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ERGODICITY, MINIMALITY AND REDUCIBILITY OF COCYCLES ON SOME COMPACT GROUPS

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Abstract. In this paper, we use the results in [4, 5, 11] to prove several interesting local or global density results of minimal, ergodic or reducible quasi-periodic linear systems on $SU(2)$ or $SO(3,\mathbb{R})$. In particular, we give a positive answer to a open problem of H. Eliasson in [3, 4].

1. Introduction

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional torus. We use the notation G and g to represent some Lie subgroup of $GL(N,\mathbb{C})$ and its Lie algebra. Let $r = 0, 1, \cdots, \infty, \omega$. A C^r quasi-periodic linear system on $\mathbb{T}^d \times G$ is a C^r map of the form

$$\Gamma : \Theta \times \mathbb{T}^d \times G \to \mathbb{T}^d \times G, \quad (t, x, y) \mapsto (x + t\alpha, \Phi(t, x)y), \tag{1.1}$$

where Θ is $\mathbb{Z}$ (discrete time) or $\mathbb{R}$ (continuous time) and $\Phi \in C^r(\Theta \times \mathbb{T}^d, G)$, satisfying the cocycle condition

$$\Phi(s + t, \cdot) = \Phi(s, \cdot + t\alpha)\Phi(t, \cdot), \quad \forall s, t \in \Theta.$$

We say that the C^r quasi-periodic linear system (1.1) is C^s ($0 \le s \le r$) conjugated to another C^r quasi-periodic linear system

$$\widetilde{\Gamma} : \Theta \times \mathbb{T}^d \times G \to \mathbb{T}^d \times G, \quad (t, x, y) \mapsto (x + t\alpha, \widetilde{\Phi}(t, x)y) \tag{1.2}$$

if there exists $B \in C^s(\mathbb{T}^d, G)$, such that

$$\widetilde{\Phi}(t, \cdot) = B(\cdot + t\alpha)\Phi(t, \cdot)B(\cdot)^{-1}, \quad t \in \Theta.$$

When $\widetilde{\Phi}$ depend only on time, we say that $\widetilde{\Gamma}$ is a constant system and Γ is C^s-reducible (or reducible simply).

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Remark 1.1. For the continuous-time case, it is obvious that $\Phi(0,\cdot)\equiv I$, where I represent the identity element.

A C^r quasi-periodic linear system can be decided by a generator. Denote by $SW^r(\mathbb{T}^d,G)$ the set of all $(\alpha,A)\in\mathbb{R}^d\times C^r(\mathbb{T}^d,G)$, which represent a C^r quasi-periodic cocycle on $\mathbb{T}^d\times G$, i.e., a diffeomorphism of $\mathbb{T}^d\times G$ of the form

$$(\alpha,A):\mathbb{T}^d\times G\to\mathbb{T}^d\times G,\quad (x,y)\mapsto(x+\alpha,A(x)y). \tag{1.3}$$

The iterations of (α,A) defines a discrete-time C^r quasi-periodic linear system. Denote by $sw^r(\mathbb{T}^d,g)$ the set of all $\{\alpha,a\}\in\mathbb{R}^d\times C^r(\mathbb{T}^d,g)$, which represent a ODE on $\mathbb{T}^d\times G$ of the form

$$\dot{x}=\alpha,\quad \dot{y}=a(x)y, \tag{1.4}$$

the flow of which defines a continuous-time C^r quasi-periodic linear system. In $SW^r(\mathbb{T}^d,G)$ ($sw^r(\mathbb{T}^d,g)$), (α,A) ($\{\alpha,A\}$) is said to be C^s ($0\le s\le r$) conjugated to another cocycle $(\alpha,\widetilde{A})$ ($\{\alpha,\widetilde{A}\}$), if the corresponding quasi-periodic linear system of (α,A) ($\{\alpha,A\}$) can be conjugated to the one of $(\alpha,\widetilde{A})$ ($\{\alpha,\widetilde{A}\}$). For cocycles, it is equivalent to say that for some $B\in C^s(\mathbb{T}^d,G)$, there is

$$B(\cdot+\alpha)A(\cdot)B(\cdot)^{-1}=\widetilde{A}(\cdot).$$

And for ODE, as $s\ge 1$, it is equivalent to say that there is $B\in C^s(\mathbb{T}^d,G)$ satisfying

$$\widetilde{A}=(\partial_\alpha B)B^{-1}-BAB^{-1}.$$

Naturally, when $\widetilde{A}$ is a constant, we say that $(\alpha,\widetilde{A})$ ($\{\alpha,\widetilde{A}\}$) is constant and (α,A) ($\{\alpha,A\}$) is C^s-reducible (or reducible simply).

It is a problem when a system in $SW^r(\mathbb{T}^d,G)$ ($sw^r(\mathbb{T}^d,g)$) is C^s ($0\le s\le r$) reducible or non-reducible (i.e., not C^0 reducible). However, in the study of dynamical system, it is usually difficult and not necessary to say something for a given system. In fact, it is more probable and important to ask: how many systems in $SW^r(\mathbb{T}^d,G)$ ($sw^r(\mathbb{T}^d,g)$) are C^s reducible or non-reducible? Usually, the word 'many' can be explicated in two meaning: measurement meaning and topology meaning. For measurable meaning, one usually consider that for a one-parameter family of systems if some particular property is of full-measure (which is usually called full-measure problem, one can refer to [1, 2, 6, 9, 10, 13] for more results). For topology meaning, we usually ask in a topological space of systems or some open set of it whether or not the subset of systems satisfying some particular property is dense, generic or open dense (we call it density problem roughly). However, as we will show, there is some relations between the full-measure problem and density problem.

Some obstructions will arise in the study of the reducibility. Firstly, the C^0 reducibility of any $(\alpha, A) \in SW^0(\mathbb{T}^d, G)$ implies that the continuous map $A : \mathbb{T}^d \to G$ is homotopic to a constant map (recall that the set of all such A and its complementary set are both open in $C^0(\mathbb{T}^d, G)$). Notice that this never occurs in the continuous-time case. Secondly, when G is a non-abelian compact Lie group, the C^0 reducibility of $(\alpha, A) \in SW^0(\mathbb{T}^d, G)$ ($\{\alpha, a\} \in sw^0(\mathbb{T}^d, g)$) always implies that the dynamics of it has foliation structure and thus has infinite close invariant set and infinite ergodic measure, and then finite close invariant set and finite ergodic measure will implies the non$-C^0-$reducibility.

We equip $C^r(\mathbb{T}^d, G)$ ($C^r(\mathbb{T}^d, g)$) with the usual C^r topology. In particular, the set $C^\omega(\mathbb{T}^d, G)$ ($C^\omega(\mathbb{T}^d, g)$) is equipped with the following topology. Denote by $C^\omega_h(\mathbb{T}^d, G)$ ($C^\omega_h(\mathbb{T}^d, g)$), where $h > 0$, the set of functions $\mathbb{T}^d \to G$ ($\mathbb{T}^d \to g$) which admit bounded holomorphic extensions in $|Im\theta| < h$, equipped with the topology induced by the norm $|\cdot|_h$ defined as

$$|F|_h = \sup_{|Im\theta|<h} |F(\theta)|,$$

where $|\cdot|$ denotes the usual matrix norm. And $C^\omega(\mathbb{T}^d, G) = \cup_{h>0} C^\omega_h(\mathbb{T}^d, G)$ ($C^\omega(\mathbb{T}^d, g) = \cup_{h>0} C^\omega_h(\mathbb{T}^d, g)$) is equipped with the inductive limit topology. In other words, in $C^\omega(\mathbb{T}^d, G)$ ($C^\omega(\mathbb{T}^d, g)$), $A^{(n)}$ converges to A if and only if there is some $h > 0$, such that $A^{(n)}$ and A are all in $C^\omega_h(\mathbb{T}^d, G)$ ($C^\omega(\mathbb{T}^d, g)$), with $A^{(n)}$ converging to A in $C^\omega_h(\mathbb{T}^d, G)$ ($C^\omega(\mathbb{T}^d, g)$), i.e., $\lim_{n\to\infty} |A^{(n)} - A|_h = 0$.

We introduce some notations for convenience. Let $\alpha \in \mathbb{R}^d$, we define some subsets of $C^r(\mathbb{T}^d, G)$, which represent the corresponding subsets of cocycles:

$$\begin{aligned}
\mathrm{RE}^r_d(\alpha, G) &:= \{A \in C^r(\mathbb{T}^d, G) : (\alpha, A) \text{ is } C^r \text{ reducible}\},\\
\mathrm{NR}^r_d(\alpha, G) &:= \{A \in C^r(\mathbb{T}^d, G) : (\alpha, A) \text{ is not } C^0 \text{ reducible}\},\\
\mathrm{UE}^r_d(\alpha, G) &:= \{A \in C^r(\mathbb{T}^d, G) : (\alpha, A) \text{ is unique ergodic}\},\\
\mathrm{M}^r_d(\alpha, G) &:= \{A \in C^r(\mathbb{T}^d, G) : (\alpha, A) \text{ is minimal}\},\\
\mathrm{E}^r_d(\alpha, G) &:= \{A \in C^r(\mathbb{T}^d, G) : (\alpha, A) \text{ has 2 ergodic probability measures at most}\},\\
\mathrm{ME}^r_d(\alpha, G) &:= \mathrm{M}^r_d(\alpha, G) \cap \mathrm{E}^r_d(\alpha, G).
\end{aligned}$$

In the same way, we define the subsets

$$\mathrm{re}^r_d(\alpha, g), \mathrm{nr}^r_d(\alpha, g), \mathrm{ue}^r_d(\alpha, g), \mathrm{m}^r_d(\alpha, g), \mathrm{e}^r_d(\alpha, G), \mathrm{me}^r_d(\alpha, g)$$

of $C^r(\mathbb{T}^d, g)$ correspondingly.

In this paper, we will mainly consider the density problem of quasi-periodic linear systems on $G = SU(2), SO(3, \mathbb{R})$ ($g = su(2), so(3, \mathbb{R})$ correspondingly).

Notice that we obviously have

$$\mathrm{UE}_d^r(\alpha, G), \mathrm{M}_d^r(\alpha, G), \mathrm{E}_d^r(\alpha, G), \mathrm{ME}_d^r(\alpha, G) \subseteq \mathrm{NR}_d^r(\alpha, G)$$

and

$$\mathrm{ue}_d^r(\alpha, g), \mathrm{m}_d^r(\alpha, G), \mathrm{e}_d^r(\alpha, G), \mathrm{me}_d^r(\alpha, G) \subseteq \mathrm{nr}_d^r(\alpha, G).$$

There is some results about reducibility of linear quasi-periodic skew-product systems on $SU(2)$ or $SO(3, \mathbb{R})$. We need to give some arithmetic condition before introducing these results. Let $|x|_{\mathbb{Z}} = inf\{|x - j| : j \in \mathbb{Z}\}$, and

$$DC_{\mathbb{R}}^d(\gamma, \tau) = \{\alpha \in \mathbb{R}^d : | < k, \alpha > | \geq \frac{\gamma}{|k|^\tau}, 0 \neq k \in \mathbb{Z}^d\};$$

$$DC_{\mathbb{Z}}^d(\gamma, \tau) = \{\alpha \in \mathbb{R}^d : | < k, \alpha > |_{\mathbb{Z}} \geq \frac{\gamma}{|k|^\tau}, 0 \neq k \in \mathbb{Z}^d\}.$$

It is well known that $DC_{\mathbb{R}}^d := \cup_{\gamma,\tau>0} DC_{\mathbb{R}}^d(\gamma, \tau)$ and $DC_{\mathbb{Z}}^d := \cup_{\gamma,\tau>0} DC_{\mathbb{Z}}^d(\gamma, \tau)$ are all of full measure.

1.1. Local Density Problem

We firstly introduce some local results (results of systems closing constants) on reducibility. It is proved by R.Krikorian in [9, 10] the following proposition:

Proposition 1.1. (R.Krikorian [9, 10]). *Let $r = \infty, \omega$, $\alpha \in DC_{\mathbb{Z}}^d$ ($\alpha \in DC_{\mathbb{R}}^d$), $G = SU(2), SO(3, \mathbb{R})$ and $g = su(2), so(3, \mathbb{R})$ correspondingly. For any $C \in G$ ($C \in g$), there exists a neighborhood $\mathcal{U}$ ($\mathcal{V}$) around C, such that the set $\mathrm{RE}_d^r(\alpha, G)$ ($\mathrm{re}_d^r(\alpha, g)$) is dense in $\mathcal{U}$ ($\mathcal{V}$).*

Remark 1.2. For C^ω case, though such density result can not be found in [9, 10], it is in fact a corollary of a result in [9]. It is proved in [9] that for any $h > 0$ and the interval $(1/2, 3/2)$, there exists $\delta = \delta(C, h, \gamma, \tau) > 0$, if $F \in C^\omega(\mathbb{T}^d, g)$ and $|F|_h < \delta$, $(\alpha, Ce^{\lambda F})$ ($\{\alpha, C + \lambda F\}$) is C^ω reducible for $a.e.$ $\lambda \in (1/2, 3/2)$. We can define $\mathcal{U}$ ($\mathcal{V}$) as

$$\mathcal{U} = \{Ce^F : \exists h > 0 \text{ s.t. } F \in C_h^\omega(\mathbb{T}^d, g) \text{ and } |F|_h < \delta\}$$

$$(\mathcal{V} = \{C + F : \exists h > 0 \text{ s.t. } F \in C_h^\omega(\mathbb{T}^d, g) \text{ and } |F|_h < \delta\})$$

Then for any $Ce^F \in \mathcal{U}$ ($C + F \in \mathcal{V}$), one can find a sequence of $\lambda_n \to 1$, such that $(\alpha, Ce^{\lambda_n F}) \in \mathcal{U}$ ($\{\alpha, C + \lambda_n F\} \in \mathcal{V}$) are C^ω reducible.

For local density problem of non-reducible systems, we have a result as follows.

Theorem 1.1. *Let $G = SU(2)$ and $g = su(2)$. For $r = \infty, \omega$, $\alpha \in DC_{\mathbb{Z}}^d(\gamma, \tau)$ ($\alpha \in DC_{\mathbb{R}}^d(\gamma, \tau)$) and $C \in G$ ($C \in g$), there exists a neighborhood $\mathcal{U} = \mathcal{U}(d, \gamma, \tau, C)$ ($\mathcal{V} = \mathcal{V}(d, \gamma, \tau, C)$) around C in $C^r(\mathbb{T}^d, G)$ ($C^r(\mathbb{T}^d, g)$), such that the set $\mathrm{ME}_d^r(\alpha, G)$ ($\mathrm{me}_d^r(\alpha, g)$) is dense in $\mathcal{U}$ ($\mathcal{V}$).*

H. Eliasson has proved the following conclusion in [4] for the case of $G = SO(3,\mathbb{R})$.

Proposition 1.2. (H. Eliasson [4]). *Let $C \in so(3,\mathbb{R})$, $\alpha \in DC^d_{\mathbb{R}}(\gamma,\tau)$ and $h > 0$. There exist $\delta = \delta(C,h,\gamma,\tau) > 0$, such that the set $\mathrm{ue}^r_d(\alpha, so(3,\mathbb{R}))$ is generic in the neighborhood $\{C + F : |F|_h < \delta\}$.*

In this paper, the following more general result can been obtained.

Theorem 1.2. *Let $G = SO(3,\mathbb{R})$ and $g = so(3,\mathbb{R})$. For $r = \infty, \omega$, $\alpha \in DC^d_{\mathbb{Z}}(\gamma,\tau)$ ($\alpha \in DC^d_{\mathbb{R}}(\gamma,\tau)$) and $C \in G$ ($C \in g$), there exists a neighborhood $\mathcal{U} = \mathcal{U}(d,\gamma,\tau,C)$ ($\mathcal{V} = \mathcal{V}(d,\gamma,\tau,C)$) around C in $C^r(\mathbb{T}^d, G)$ ($C^r(\mathbb{T}^d, g)$), such that the set $\mathrm{UE}^r_d(\alpha, G)$ ($\mathrm{ue}^r_d(\alpha, g)$) is generic in $\mathcal{U}$ ($\mathcal{V}$).*

Now for $G = SO(3,\mathbb{R})$, $r = \infty, \omega$, and $\alpha \in DC^d_{\mathbb{Z}}$ ($\alpha \in DC^d_{\mathbb{R}}$), we have such a local picture of dynamics: for any constant in $C^r(\mathbb{T}^d, G)$ ($C^r(\mathbb{T}^d, g)$), there is some neighborhood around it, such that $\mathrm{RE}^r_d(\alpha, G)$ ($\mathrm{re}^r_d(\alpha, G)$) and $\mathrm{UE}^r_d(\alpha, G)$ (or $\mathrm{ue}^r_d(\alpha, G)$) are both dense in it.

1.2. Global Density Problem

When a problem is on systems which are not necessarily close to any constant system, we call it a global problem. Global problems are more interesting while there are few results.

Let us firstly consider $G = SU(2)$. In [8], a C^0-density result is given as follows.

Proposition 1.3. (R. Krikorian [8]). *For any irrational $\alpha \in \mathbb{T}^1$, $\mathrm{NR}^0_1(\alpha, SU(2))$ is dense in $C^0(\mathbb{T}^1, SU(2))$.*

Let

$$\Sigma := \cap_{N=1}^{\infty} \cup_{k \geq N} \left\{ \alpha \in [0,1) : F^k(\alpha), F^{k+1}(\alpha) \in DC^1_{\mathbb{Z}}(\gamma,\tau) \cap \left(\frac{1}{5}, \frac{1}{4}\right] \right\}, \tag{1.5}$$

in which $F : [0,1) \to [0,1), x \mapsto \{1/x\}$ ($\{\cdot\}$ represent the fractional part) be the Gauss map. One can prove Σ is a full measure set of $[0,1)$. R.Krikorain has also proved the following result.

Proposition 1.4. (R.Krikorian [11]). *For any $\alpha \in \Sigma$, $\mathrm{RE}^\infty_1(\alpha, SU(2))$ is dense in $C^\infty(\mathbb{T}^1, SU(2))$.*

The above result has been proved by a so-called Renormalization method (or more precisely, R.Krikorian's Renormalization scheme). Unfortunately, it seems that such R.Krikorian's Renormalization method is bounded to 1-dimensional (it means that the torus is 1-dimensional) discrete-time case.

In this paper, we can obtain a interesting conclusion as follows.

Theorem 1.3. *For any* $\alpha \in \Sigma \cap DC^1_{\mathbb{Z}}$*,* $\mathrm{ME}_1^\infty(\alpha, SU(2))$ *is dense in* C^∞ $(\mathbb{T}^1, SU(2))$.

It is obvious that, for a.e. α, Theorem 1.3 is much more stronger than Proposition 1.3. Moreover, Proposition 1.4 and Theorem 1.3 together give a global picture for dynamics of cocycles in $SW^\infty(\mathbb{T}^1, SU(2))$: for $a.e.\alpha$, RE_1^∞ and ME_1^∞ are both dense in $C^\infty(\mathbb{T}^1, SU(2))$.

Now we consider the case of $SO(3, \mathbb{R})$. In [5], R. Fahti and M.Herman have proved a conclusion as follows.

Proposition 1.5. (R. Fahti and M. Herman [5]). *For any irrational* $\alpha \in \mathbb{T}^d$*, the set* $\mathrm{UE}_d^0(\alpha, SO(3,\mathbb{R}))$ *(*$\mathrm{ue}_d^0(\alpha, so(3,\mathbb{R}))$*) is a* G_δ *set.*

In [3, 4], H.Eliasson has ask such a open problem : if the local picture of the dynamics of cocycles on $SO(3,\mathbb{R})$ is still true in global case? Let $C^r_{hom}(\mathbb{T}^1, SO(3,\mathbb{R}))$ denotes the set of all maps in $C^r(\mathbb{T}^1, SO(3,\mathbb{R}))$ which are homotopic to constant maps. Thus $C^r(\mathbb{T}^1, SO(3,\mathbb{R}))$ has two connected components: $C^r_{hom}(\mathbb{T}^1, SO(3,\mathbb{R}))$ and its complementary set. The following theorem give a positive answer to the open problem given of H.Eliasson (at least for discrete-time case).

Theorem 1.4. *Let* $G = SO(3,\mathbb{R})$*, we have conclusions as follows.*

(a) *For any* $\alpha \in \Sigma$*,* $\mathrm{RE}_1^\infty(\alpha, G)$ *is dense in* $C^\infty_{hom}(\mathbb{T}^1, G)$*.*

(b) *For any* $\alpha \in \Sigma \cap DC^1_{\mathbb{Z}}$*,* $\mathrm{UE}_1^\infty(\alpha, G)$ *is generic in* $C^\infty_{hom}(\mathbb{T}^1, G)$.

Outline of the paper: In the remainder of the paper, we will prove Theorem 1.2 in section 2, then we use the resuls of Theorem 1.2 to prove Theorem 1.4 in section 3, and we prove Theorem 1.1, 1.3 in section 4 in the the end.

2. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. We will consider it for different cases: continuous-time C^ω case, continuous-time C^∞ case and discrete-time case.

The proof of continuous-time C^ω case is easy.

Proof. (Continuous-time C^ω case). Let $\delta = \delta(h)$ (we keep the dependence on γ, τ, C implicit) be the same on as in Proposition 1.2, we define the open set

$$\mathcal{V} = \{C + F : \exists h > 0 \text{ s.t. } F \in C_h^\omega(\mathbb{T}^d, g) \text{ and } |F|_h < \delta(h)\}.$$

By Proposition 1.2, $ue_d^\omega(\alpha, so(3,\mathbb{R}))$ is dense in $\mathcal{V}$. By Proposition 1.5, $ue_d^\omega(\alpha, so(3,\mathbb{R}))$ is a G_δ set w.r.t. C^0 topology, so is it w.r.t. C^ω topology. Thus $ue_d^\omega(\alpha, so(3,\mathbb{R}))$ is generic in $\mathcal{V}$. ∎

Now we can use the conclusion of C^ω case to prove the C^∞ case.

Proof. (Continuous-time C^∞ case). By Proposition 1.1, there exist $\mathcal{V}$ around C, such that the set of $re_d^\infty(\alpha, so(3,\mathbb{R}))$ is a dense in $\mathcal{V}$ (w.r.t. C^∞ topology). Thus for any $A \in \mathcal{V}$, one can find a sequence of $A_0^{(n)} \in re_d^\infty(\alpha, so(3,\mathbb{R}))$, such that $A_0^{(n)} \to A$ as $n \to \infty$ w.r.t. C^∞ topology. For $A_0^{(n)} \in re_d^\infty(\alpha, so(3,\mathbb{R}))$, $A_0^{(n)}$ can be C^∞ conjugated to a sequence of constants $C^{(n)}$. By the conclusion of Theorem 1.2 for continuous-time C^ω case, for every $C^{(n)}$, one can find a sequence of $A_1^{(m,n)} \in ue_d^\omega(\alpha, so(3,\mathbb{R}))$, satisfying $A_1^{(m,n)} \to C^{(n)}$ w.r.t. C^ω topology as $m \to \infty$. It is obvious that $A_1^{(m,n)} \to C^{(n)}$ as $m \to \infty$ w.r.t. C^∞ topology for every n. By conjugacies, for every n, there exists a sequence of $A_0^{(m,n)} \in ue_d^\omega(\alpha, so(3,\mathbb{R}))$, satisfying $A_0^{(m,n)} \to A_0^{(n)}$ as $m \to \infty$ w.r.t. C^∞ topology. Thus we can select a subsequence of $\widetilde{A}^{(n)} \in ue_d^\infty(\alpha, so(3,\mathbb{R}))$ such that it converges to A as $n \to \infty$ w.r.t. C^∞ topology. The proof for continuous-time C^∞ case is thus completed. ■

In order to deduce the discrete-time case, we need some preparing.

Let $\alpha \in \mathbb{T}^d$ and $a \in C^r(\mathbb{T}^{d+1}, G)$. For any $x = (x_1, \cdots x_d) \in \mathbb{T}^d$ and $w \in \mathbb{R}$, we define $x \oplus w = (x_1, \cdots x_d, w) \in \mathbb{T}^{d+1}$. Assume that $((\alpha \oplus 1)t, \Phi(t,\cdot))$ be the flow of system $\{\alpha \oplus 1, a\} \in sw^r(\mathbb{T}^{d+1}, G)$. There is a cocycle (α, A) defined as

$$\begin{aligned}(\alpha, A): \quad & \mathbb{T}^d \times G \to \mathbb{T}^d \times G \\ & (x,y) \mapsto (x+\alpha, \Phi(1, x\oplus 0)y).\end{aligned}$$

We call that (α, A) is a Poincaré cocycle of $\{\alpha\oplus 1, a\}$ and there is close relationships between the dynamics of them. Particularly, the unique ergodicity of a Poincaré cocycle of a linear quasi-periodic skew-product ODE can be deduced from the unique ergodicity of the continuous-time system. Recall that a general dynamical system Γ^t ($t \in \mathbb{R}, \mathbb{Z}$) on a compact topological space X is unique ergodic if and only if for any $f \in C^0(X, \mathbb{R})$

$$\lim_{N\to+\infty} \frac{1}{N+1} \sum_{k=0}^{N} f\circ\Gamma^k(x) \quad (\text{as } t \in \mathbb{Z}) \quad \textit{or} \quad \lim_{T\to+\infty} \frac{1}{T} \int_0^T f\circ\Gamma^t(x)dt \quad (\text{as } t\in\mathbb{R})$$

converges uniformly to some constant (see [Wa81]).

Lemma 2.1. *Let Ω be a compact topological space and Φ^t ($t \in \mathbb{R}$) is a continuous skew-product dynamic system defined as*

$$\begin{aligned}\Phi^t: \quad & \mathbb{T}^1 \times \Omega \to \mathbb{T}^1 \times \Omega \\ & (x,y) \mapsto (x+t, \varphi(t,x,y))\end{aligned}$$

is unique ergodic, where $\varphi \in C^0(\mathbb{R} \times \mathbb{T}^1 \times \Omega, \Omega)$ *and* $\varphi(t, x, \cdot)$ *is a diffeomophism of* Ω *for any fixed* $(t, x) \in \mathbb{R} \times \mathbb{T}^1$*. Then the homeomorphism*

$$\psi(\cdot) = \varphi(1, 0, \cdot) = \varphi(1, 1, \cdot) : \Omega \to \Omega$$

is unique ergodic also.

Proof. Let π_i $(i = 1, 2)$ be the standard projection of $\mathbb{T}^1 \times \Omega$ to its i^{th} component. Choose $\xi \in C^0([0, 1])$ such that

$$\xi(0) = \xi(1) = 0, \quad \int_0^1 \xi(t)dt = 1.$$

For any $f \in C^0(\Omega)$, define

$$\widetilde{f}(x, y) = \xi(x) f \circ \pi_2 \circ \Phi^{-x}(x, y), \quad 0 \le x < 1, y \in \Omega.$$

$\widetilde{f}$ can be extended to a continuous function on $\mathbb{T} \times \Omega$ and

$$\widetilde{f} \circ \Phi^t(0, y) = \xi(t - [t]) f \circ \psi^{\circ [t]}(y), \quad t \ge 0.$$

Thus we have

$$\int_n^{n+1} \widetilde{f} \circ \Phi^t(0, y)dt = f \circ \psi^{\circ n}(y), \quad n = 0, 1, \cdots.$$

So

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f \circ \psi^{\circ n}(y) = \lim_{N\to\infty} \frac{1}{N+1} \int_0^N \widetilde{f} \circ \Phi^t(0, y)dt$$

converge uniformly to some constant.

Then we can use the above facts to complete the proof of Theorem 1.2. ■

Proof. (Discrete-time C^r case for $r = \infty, \omega$) For any $\alpha \in DC_{\mathbb{Z}}^d(\gamma, \tau)$, $r = \infty, \omega$ and $C \in SO(3, \mathbb{R})$, one can find $c \in so(3, \mathbb{R})$ satisfying $C = e^c$ and a sequence of $a_n \in \mathrm{ue}_{d+1}^r(\widetilde{\alpha}, so(3, \mathbb{R}))$ converging to c, where $\widetilde{\alpha} = \alpha \oplus 1$. Let $(\widetilde{\alpha}t, \Psi_n(t, \cdot, \cdot))$ be the flow of $\{\widetilde{\alpha}, a_n\}$ and $A_n(\cdot) = \Psi_n(1, \cdot, 0)$ It is obvious that A_n converges to C. By Lemma 2.1, $A_n \in \mathrm{UE}_d^r(\alpha, SO(3, \mathbb{R}))$.

Thus for any $\widetilde{A} \in \mathrm{RE}_d^r(\alpha, SO(3, \mathbb{R}))$, one can find $\widetilde{A}_n \in \mathrm{UE}_d^r(\alpha, SO(3, \mathbb{R}))$ converging to $\widetilde{A}$. Note that for any constant $D \in SO(3, \mathbb{R})$, by Proposition 1.1 one can find a neighborhood $\mathcal{U} = \mathcal{U}(d, \gamma, \tau, D)$ around D in $C^r(\mathbb{T}^d, SO(3, \mathbb{R}))$ such that $\mathcal{U} \cap \mathrm{RE}_d^r(\alpha, SO(3, \mathbb{R}))$ is dense in $\mathcal{U}$. Then it is obvious that $\mathcal{U} \cap \mathrm{UE}_d^r(\alpha, SO(3, \mathbb{R}))$ is dense in $\mathcal{U}$. By Proposition 1.5, $\mathcal{U} \cap \mathrm{UE}_d^r(\alpha, SO(3, \mathbb{R}))$ is then generic in $\mathcal{U}$.

Thus the proof of Theorem 1.2 is completed. ■

3. Proof of Theorem 1.4

It is well-known that there exists a smooth covering homomorphism p from $SU(2)$ to $SO(3,\mathbb{R})$, i.e., p is a Lie group homomorphism and smooth covering projection (the covering number is 2) satisfying

$$p(y) = p(-y), \quad y \in SU(2).$$

Then any $A \in C^{\infty}_{hom}(\mathbb{T}^1, SO(3,\mathbb{R}))$ can be lifted to $\widetilde{A} \in C^{\infty}(\mathbb{T}^1, SU(2))$ satisfying $A = p \circ \widetilde{A}$ and p does not change C^{∞} reducibility, i.e., (α, A) is C^{∞} reducible if and only if $(\alpha, \widetilde{A})$ is C^{∞} reducible.

Proof. (Proof of Theorem 1.4) From Proposition 1.4, Theorem 1.4. a) is obvious. Then by Theorem 1.2 and Theorem 1.4. a), for any $\alpha \in \Sigma \cap DC^1_{\mathbb{Z}}$, $\mathrm{UE}^{\infty}_1(\alpha, G)$ is dense in $C^{\infty}_{hom}(\mathbb{T}^1, G)$ (w.r.t. C^{∞} topology). By Proposition 1.5, $\mathrm{UE}^{0}_1(\alpha, G)$ is a G_δ set w.r.t. C^0 topology, which implies that $\mathrm{UE}^{\infty}_1(\alpha, G)$ is a G_δ set w.r.t. C^{∞} topology. So $\mathrm{UE}^{\infty}_1(\alpha, G)$ is generic in $C^{\infty}_{hom}(\mathbb{T}^1, G)$ (w.r.t. C^{∞} topology). ■

4. Proof of Theorem 1.3

Let λ, $\widetilde{\lambda}$ and m_d be the Haar measure of $SO(3,\mathbb{R})$, $SU(2)$ and $\mathbb{T}^d$ respectively. Thus the product measure $\mu_d := m \times \lambda$ and $\widetilde{\mu}_d := m \times \widetilde{\lambda}$ be the Haar measure of $\mathbb{T}^d \times SO(3,\mathbb{R})$ and $\mathbb{T}^d \times SU(2)$ respectively.

Recall that there exists a smooth covering homomorphism

$$p : SU(2) \to SO(3,\mathbb{R})$$

satisfying

$$p^{-1}(\{p(y)\}) = \{y, -y\}, \quad y \in SU(2).$$

Thus

$$\Pi_d := id_{\mathbb{T}^d} \times p : \mathbb{T}^d \times SU(2) \to \mathbb{T}^d \times SO(3,\mathbb{R})$$

is also a smooth covering homomorphism, where $id_{\mathbb{T}^d}$ represents the identity map of $\mathbb{T}^d$. We use the notation $\widehat{z}$ represents $(x, -y)$ for any $z = (x, y) \in \mathbb{T}^d \times SU(2)$ and $\widehat{W}$ represents the set

$$\{z \in \mathbb{T}^d \times SU(2) : \widehat{z} \in W\}$$

for any $W \subseteq \mathbb{T}^d \times SU(2)$. One can see

$$\Pi_d^{-1}(\Pi_d(W)) = W \cup \widehat{W}$$

for any $W \subseteq \mathbb{T}^d \times SU(2)$, and particularly

$$\Pi_d^{-1}(\{\Pi_d(z)\}) = \{z, \widehat{z}\}$$

for any $z \in \mathbb{T}^d \times SU(2)$.

We use the notation M_d ($\widetilde{M}_d$) to represent the space of all probability Borel measures of $\mathbb{T}^d \times SO(3,\mathbb{R})$ ($\mathbb{T}^d \times SU(2)$), and equip it with week* topology (see [14]). For any $\nu \in \widetilde{M}_d$, $(\Pi_d)_*(\nu) \in M_d$ is defined as

$$(\Pi_d)_*(\nu)(W) = \nu(\Pi_d^{-1}(W)),$$

for any Borel set $W \subseteq \mathbb{T}^d \times SO(3,\mathbb{R})$. It is obvious that

$$(\Pi_d)_*(\widetilde{\mu}_d) = \mu_d,$$

which implies that for any Borel set $W \subseteq \mathbb{T}^d \times SO(3,\mathbb{R})$, there is

$$\mu_d(\Pi_d(W)) = \widetilde{\mu}_d(\Pi_d^{-1}(\Pi_d(W))) = \widetilde{\mu}_d(W \cup \widehat{W}).$$

Let us consider the discrete-time case firstly. We use the notation $M_d(\alpha, A)$ ($\widetilde{M}_d(\alpha, \widetilde{A})$) to represent the space of all probability measures of

$$\mathbb{T}^d \times SO(3,\mathbb{R}) \quad (\mathbb{T}^d \times SU(2))$$

which are invariant under the iterations of $(\alpha, A) \in SW^0(\mathbb{T}^d, SO(3,\mathbb{R}))$ ($(\alpha, \widetilde{A}) \in SW^0(\mathbb{T}^d, SU(2))$).

We obviously have the following Lemma.

Lemma 4.1. *For any $(\alpha, \widetilde{A}) \in SW^0(\mathbb{T}^d, SU(2))$, we have:*

- (a) *For any Borel set $W \subseteq \mathbb{T}^d \times SO(3,\mathbb{R})$. it is a invariant set of $(\alpha, \widetilde{A})$ if and only if $p(W)$ is a invariant set of $(\alpha, p \circ \widetilde{A})$;*
- (b) *For any $\nu \in \widetilde{M}_d$, $\nu \in \widetilde{M}_d(\alpha, \widetilde{A}) \Leftrightarrow (\Pi_d)_*(\nu) \in M_d(\alpha, p \circ A)$.*

Now we can prove the following conclusion.

Lemma 4.2. *Let $\alpha \in \mathbb{R}^d$ be irrational and $\widetilde{A} \in C^r(\mathbb{T}^d, SU(2))$ satisfying $p \circ \widetilde{A} \in UE_d^r(\alpha, SU(2))$. Then we have $\widetilde{A} \in \mathrm{ME}_d^r(\alpha, SU(2))$.*

Proof. Let K be a nonempty invariant compact minimal set of $(\alpha, \widetilde{A})$. Then so is $\widehat{K}$ and $K \cup \widehat{K}$ is a nonempty invariant compact set. For $\Pi_d(K \cup \widehat{K})$ is a nonempty invariant compact set, we have

$$\Pi_d(K \cup \widehat{K}) = \mathbb{T}^d \times SO(3,\mathbb{R}),$$

which implies that

$$K \cup \widehat{K} = (K \cup \widehat{K}) \cup (\widehat{K \cup \widehat{K}}) = \Pi_d^{-1}(\Pi_d(K \cup \widehat{K})) = \mathbb{T}^d \times SU(2).$$

For $\mathbb{T}^d \times SU(2)$ is connect, we have

$$K \cap \widehat{K} \neq \emptyset,$$

otherwise $K \cup \widehat{K}$ would be a partition of $\mathbb{T}^d \times SU(2)$. By minimality of K one has $K = \widehat{K}$, which means

$$K = \widehat{K} = K \cup \widehat{K} = \mathbb{T}^d \times SU(2).$$

We then complete the proof of minimality.

Now we prove that there are at most two ergodic measures. We assume that $(\alpha, \widetilde{A})$ is not unique ergodic without lose of generality. Then there exists a invariant Borel set W of $(\alpha, \widetilde{A})$ satisfying $0 < \widetilde{\mu}_d(W) < 1$. We then have

$$\widetilde{\mu}_d(W \cap \widehat{W}) = 0.$$

Otherwise, $\widetilde{\mu}_d(W \cap \widehat{W}) > 0$, for $W \cap \widehat{W} \geq \widehat{W \cap \widehat{W}}$, we have

$$\widetilde{\mu}_d(W \cap \widehat{W}) = \mu_d(\Pi_d(W \cap \widehat{W})) = 1,$$

for $\Pi_d(W \cap \widehat{W})$ is a invariant set (by Lemma 4.1) and μ is ergodic, which is contradict to

$$1 > \widetilde{\mu}_d(W) \geq \widetilde{\mu}_d(W \cap \widehat{W})$$

In the same way, we also have $\widetilde{\mu}_d((W \cup \widehat{W})^c) = 0$. Let

$$W_1 = W \backslash \widehat{W}, \quad W_2 = \widehat{W_1} = \widehat{W} \backslash W, \quad and \quad W_* = (W \cup \widehat{W})^c \cup (W \cap \widehat{W}).$$

We define

$$\widetilde{\mu}_i(X) = \widetilde{\mu}(X \cap W_i)/\widetilde{\mu}(W_i), \quad i = 1, 2, \quad \text{for any Borel set } X.$$

Then $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are two ergodic measures, and there is no other ergodic measures. In fact, if ν is such a measure, one must has

$$\nu(W_1 \cup W_2) = 0 \quad and \quad \nu(W_*) = 1.$$

But it is contradict to the fact that $(\Pi_d)_*\nu = (\Pi_d)_*\widetilde{\mu} = \mu$ (by Lemma 4.1, notice that $(\alpha, p \circ \widetilde{A})$ is unique ergodic). ■

In the same way, one can prove a similar conclusion for continuous-time case.

Lemma 4.3. *Let* $\alpha \in \mathbb{R}^d$ *be irrational and* $\widetilde{A} \in C^r(\mathbb{T}^d, su(2))$ *satisfying* $p \circ \widetilde{A} \in ue_d^r(\alpha, su(2))$. *Then we have* $\widetilde{A} \in \mathrm{me}_d^r(\alpha, su(2))$.

Now the proofs of Theorem 1.1, 1.3 are obvious.

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