceil \frac{(\Delta+r+1)r}{2} \rceil$ edges, then $\lambda_{\mathcal{F}}(G) = \lceil \frac{(\Delta+r+1)r}{2} \rceil$ for any tree G of maximum degree Δ.*

Proof. Suppose $H \in \Omega(\mathcal{F}, G)$ and $\lambda_{\mathcal{F}}(G) = |E(H)|$. Let f be an $L(2,1)$-H-labeling of G. Choose a vertex x of degree Δ in G with $N_G(x) = \{y_1, y_2, \ldots, y_\Delta\}$. By definition, $f(x), f(y_1), f(y_2), \ldots, f(y_\Delta)$ are distinct and $N_H(f(x)) \cap \{f(y_1), f(y_2), \ldots, f(y_\Delta)\} = \emptyset$. As H is r-regular, it has at least $\Delta+1+|N_H(f(x))| = \Delta+r+1$ vertices and hence at least $\lceil \frac{(\Delta+r+1)r}{2} \rceil$ edges. This gives that $\lambda_{\mathcal{F}}(G) \ge \lceil \frac{(\Delta+r+1)r}{2} \rceil$.

On the other hand, suppose G is a tree of maximum degree Δ. Order the vertices of G into $v_1, v_2, \ldots, v_n$ such that for each $i>1$ there is exactly one $j<i$ with $v_iv_j \in E(G)$. Since H has $\lceil \frac{(\Delta+r+1)r}{2} \rceil$ edges, it has at least $\Delta+r+1$ vertices. We label the vertices of G one by one from v_1 to v_n. After $v_1, v_2, \ldots, v_{i-1}$ having been labeled, it is possible to choose a vertex of H to label v_i so that the distance constraints hold. This is always possible because the only labeled neighbor of v_i forbids at most $r+1$ labels and the labeled distance-two neighbors of v_i forbidden at most $\Delta-1$ labels, and so totally at most $\Delta+r$ labels forbidden while H has at least $\Delta+r+1$ vertices. ∎

We remark that the same arguments as in the proof of Proposition 8 also lead to a slightly general result for the lower bound part: If $\mathcal{F}$ is a family of graphs of minimum degree r, then $\lambda_{\mathcal{F}}(G) \geq \lceil \frac{(\Delta+r+1)r}{2} \rceil$.

Theorem 9. ([16]). *If G is a tree of maximum degree Δ, then $\lambda_{\mathcal{C}}(G) = \Delta+3$.*

Proof. The result follows from Proposition 8 by using $r = 2$. ∎

Theorem 10. *If G is a tree with maximum degree Δ, then $\lambda_{\mathcal{T}}(G) = \lambda_{\mathcal{S}}(G) = \Delta + 1$.*

Proof. By Propositions 1 and 2, $\Delta+1 \leq \lambda_{\mathcal{T}}(G) \leq \lambda_{\mathcal{S}}(G)$. On the other hand, suppose G is a tree of maximum degree Δ. Order the vertices of G into $v_1, v_2, \ldots, v_n$ such that for each $i > 1$ there is exactly one $j < i$ with $v_i v_j \in E(G)$. Consider the tree $K_{1,\Delta+1} \in \mathcal{S}$. We label the vertices of G one by one from v_1 to v_n by using the leaves of $K_{1,\Delta+1}$. After $v_1, v_2, \ldots, v_{i-1}$ having been labeled, it is possible to choose a leaf of $K_{1,\Delta+1}$ to label v_i so that the distance constraints hold. This is always possible because the only labeled neighbor or distance-two neighbors of v_i forbidden at most Δ leaves of $K_{1,\Delta+1}$ while $K_{1,\Delta+1}$ has $\Delta + 1$ leaves. ∎

For any positive integer k, the *kth power* of a graph G is the graph G^k with $V(G^k) = V(G)$ and $E(G^k) = \{uv : 1 \leq d_G(u,v) \leq k\}$. The chromatic number $\chi(G)$ of G is the minimum number of colors needed to color the vertices of G so that adjacent vertices receive different colors.

Theorem 11. *If G is a graph, then $\lambda_{\mathcal{S}}^{(G)} = \chi(G^2)$.*

Proof. Suppose $c : V(G^2) \to \{1, 2, \ldots, \chi(G^2)\}$ is a proper $\chi(G^2)$-coloring of G^2. Choose $K_{1,\chi(G^2)} \in \mathcal{S}$ with leaves $v_1, v_2, \ldots, v_{\chi(G^2)}$. The mapping f on $V(G)$ defined by $f(x) = v_{c(x)}$ for $x \in V(G)$ is then an $L(2,1)$-$K_{1,\chi(G^2)}$-labeling. Hence, $\lambda_{\mathcal{S}}(G) \leq \chi(G^2)$.

On the other hand, for any $L(2,1)$-$K_{1,\lambda_{\mathcal{S}}(G)}$-labeling f of G, we may assume without loss of generality that $f(x)$ is a leaf of $K_{1,\lambda_{\mathcal{S}}(G)}$. It is then easy to see that f is a proper $\lambda_{\mathcal{S}}(G)$-coloring of G^2, if we identify the names of the vertices in G^2 with the names of the leaves in $K_{1,\chi(G^2)}$. This gives that $\chi(G^2) \leq \lambda_{\mathcal{S}}(G)$. ∎

3. Hamilitonicity

Graph hamiltonicity has a close relation with distance-two labeling. We now establish relation between hamiltonicity and $L(2,1)$-$\mathcal{F}$-labeling.

Proposition 12. *If $\mathcal{F}$ is a family of graphs and G is a graph, then for any $H \in \mathcal{F}$ with $|V(H)| = |V(G)|$ we have that $H \in \Omega'(\mathcal{F}, G)$ if and only if H is a spanning subgraph of G^c.*

Proof. Suppose $H \in \Omega'(\mathcal{F}, G)$. Since $|V(H)| = |V(G)|$, G has a bijective $L'(2,1)$-H-labeling f. For any edge uv in H, $f^{-1}(u)f^{-1}(v)$ is not an edge in G, or equivalently, $f^{-1}(u)f^{-1}(v)$ is an edge in G^c. Thus, H is a spanning subgraph of G^c if we consider the mapping f^{-1}.

Suppose H is a spanning subgraph of G^c. Define mapping $f : V(G) \to V(H)$ by $f(v) = v$ for $v \in V(G)$. It is easy to check that f is an $L'(2,1)$-H-labeling of G. ∎

As $\lambda'_{\mathcal{P}}(G) = \lambda'(G)$ and $\lambda'_{\mathcal{C}}(G) = \sigma'(G)$, Proposition 12 then gives the following two consequences.

Theorem 13. ([3]). *For any graph G, we have that $\lambda'(G) = |V(G)| - 1$ if and only if G^c has a Hamiltonian path.*

Theorem 14. ([16]). *For any graph G, we have that $\sigma'(G) = |V(G)|$ if and only if G^c has a hamiltonian cycle.*

Suppose f is an $L(2,1)$-H-labeling of G. An H-*hole* of f is a vertex v in H with $f^{-1}(v) = \emptyset$.

Proposition 15. *If $\mathcal{F}$ is a family of connected graphs of maximum degree 2 and G is a graph, then $H \in \Omega(\mathcal{F}, G)$ with $|V(H)| \geq |V(G)|$ if and only if $H \in \Omega'(\mathcal{F}, G)$.*

Proof. We only need to prove the only if part. Suppose f is an $L(2,1)$-H-labeling with fewest H-holes. Let $M = \{u \in V(H) : |f^{-1}(u)| \geq 2\}$. In order to get $H \in \Omega'(\mathcal{F}, G)$, we only have to prove that $M = \emptyset$. Suppose to the contrary that M is not empty. Since $|V(H)| \geq |V(G)|$, we know that there also exist H-holes.

Claim 1. Any H-hole v has exactly two neighbors v' and v'' which are in M.

Proof of Claim 1. Suppose to the contrary that either v has exactly one neighbor v' or else v has two neighbors v' and v'' with $v'' \notin M$. Choose a vertex $u \in M$ nearest to v, and a shortest v-u path $v_0, v_1, \ldots, v_m$, where $v_0 = v$, $v_1 = v'$ and $v_m = u$. Notice that for the case when v has two neighbors we may assume $v_1 = v'$ for otherwise just interchanging the role of v' and v''. Also, it is possible that $m = 1$ and $v' = u$.

First, any vertex $x \in f^{-1}(u)$ is adjacent/equal to a vertex in $f^{-1}(v')$ (or $f^{-1}(v'')$ if v'' exists), for otherwise re-labeling x to v give a new $L(2,1)$-H-labeling with fewer H-holes. Also, no two vertices in $f^{-1}(u)$ are adjacent/equal to a same vertex in $f^{-1}\{v', v''\}$. So, in fact v'' exists and $f^{-1}(u) = \{u_1, u_2\}$ such that u_1 is adjacent/equal to a vertex in $f^{-1}(v')$ but no vertex in $f^{-1}(v'')$ and u_2 is adjacent/equal to a vertex in $f^{-1}(v'')$ but no vertex in $f^{-1}(v')$.

Now define a function $f' : V(G) \to V(H)$ by

$$f'(w) = \begin{cases} v_0, & \text{if } w = u_1; \\ v_{m-i}, & \text{if } f(w) = v_i \text{ with } 1 \le i \le m-1; \\ f(w), & \text{otherwise.} \end{cases}$$

We now check that f' is an $L(2,1)$-$\mathcal{F}$-labeling. Suppose f' is not an $L(2,1)$-$\mathcal{F}$-labeling. Then there are vertices x and y in G such that either $d_G(x,y) = 1$ but $d_H(f'(x), f'(y)) \le 1$ or $d_G(x,y) = 2$ but $d_H(f'(x), f'(y)) = 0$. By the definition of f', $d_H(f'(x), f'(y)) = 0$ implies $d_H(f(x), f(y)) = 0$ which further implies that $d_G(x,y) \ge 3$. Therefore, it is only possible that $d_G(x,y) = d_H(f'(x), f'(y)) = 1$ and so $d_H(f(x), f(y)) \ge 2$. By the definition of f' again, one of $f'(x)$ and $f'(y)$, say $f'(x)$, is in $\{v_0, v_1, v_2, \ldots, v_{m-1}\}$ while $f'(y)$ is not. So, either $f'(x) = v_0 = v$ with $f'(y) = v''$ or $f'(x) = v_{m-1}$ with $f'(y) = v_m = u$. For the former case, $u_1 = x$ is adjacent to vertex $y \in f^{-1}(v'')$, a contradiction. For the latter case, $u_2 = y$ is adjacent to to vertex $x \in f^{-1}(v')$, a contradiction. ∎

Since $|V(H)| \ge |V(G)|$, every H-hole has exactly two neighbors in M (by Claim 1), and every vertex in M has at most two neighbors that are H-holes (since the maximum degree of H is 2), we have that H has exactly $|M|$ holes and $|f^{-1}(u)| = 2$ for each $u \in M$. Consequently, H is a cycle $v_1, v_2, \ldots, v_{2n}, v_1$ such that v_i is an H-hole for each odd i and $f^{-1}(v_j) = \{x_{j-1}, x_j\}$ for each even j, where indices are taken modulo $2n$. For $j = 2, 4, \ldots, 2n-2$, since neither x_{j-1} nor x_j is adjacent to both x_{j+1} and x_{j+2} in G, without loss of generality we may assume that x_{j-1} is not adjacent to x_{j+1} and x_j is not adjacent to x_{j+2}. Define function $f'' : V(G) \to V(H)$ by

$$f''(x_i) = \begin{cases} v_{\lceil \frac{i}{2} \rceil}, & \text{if } i \text{ is odd;} \\ v_{2n-\frac{i}{2}+1}, & \text{if } i \text{ is even.} \end{cases}$$

It is straightforward to check that f'' is an $L'(2,1)$-$\mathcal{F}$-labeling from $V(G)$ into $V(H)$, and so $H \in \Omega'(\mathcal{F}, G)$. ∎

Applying Propositions 12 and 15 to the family $\mathcal{P}$ (respectively, $\mathcal{C}$) we have the following result in [11] (respectively, [16]). Notice that part (1) of these two theorems are directly from the propositions. Part (2) of the theorems follows from part (1). For Theorem 16, this is because "the smallest number of vertex-disjoint paths needed to cover the vertices of G^c is r" if and only if "$G^c \wedge (r-1)K_1$ has a Hamiltonian path but $G^c \wedge (r-2)K_1$ does not, where $G \wedge H$ is the join of G and H. We can use similar arguments for Theorem 17.

Theorem 16. ([11]). (1) $\lambda_{\mathcal{P}}^{(G)} \leq |V(G)| - 1$ *if and only if* G^c *has a Hamiltonian path.*

(2) *For integer* $r > 1$, *we have* $\lambda_{\mathcal{P}}^{(G)} = |V(G)| + r - 2$ *if and only if the smallest number of vertex-disjoint paths needed to cover the vertices of* G^c *is* r.

Theorem 17. ([16]). (1) $\lambda_{\mathcal{P}}^{(G)} \leq |V(G)|$ *if and only if* G^c *is Hamiltonian.*

(2) *For integer* $r > 0$, *we have* $\lambda\mathcal{C}(G) = |V(G)| + r$ *if and only if* G^c *is not Hamiltonian and the smallest number of vertex-disjoint paths needed to cover the vertices of* G^c *is* r.

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Gerard J. Chang
Department of Mathematics
National Taiwan University
and
Institute for Mathematical Sciences
National Taiwan University
and
National Center for Theoretical Sciences, Taipei Office
Taipei 10617, Taiwan
E-mail: gjchang@math.ntu.edu.tw

Changhong Lu
Department of Mathematics
East China Normal University
Shanghai 200241
P. R. China
E-mail: chlu@math.ecnu.edu.cn