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WELL-POSEDNESS OF HEMIVARIATIONAL INEQUALITIES AND INCLUSION PROBLEMS

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Abstract. In the present paper, we generalize the concept of well-posedness to a hemivariational inequality, give some metric characterizations of the well-posed hemivariational inequality, and derive some conditions under which the hemivariational inequality is strongly well-posed in the generalized sense. Also, we show that the well-posedness of the hemivariational inequality is equivalent to the well-posedness of the corresponding inclusion problem.

1. INTRODUCTION

Well-posedness for a minimization problem is a classical notion which first was introduced by Tykhonov [41] in 1966 and plays a crucial role in the theory of optimization problems. A minimization problem is said to be well-posed if there exists a unique minimizer and every minimizing sequence converges to the unique minimizer. Clearly, the concept of the well-posedness is inspired by numerical methods producing optimizing sequences for optimization problems. Because of its importance in optimization problems, various concept of the well-posedness have been introduced and studied for optimization problems. For details, we refer to [11, 12, 16, 19, 29, 31] and the reference therein.

It is well known that a differentiable minimization problem is closely related to a variational inequality of differential type (see [21, 46]). This fact motivates researchers to study well-posedness of variational inequalities. By means of Ekeland's variational principle, Lucchetti and Patrone [29] introduced a notion of well-posedness for a variational inequality and proved some related results. In [11],

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Crespi, Guerraggio and Rocca gave notions of well posedness for a vector optimization problem and a vector variational inequality of the differential type, explored their basic properties and investigated their links. For other more results on the well-posedness of variational inequalities, we can refer to [5-7, 13-15, 17, 24, 29, 30, 40] and the references therein. In recent years, the concept of well-posedness has been generalized to other related problems. Such as the saddle point problems [4], equilibrium problems [1, 20, 26, 28, 30, 34], inclusion problems [14, 24] and fixed point problems [14, 24, 39]. It is interesting and important to establish their metric characterization, to find conditions under which these problems are well-posed, and to discuss their links. In [34], Margiocco, Patrone and Pusillo considered the Tikhonov well-posedness for concave games and Cournot oligopoly games. Lemaire [24] discussed the relations among the well-posedness of minimization problems, inclusion problems and fixed point problems. Recently, Lemaire et al. [27] further extended the results in [24] by considering perturbations. Very recently, Fang, Huang and Yao [14] generalized the concept of well-posedness to a generalized mixed variational inequality which includes as a special case the classical variational inequality, discussed its links with well-posedness of corresponding inclusion problem and the well-posedness of corresponding fixed point problem, and derived some conditions under which a mixed variation inequality is well-posed.

As an important and useful generalization of variational inequality, hemivariational inequality was first introduced in order to formulate variational principles connected to energy functions which are neither convex nor smooth, and investigated by Panagiotopoulos [37] using the mathematical notion of the generalized gradient of Clarke for nonconvex and nondifferentiable functions [10]. The hemivariational inequalities have been proved very efficient to describe a variety of mechanical problems such as unilateral contact problems in nonlinear elasticity, problems describing the adhesive and frictional effects, and nonconvex semipermeability problems (see, for example, [33, 35-37]). Recently, the hemivariational inequality theory with applications have been intensively studied by many authors (see, for example, [3, 8, 9, 25, 32, 33, 35-37, 39, 42-45]).

However, compared with variational inequalities, the study of well-posedness for hemivariational inequalities is very limited. In 1995, Goeleven and Mentagui [18] considered the following hemivariational inequality in Banach space V: Find $u \in K \subset V$ such that

$$(GM) : \langle Au + Tu, v - u\rangle + \int_\Omega j^\circ(x, u(x); v(x) - u(x))d\Omega \geq \langle f, v - u\rangle,\ \forall v \in K.$$

They introduced the well-posedness for the hemivariational inequality (GM) as follows:

Definition 1.1. ([18]). The hemivariational inequality (GM) is well-posed if

(i) for any $\epsilon > 0$, $G(\epsilon) \neq \emptyset$,

(ii) $diam(G(\epsilon)) \to 0$ as $\epsilon \to 0$,

where

$$G(\epsilon) = \{u \in K : \langle Au + Tu - f, v - u\rangle + \int_\Omega j^\circ(x, u(x); v(x) - u(x))d\Omega \geq -\epsilon\|v - u\|, \forall v \in K\}.$$

By using the notion of well-posedness, they gave some sufficient conditions of well-posedness for the hemivariational inequality (GM). They also showed some relations between the well-posedness and the solution for the hemivariational inequality (GM).

Inspired by the works mentioned above, in this paper, we generalize the concept of well-posedness for variational inequalities to a class of hemivariational inequality which include as special cases the mixed variational inequality and the classical variational equation. By using the methods presented in the paper due to Fang, Huang and Yao [14], we give some metric characterizations of the well-posed hemivariational inequality and derive some conditions under which the hemivariational inequality is strongly well-posed in the generalized sense. We also show that the well-posedness of the hemivariational inequality is equivalent to the well-posedness of the corresponding inclusion problem.

2. Preliminaries

Let V be a real reflexive Banach space with its dual V^*. We denote the duality between V and V^* by $\langle\cdot,\cdot\rangle$, and the norms of Banach space V and V^* by $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$, respectively. We suppose in what follows that $A : V \to V^*$ is a mapping and $f \in V^*$ is some given element. Consider the following hemivariational inequality associated with (A, f, J):

(2.1) $$\text{HVI}(A, f, J) : \text{ find } u \in V \text{ such that } \langle Au - f, v - u\rangle + J^\circ(u, v - u) \geq 0, \ \forall v \in V,$$

where $J^\circ(u, v)$ denotes the generalized directional derivative in the sense of Clarke for a locally Lipschitz functional $J : V \to \mathbb{R}$ at u in the direction v (see [10]) given by

$$J^\circ(u, v) = \limsup_{w \to u \ \lambda \downarrow 0} \frac{J(w + \lambda v) - J(w)}{\lambda}.$$

It is worth mentioning that HVI(A, f, J) is different from the hemivariational inequality considered by Goeleven and Mentagui [18].

Let $\partial J(u) : V \to 2^{V^*}\backslash\{\emptyset\}$ denotes Clarke's generalized gradient of locally Lipschitz functional J (see [10]), which defined by

$$\partial J(u) = \{\omega \in V^* : J^\circ(u,v) \geq \langle \omega, v\rangle, \quad \forall v \in V\}.$$

An equivalent multivalued formulation of HVI(A, f, J) is given by the following lemma.

Lemma 2.1. ([3]). *$u \in V$ is a solution of Hemivariational inequality HVI(A, f, J) if and only if u is a solution of the following inclusion problem*

(2.2) *Find $u \in V$ such that $Au - f + \partial J(u) \ni 0$.*

For Clarke's generalized directional derivative and Clarke's generalized gradient, we have the following basic properties (see [10]).

Lemma 2.2. *Let $u, v \in V$ and J be a locally Lipschitz functional defined on V. Then*

(1) *The function $v \to J^\circ(u,v)$ is finite, positively homogeneous, subadditive and then convex on V.*

(2) *$J^\circ(u,v)$ is upper semicontinuous as a function of (u,v), as a function of v alone, is Lipschitz continuous on V.*

(3) *$J^\circ(u,-v) = (-J)^\circ(u,v)$.*

(4) *$\partial J(u)$ is a nonempty, convex, bounded, weak*-compact subset of V^*.*

(5) *For every $v \in V$, one has*

$$J^\circ(u,v) = \max\{\langle \xi, v\rangle : \xi \in \partial J(u)\}.$$

Definition 2.1. ([47]). Let V be a real Banach space with its dual V^* and T be an operator from V to its dual space V^*. T is said to be monotone if and only if

$$\langle Tu - Tv, u - v\rangle \geq 0, \quad \forall u, v \in V.$$

Definition 2.2. ([47]). A mapping $T : V \to V^*$ is said to be hemicontinuous if for any $u, v \in V$, the function $t \mapsto \langle T(u + t(v - u)), v - u\rangle$ from $[0, 1]$ into $(-\infty, +\infty)$ is continuous at 0_+.

Remark 2.1. Clearly, the continuity implies the hemicontinuity, but the converse is not true in general.

Definition 2.3. ([23]). Let S be a nonempty subset of V. The measure of noncompactness μ of the set S is defined by

$$\mu(S) = \inf\{\epsilon > 0 : S \subset \cup_{i=1}^n S_i,\ \text{diam}(S_i) < \epsilon, i = 1, 2, \cdots, n\},$$

where diam(S_i) means the diameter of set S_i.

Definition 2.4. ([23]). Let A, B be nonempty subset of V. The Hausdorff metric $\mathcal{H}(\cdot,\cdot)$ between A and B is defined by

$$\mathcal{H}(A,B) = \max\{e(A,B), e(B,A)\},$$

where $e(A,B) = \sup_{a\in A} d(a,B)$ with $d(a,B) = \inf_{b\in B}\|a-b\|_V$.

Let $\{A_n\}$ be a sequence of nonempty subset of V. We say that A_n converges to A in the sense of Hausdorff metric if $\mathcal{H}(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all selection $a_n \in A_n$. For more details on this topic, we refer the reader to [23, 22]. In order to obtain our results, the following lemma is crucial to us.

Lemma 2.3. ([17]). *Let $C \subset V$ be nonempty, closed and convex, $C^* \subset V^*$ be nonempty, closed, convex and bounded, $\varphi : V \to \mathbb{R}$ be proper, convex and lower semi-continuous and $y \in C$ be arbitrary. Assume that, for each $x \in C$, there exists $x^*(x) \in C^*$ such that*

$$\langle x^*(x), x-y\rangle \geq \varphi(y) - \varphi(x).$$

Then, there exists $y^* \in C^*$ such that

$$\langle y^*, x-y\rangle \geq \varphi(y) - \varphi(x), \ \ \forall x \in C.$$

3. Main Results

3.1. Well-posedness of HVI with metric characterizations

In this subsection we introduce some concepts of well-posedness for the hemivariational inequality HVI(A, f, J), establish its metric characterizations and give some conditions under which the hemivariational inequality is strongly well-posed in the generalized sense.

Definition 3.1. A sequence $\{u_n\} \subset V$ is said to be an approximating sequence for HVI(A, f, J) if there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\langle Au_n - f, v-u_n\rangle + J^\circ(u_n, v-u_n) \geq -\epsilon_n\|v-u_n\|_V, \quad \forall v \in V. \tag{3.1}$$

Definition 3.2. HVI(A, f, J) is said to be strongly (resp. weakly) well-posed if HVI(A, f, J) has a unique solution in V and every approximating sequence converges strongly (resp. weakly) to the unique solution.

Remark 3.1. Strong well-posedness implies weak well-posedness, but the converse is not true in general.

Definition 3.3. HVI(A, f, J) is said to be strongly (resp. weakly) well-posed in the generalized sense if HVI(A, f, J) has a nonempty solution set S in V and every approximating sequence has a subsequence which converges strongly (resp. weakly) to some point of solution set S.

Remark 3.2. Strong well-posedness in generalized sense implies weak well-posedness in generalized sense, but the converse is not true in general.

Remark 3.3. The concepts of strong and weak well-posedness for the hemivariational inequalities introduced in this paper are quite different from Definition 1.1 introduced by Goeleven and Mentagui [18].

For any $\epsilon > 0$, we define the following two sets:

$$\Omega(\epsilon) = \{u \in V : \langle Au - f, v - u\rangle + J^{\circ}(u, v - u) \geq -\epsilon\|v - u\|_V, \forall v \in V\}$$

and

$$\Psi(\epsilon) = \{u \in V : \langle Av - f, v - u\rangle + J^{\circ}(u, v - u) \geq -\epsilon\|v - u\|_V, \forall v \in V\}.$$

Lemma 3.1. *Suppose that $A : V \to V^*$ is a monotone and hemicontinuous mapping. Then, $\Omega(\epsilon) = \Psi(\epsilon)$ for all $\epsilon > 0$.*

Proof. By the monotonicity of mapping A, it is easy to get the inclusion $\Omega(\epsilon) \subset \Psi(\epsilon)$. Now we prove that $\Psi(\epsilon) \subset \Omega(\epsilon)$. In fact, for any $u \in \Psi(\epsilon)$, we have

$$\langle Av - f, v - u\rangle + J^{\circ}(u, v - u) \geq -\epsilon\|v - u\|_V, \forall v \in V. \tag{3.2}$$

For any $w \in V$ and $t \in [0, 1]$, letting $v = tw + (1 - t)u = u + t(w - u)$ in (3.2), we obtain, by the positive homogeneousness of $J^{\circ}(u, v)$ with respect to v, that

$$\langle A(tw + (1 - t)u) - f, w - u\rangle + J^{\circ}(u, w - u) \geq -\epsilon\|w - u\|_V. \tag{3.3}$$

Taking the limit $t \to 0^+$ in (3.3), we get from the hemicontinuity of mapping A that

$$\langle Au - f, w - u\rangle + J^{\circ}(u, w - u) \geq -\epsilon\|w - u\|_V.$$

Since $w \in V$ is arbitrary, it follows that $u \in \Omega(\epsilon)$, which completes the proof of Lemma 3.1.

Lemma 3.2. *Suppose that $A : V \to V^*$ is a monotone and hemicontinuous mapping. Then, $\Psi(\epsilon)$ is closed in V for all $\epsilon > 0$.*

Proof. Let $\{u_n\} \subset \Psi(\epsilon)$ be a sequence such that $u_n \to u$ in V. Then

$$\langle Av - f, v - u_n\rangle + J^\circ(u_n, v - u_n) \geq -\epsilon\|v - u_n\|_V, \quad \forall v \in V. \tag{3.4}$$

It follows from the upper semicontinuity of Clarke's generalized directional derivative $J^\circ(u, v)$ with respect to (u, v) that

$$\limsup_{n\to\infty} J^\circ(u_n, v - u_n) \leq J^\circ(u, v - u).$$

Taking $\limsup$ at both sides of (3.4), we have

$$\langle Av - f, v - u\rangle + J^\circ(u, v - u) \geq -\epsilon\|v - u\|_V, \quad \forall v \in V,$$

which implies that $u \in \Psi(\epsilon)$. Thus, $\Psi(\epsilon)$ is closed in V. This completes the proof of Lemma 3.2.

Theorem 3.1. *Suppose that $A : V \to V^*$ is a monotone and hemicontinuous mapping. Then, HVI(A, f, J) is strongly well-posed if and only if*

$$\Omega(\epsilon) \neq \emptyset \; \forall \epsilon > 0 \quad \textit{and} \quad diam(\Omega(\epsilon)) \to 0 \textit{ as } \epsilon \to 0. \tag{3.5}$$

Proof. "Necessity": Suppose that HVI(A, f, J) is strongly well-posed. Then HVI(A, f, J) has a unique solution which belongs to $\Omega(\epsilon)$ and so $\Omega(\epsilon) \neq \emptyset$ for all $\epsilon > 0$. If diam$(\Omega(\epsilon))$ does not converge to 0 as $\epsilon \to 0$, then there exist a constant $l > 0$, a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ and $u_n, v_n \in \Omega(\epsilon_n)$ such that

$$\|u_n - v_n\|_V > l \quad n = 1, 2, \cdots. \tag{3.6}$$

Since $u_n, v_n \in \Omega(\epsilon_n)$, both $\{u_n\}$ and $\{v_n\}$ are approximating sequence for HVI(A, f, J). It follows from strong well-posedness of HVI(A, f, J) that both $\{u_n\}$ and $\{v_n\}$ converge strongly to the unique solution of HVI(A, f, J), which is a contradiction to (3.6).

"Sufficiency": Let $\{u_n\} \subset V$ be an approximating sequence for HVI(A, f, J). Then, there exists a nonnegative sequence ϵ_n with $\epsilon_n \to 0$ such that

$$\langle Au_n - f, v - u_n\rangle + J^\circ(u_n, v - u_n) \geq -\epsilon_n\|v - u_n\|_V, \quad \forall v \in V, n = 1, 2, \cdots, \tag{3.7}$$

which implies that $u_n \in \Omega(\epsilon_n)$. It follows from (3.5) that $\{u_n\}$ is a Cauchy sequence and so u_n converges strongly to some point $u \in V$. Since the mapping A is monotone and Clarke generalized directional derivative $J^\circ(u, v)$ is upper semicontinuous with respect to (u, v), by (3.7),

$$\begin{aligned} \langle Av - f, v - u\rangle + J^\circ(u, v - u) &\geq \limsup\{\langle Av - f, v - u_n\rangle + J^\circ(u_n, v - u_n)\} \\ &\geq \limsup\{\langle Au_n, v - u_n\rangle + J^\circ(u_n, v - u_n)\} \\ &\geq \limsup -\epsilon_n\|v - u_n\|_V \\ &= 0, \quad \forall v \in V. \end{aligned} \tag{3.8}$$

For any $w \in V$ and $t \in [0,1]$, letting $v = tw + (1-t)u = u + t(w-u)$ in (3.8), we obtain, by the positive homogeneousness of $J^{\circ}(u, v)$ with respect to v, that

$$\langle A(tw + (1-t)u) - f, w - u\rangle + J^{\circ}(u, w-u) \geq 0. \tag{3.9}$$

Taking the limit $t \to 0^+$ in (3.9) and using the hemicontinuity of mapping A, we obtain

$$\langle Au - f, w - u\rangle + J^{\circ}(u, w-u) \geq 0.$$

Since $w \in V$ is arbitrary, it follows that u solves HVI(A, f, J).

To complete proof of Theorem 3.1, we need only to prove HVI(A, f, J) has a unique solution. Assume by contradiction that HVI(A, f, J) has two distinct solution u_1 and u_2. Then it's easy to see that $u_1, u_2 \in \Omega(\epsilon)$ for all $\epsilon > 0$ and

$$0 < \|u_1 - u_2\|_V \leq diam(\Omega(\epsilon)) \to 0,$$

which is a contradiction. Therefore, HVI(A, f, J) has a unique solution. This completes the proof of Theorem 3.1.

Theorem 3.2. *Suppose that $A : V \to V^*$ is a monotone and hemicontinuous mapping. Then, HVI(A, f, J) is strongly well-posed in the generalized sense if and only if*

$$\Omega(\epsilon) \neq \emptyset \ \forall \epsilon > 0 \quad \textit{and} \quad \mu(\Omega(\epsilon)) \to 0 \ \textit{as} \ \epsilon \to 0. \tag{3.10}$$

Proof. "Necessity": Suppose that HVI(A, f, J) is strongly well-posed in the generalized sense. Then the solution set S of HVI(A, f, J) is nonempty and $S \subset \Omega(\epsilon)$ for any $\epsilon > 0$. Furthermore, the solution set S of HVI(A, f, J) also is compact. In fact, for any sequence $\{u_n\} \subset S$, it follows from $S \subset \Omega(\epsilon)$ for any $\epsilon > 0$ that $\{u_n\} \subset S$ is an approximating sequence for HVI(A, f, J). Since HVI(A, f, J) is strongly well-posed in the generalized sense, $\{u_n\}$ has a subsequence which converges to some point of solution set S. Thus, the solution set S of HVI(A, f, J) is compact. Now we show that $\mu(\Omega(\epsilon)) \to 0$ as $\epsilon \to 0$. From $S \subset \Omega(\epsilon)$ for any $\epsilon > 0$, we get

$$\mathcal{H}(\Omega(\epsilon), S) = \max\{e(\Omega(\epsilon), S), e(S, \Omega(\epsilon))\} = e(\Omega(\epsilon), S). \tag{3.11}$$

Taking into account the compactness of solution set S, we obtain by (3.11) that

$$\mu(\Omega(\epsilon)) \leq 2\mathcal{H}(\Omega(\epsilon), S) = 2e(\Omega(\epsilon), S).$$

So, to prove $\mu(\Omega(\epsilon)) \to 0$ as $\epsilon \to 0$, it sufficient to show $e(\Omega(\epsilon), S) \to 0$ as $\epsilon \to 0$. Assume by contradiction that $e(\Omega(\epsilon), S) \nrightarrow 0$ as $\epsilon \to 0$. Then there exist a constant $l > 0$, a sequence $\{\epsilon_n\} \subset R_+$ with $\epsilon_n \to 0$ and $u_n \in \Omega(\epsilon_n)$ such that

$$u_n \notin S + B(0, l), \tag{3.12}$$

where $B(0, l)$ is the closed ball centered at 0 with radius l. Since $u_n \in \Omega(\epsilon_n)$, $\{u_n\}$ is an approximating sequence for HVI(A, f, J). So, there exists a subsequence u_{n_k} which converges to some point $u \in S$ due to the strong well-posedness in the generalized sense of HVI(A, f, J), This is a contradiction to (3.12). Then $\mu(\Omega(\epsilon)) \to 0$ as $\epsilon \to 0$.

"Sufficiency": Assume condition (3.10) holds. By Lemma 3.1 and Lemma 3.2, we get $\Omega(\epsilon)$ is nonempty and closed for all $\epsilon > 0$. Observe that

$$S = \cap_{\epsilon>0}\Omega(\epsilon). \tag{3.13}$$

Since $\mu(\Omega(\epsilon)) \to 0$ as $\epsilon \to 0$, by applying the Theorem on page 412 of [23], one easily concludes that S is nonempty and compact with

$$e(\Omega(\epsilon), S) = \mathcal{H}(\Omega(\epsilon), S) \to 0 \text{ as } \epsilon \to 0. \tag{3.14}$$

Letting $\{u_n\} \subset V$ be an approximating sequence for HVI(A, f, J), there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ such that

$$\langle Au_n - f, v - u_n\rangle + J^\circ(u_n, v - u_n) \geq -\epsilon_n\|v - u_n\|_V, \quad \forall v \in V, n = 1, 2, \cdots$$

and so $u_n \in \Omega(\epsilon_n)$ by the definition of $\Omega(\epsilon_n)$. It follows from (3.14) that

$$d(u_n, S) \leq e(\Omega(\epsilon_n), S) \to 0.$$

Since the solution set S is compact, there exists $\overline{u}_n \in S$ such that

$$\|u_n - \overline{u}_n\|_V = d(u_n, S) \to 0. \tag{3.15}$$

Again from the compactness of solution set S, $\overline{u}_n$ has a subsequence $\overline{u}_{n_k}$ converging strongly to some $\overline{u} \in S$. It follows from (3.15) that

$$\|u_{n_k} - \overline{u}\|_V \leq \|u_{n_k} - \overline{u}_{n_k}\|_V + \|\overline{u}_{n_k} - \overline{u}\|_V \to 0,$$

which implies that u_{n_k} converges strongly to $\overline{u}$. Therefore, HVI(A, f, J) is strongly well-posed in generalized sense. This completes the proof of Theorem 3.2.

The following theorem give some conditions under which the hemivariational inequality is strongly well-posed in the generalized sense in Euclidean space $\mathbb{R}^n$.

Theorem 3.3. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a hemicontinuous and monotone mapping. If there exists some $\epsilon > 0$ such that $\Omega(\epsilon)$ is nonempty and bounded. Then hemivariational inequality HVI(A, f, J) is strongly well-posed in the generalized sense.*

Proof. Suppose that $\{u_n\}$ is an approximating sequence for HVI(A, f, J). Then there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$(3.16)\quad \langle Au_n - f, v - u_n\rangle + J^\circ(u_n, v - u_n) \geq -\epsilon_n \|v - u_n\|, \quad \forall v \in \mathbb{R}^n.$$

Let $\epsilon > 0$ be such that $\Omega(\epsilon)$ is nonempty and bounded. Then there exists n_0 such that $u_n \in \Omega(\epsilon)$ for all $n > n_0$ and this implies that $\{u_n\}$ is bounded in $\mathbb{R}^n$ by the boundedness of $\Omega(\epsilon)$. Thus, there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to \overline{u}$ as $k \to \infty$. Since mapping A is monotone and Clarke generalized directional derivative $J^\circ(u, v)$ is upper semicontionuous with respect to (u, v), it follows from (3.16) that

$$(3.17)\quad \begin{aligned} & \langle Av, v - \overline{u}\rangle + J^\circ(\overline{u}, v - \overline{u}) \\ \geq\ & \limsup\{\langle Av, v - u_{n_k}\rangle + J^\circ(u_{n_k}, v - u_{n_k})\} \\ \geq\ & \limsup\{\langle Au_{n_k}, v - u_{n_k}\rangle + J^\circ(u_{n_k}, v - u_{n_k})\} \\ \geq\ & \limsup -\epsilon_{n_k}\|v - u_{n_k}\| \\ =\ & 0, \quad \forall v \in \mathbb{R}^n. \end{aligned}$$

For any $w \in \mathbb{R}^n$ and $t \in [0, 1]$, letting $v = tw + (1-t)\overline{u} = \overline{u} + t(w - \overline{u})$ in (3.17), the positive homogeneousness of $J^\circ(u, v)$ with respect to v implies that

$$(3.18)\quad \langle A(tw + (1-t)\overline{u}) - f, w - \overline{u}\rangle + J^\circ(\overline{u}, w - \overline{u}) \geq 0.$$

Taking the limit $t \to 0^+$ in (3.18), we obtain, by the hemicontinuity of mapping A, that

$$\langle A\overline{u} - f, w - \overline{u}\rangle + J^\circ(\overline{u}, w - \overline{u}) \geq 0.$$

Since $w \in \mathbb{R}^n$ is arbitrary, it follows that $\overline{u}$ solves HVI(A, f, J). Therefore, HVI(A, f, J) is strongly well-posed in the generalized sense. This completes the proof of Theorem 3.3.

3.2. Relations of Well-posedness Between HVI and IP

In this subsection, we introduce the concept of well-posedness for the inclusion problem and investigate the relations between the well-posedness of hemivariational inequality and the well-posedness of inclusion problem. In what follows we always let T be a set-valued mapping from the real reflexive Banach space V to its dual space V^*. The inclusion problem associated with mapping T is defined by

$$(3.19)\quad \text{IP(T):}\quad \text{find } x \in V \text{ such that } 0 \in T(x).$$

Definition 3.4. ([24, 27]). A sequence $\{u_n\} \subset V$ is called an approximating sequence for inclusion problem IP(T) if $d(0, T(u_n)) \to 0$, or equivalently, there exists a sequence $w_n \in T(u_n)$ such that $\|w_n\|_{V^*} \to 0$ as $n \to \infty$.

Definition 3.5. ([24, 27]). We say that IP(T) is strongly (resp. weakly) well-posed if it has a unique solution and every approximating sequence converges strongly (resp. weakly) to the unique solution of IP(T).

Definition 3.6. ([24, 27]). We say that IP(T) is strongly (resp. weakly) well-posed in the generalized sense if the solution set S of IP(T) is nonempty and every approximating sequence has a subsequence which converges strongly (resp. weakly) to some point of solution set S for IP(T).

The following two theorems establish the relations between the strong (resp. weak) well-posedness of hemivariational inequality and the strong (resp. weak) well-posedness of inclusion problem.

Theorem 3.4. *Hemivariational inequality HVI(A, f, J) is strongly (resp. weakly) well-posed if and only if inclusion problem IP$(A-f+\partial J)$ is strongly (resp. weakly) well-posed.*

Proof. "Necessity": Supposed that HVI(A, f, J) is strongly (resp. weakly) well-posed. Then HVI(A, f, J) has a unique solution u^*. By Lemma 2.1, u^* also is the unique solution of inclusion problem IP$(A - f + \partial J)$. Let $\{u_n\}$ be an approximating sequence for IP$(A - f + \partial J)$. Then there exists a sequence $\omega_n \in Au_n - f + \partial J(u_n)$ such that $\|\omega_n\|_{V^*} \to 0$ as $n \to \infty$. It follows that

$$J^\circ(u_n, v - u_n) \geq \langle -Au_n + f + \omega_n, v - u_n\rangle, \quad \forall v \in V, n = 1, 2, \cdots$$

and so

$$\begin{aligned} &\langle Au_n - f, v - u_n\rangle + J^\circ(u_n, v - u_n) \\ &\geq \langle \omega_n, v - u_n\rangle \geq -\|\omega_n\|_{V^*}\|v - u_n\|_V, \quad \forall v \in V. \end{aligned} \tag{3.20}$$

By letting $\epsilon_n = \|\omega_n\|_{V^*}$, we obtain $\{u_n\}$ is an approximating sequence for HVI(A, f, J) from $\|\omega_n\|_{V^*} \to 0$ as $n \to \infty$. Therefore, it follows from the strong (resp. weak) well-posedness of HVI(A, f, J) that u_n converges strongly (resp. weakly) to the unique solution u^*. So, the inclusion problem IP$(A - f + \partial J)$ is strongly (resp. weakly) well-posed.

"Sufficiency": Conversely, suppose that inclusion problem IP$(A - f + \partial J)$ is strongly (resp. weakly) well-posed. Then IP$(A - f + \partial J)$ has a unique solution u^*, which implies u^* is the unique solution for HVI(A, f, J) by Lemma 2.1. Let $\{u_n\}$

be an approximating sequence for HVI(A, f, J). Then there exists a sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to 0$ such that

$$\langle Au_n - f, v - u_n\rangle + J^\circ(u_n, v - u_n) \geq -\epsilon_n \|v - u_n\|_V, \quad \forall v \in V.$$

From the fact that

$$J^\circ(u_n, v - u_n) = \max\{\langle \omega, v - u_n\rangle : \omega \in \partial J(u_n)\},$$

we get that there exists a $\omega(u_n, v) \in \partial J(u_n)$ such that

$$\langle Au_n - f, v - u_n\rangle + \langle \omega(u_n, v), v - u_n\rangle \geq -\epsilon_n \|v - u_n\|_V, \quad \forall v \in V. \tag{3.21}$$

By virtue of Lemma 2.2, $\partial J(u_n)$ is a nonempty convex and bounded subset in V^* which implies that $\{Au_n - f + \omega : \omega \in \partial J(u_n)\}$ is nonempty, convex and bound in V^*. So, it follows from Lemma 2.3 with $\varphi(u) = \epsilon_n \|u - u_n\|$ and (3.21) that there exists $\omega(u_n) \in \partial J(u_n)$ such that

$$\langle Au_n - f, v - u_n\rangle + \langle \omega(u_n), v - u_n\rangle \geq -\epsilon_n \|v - u_n\|_V, \quad \forall v \in V. \tag{3.22}$$

For the sake of simplicity in writing we denote $\omega_n = \omega(u_n)$, it follows from (3.22) that

$$\langle Au_n - f + \omega_n, v\rangle \leq \epsilon_n \|v\|_V, \quad \forall v \in V,$$

which implies that

$$\|Au_n - f + \omega_n\|_{V^*} \leq \epsilon_n \to 0. \tag{3.23}$$

It follows from $Au_n - f + \omega_n \in Au_n - f + \partial J(u_n)$ and (3.23) that $\{u_n\}$ is an approximating sequence for IP$(A - f + \partial J)$. Since inclusion problem IP$(A - f + \partial J)$ is strongly (resp. weakly) well-posed, we obtain $\{u_n\}$ converges strongly (resp. weakly) to the unique solution u^*. Therefore, HVI(A, f, J) is strongly (resp. weakly) well-posed. This completes the proof of Theorem 3.4.

Theorem 3.5. *Hemivariational inequality HVI*(A, f, J) *is strongly* (*resp. weakly*) *well-posed in the generalized sense if and only if inclusion problem IP*$(A - f + \partial J)$ *is strongly* (*resp. weakly*) *well-posed in the generalized sense.*

Proof. The proof is similar to the proof of Theorem 3.4 and so we omit it here.

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