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GENERALIZED PROJECTION ALGORITHMS FOR MAXIMAL MONOTONE OPERATORS AND RELATIVELY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we prove strong convergence theorems of modified Halpern's iteration for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using two hybrid methods. Using these results, we obtain new convergence results for resolvents of maximal monotone operators and relatively nonexpansive mappings in Banach spaces.

1. Introduction

Let E be a real Banach space with $\|\cdot\|$ and let E^* be the dual space of E. Let A be a maximal monotone operator from E to E^*. It is well-known that many problems in nonlinear analysis and optimization can be formulated as follows: Find a point $u \in E$ satisfying

$$0 \in Au.$$

We denote by $A^{-1}0$ the set of all points $u \in C$ such that $0 \in Au$. Such a problem contains numerous problems in economics, optimization and physics. A well-known method for solving this problem is called the *proximal point algorithm*: $x_0 \in E$ and

$$x_{n+1} = J_{r_n} x_n, \;\; n = 0, 1, 2, 3, ...,$$

where $\{r_n\} \subset (0, \infty)$ and J_{r_n} are the resovents of A. Many researchers have studied this algorithm in a Hilbert space; see, for instance, [5, 6, 18, 19] and in a Banach space; see, for instance, [7, 8]. Let C be a nonempty closed convex subset of E. Recall that a self-mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$

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for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$.

Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one was introduced in 1953 by Mann [10] which is well-known as Mann's iteration process and is defined as follows: Take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \ \ n \geq 0, \tag{1.1}$$

where the sequence $\{\alpha_n\}$ is in the interval $[0, 1]$. Fourteen year later, Halpern [3] proposed the new innovation iteration process which is resemble in Mann's iteration (1.1). It is defined as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \ \ n \geq 0, \tag{1.2}$$

where $u \in C$ is an arbitrary (but fixed) element, the initial guess x_0 is taken in C and the sequence $\{\alpha_n\}$ is in the interval $[0, 1]$.

Next, we recall that for all $x \in E$ and $x^* \in E^*$, we denote the value of x^* at x by $\langle x, x^*\rangle$. Then, the normalized duality mapping J on E is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \ \forall x \in E.$$

By the Hahn-Banach theorem, Jx is nonempty. We know that if E is smooth, then the duality mapping J is single-valued. Next, we assume that E is a smooth Banach space and define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \ \forall y, x \in E.$$

A point $u \in C$ is said to be an *asymptotic* fixed point of T [16] if C contains a sequence $\{x_n\}$ which converges weakly to u and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$. Following Matsushita and Takahashi [12], a mapping $T : C \to C$ is said to be *relatively nonexpansive* if $\widehat{F}(T) = F(T) \neq \emptyset$ and $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$ and $x \in C$.

In 2004, Mastsushita and Takahashi [13] proposed the following modification of the Mann iteration method for a relatively nonexpansive mapping T in a Banach space E: Take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ by

$$\begin{cases} u_n &= J^{-1}\big(\alpha_n Jx_n + (1 - \alpha_n)JTx_n\big), \\ C_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0 \end{cases} \tag{1.3}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0,1]$. In particular, in a Hilbert space the iteration processes (1.3) was considered by Nakajo and Takahashi [14].

Recently, Qin and Su [15] has adapted Mastsushita and Takahashi's idea [13] to modify the process (1.2) for a relatively nonexpansive mapping T in a Banach space E: Take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$(1.4) \quad \begin{cases} u_n &= J^{-1}\big(\alpha_n Jx_0 + (1-\alpha_n)JTx_n\big), \\ C_n &= \{z \in C : \phi(z,u_n) \leq \alpha_n \phi(z,x_0) + (1-\alpha_n)\phi(z,x_n)\}, \\ Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0 \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0,1]$. In particular, in a Hilbert space the iteration processes (1.4) was considered by Martinez-Yanes and Xu [11].

Very recently, Inoue, Takahashi and Zembayashi [4] proved the following strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping by using the hybrid method:

Theorem 1.1. (Inoue, Takahashi and Zembayashi [4]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $T : C \to C$ be a relatively nonexpansive mapping such that $F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and*

$$\begin{cases} u_n &= J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JTJ_{r_n}x_n\big), \\ C_n &= \{z \in C : \phi(z,u_n) \leq \phi(z,x_n)\}, \\ Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0 \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [a,\infty)$ for some $a > 0$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap A^{-1}0}x_0$, where $\Pi_{F(T)\cap A^{-1}0}$ is the generalized projection of E onto $F(T) \cap A^{-1}0$.

Let us call the hybrid method in Theorem 1.1 the normal hybrid method. Inoue, Takahashi and Zembayashi also proved the following theorem by using another hybrid method called the shrinking projection method.

Theorem 1.2. (Inoue, Takahashi and Zembayashi [4]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $T : C \to C$ be a relatively nonexpansive mapping such that $F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$ and*

$$\begin{cases} u_n &= J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JTJ_{r_n}x_n\big), \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}}x_0 \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap A^{-1}0}x_0$, where $\Pi_{F(T)\cap A^{-1}0}$ is the generalized projection of E onto $F(T) \cap A^{-1}0$.

The purpose of this paper is to employ Inoue, Takahashi and Zembayashi's idea [4] to modify the process (1.4) for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping by using the normal hybrid method and the shrinking projection method. We have two strong convergence theorems in a Banach space and using these results, we obtain new convergence results for resolvents of maximal monotone operators and relatively nonexpansive mappings in a Banach space.

2. Preliminaries

Throughout this paper, all linear spaces are real. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of all positive integers and real numbers, respectively. Let E be a Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the *weak* convergence of $\{x_n\}$ to x and the *strong* convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Let E be a Banach space. Then the duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \ \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of E. Then the space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in S(E)$. A Banach space E is said to be *strictly convex* if

$\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| < 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \epsilon$. We know the following; see [20]:

(i) If E in smooth, then J is single-valued;

(ii) if E is reflexive, then J is onto;

(iii) if E is strictly convex, then J is one-to-one;

(iv) if E is strictly convex, then J is strictly monotone;

(v) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

A Banach space E is said to have *Kadec-Klee* property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E uniformly convex, then E has the Kadec-Klee property; see [20, 21] for more details. Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed convex subset of E. Throughout this paper, define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \ \forall y, x \in E. \tag{2.5}$$

Observe that, in a Hilbert space H, (2.5) reduces to $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Following Alber [1], the generalized projection Π_C from E onto C is the map that assigns to an arbitrary point $x \in E$ the minimum point $\bar{x}$ of the functional $\phi(y, x)$, that is, $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x).$$

Existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J. In a Hilbert space, Π_C is the metric projection of H onto C. We need the following lemmas for the proofs of our main results.

Lemma 2.3. (Kamimura and Takahashi [6]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.4. (Matsushita and Takahashi [13]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E and let T be a relatively nonexpansive mapping from C into itself. Then F(T) is closed and convex.*

Lemma 2.5. (Alber [1])., Kamimura and Takahashi [6]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space, $x \in E$ and let $z \in C$. Then, $z = \Pi_C x$ if and only if $\langle y - z, Jx - Jz\rangle \leq 0$ for all $y \in C$.*

Lemma 2.6. (Alber [1], Kamimura and Takahashi [6]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y),\ \forall x \in C,\ y \in E.$$

Let E be a smooth, strictly convex and reflexive Banach space, and let A be a set-valued mapping from E to E^* with graph $G(A) = \{(x, x^*) : x^* \in Ax\}$, domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$. We denote a set-valued operator A from E to E^* by $A \subset E \times E^*$. A is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \ge 0,\ \forall (x, x^*), (y, y^*) \in G(A)$. A monotone operator $A \subset E \times E^*$ is said to be *maximal monotone* if its graph is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}0$ is closed and convex; see [20] for more details. The following theorem is well-known.

Lemma 2.7. (Rockafellar [17]). *Let E be a smooth, strictly convex and reflexive Banach space and let $A \subset E \times E^*$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let $A \subset E \times E^*$ be a monotone operator satisfying

$$D(A) \subset C \subset J^{-1}(\cap_{r>0} R(J + rA)).$$

Then we can define the resolvent $J_r : C \to D(A)$ of A by

$$J_r x = \{z \in D(A) : Jx \in Jz + rAz\},\ \forall x \in C.$$

We know that $J_r x$ consists of one point. For $r > 0$, the Yosida approximation $A_r : C \to E^*$ is defined by $A_r x = \frac{Jx - JJ_r x}{r}$ for all $x \in C$.

Lemma 2.8. (Kohsaka and Takahashi [9]). *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let $A \subset E \times E^*$ be a monotone operator satisfying*

$$D(A) \subset C \subset J^{-1}(\cap_{r>0} R(J + rA)).$$

Let $r > 0$ and J_r and A_r be the resolvent and the Yosida approximation of A, respectively. Then, the following hold:

(*i*) $\phi(u, J_r x) + \phi(J_r x, x) \le \phi(u, x),\ \forall x \in C,\ y \in A^{-1}0$;

(*ii*) $(J_r x, A_r x) \in A,\ \forall x \in C$;

(*iii*) $F(J_r) = A^{-1}0$.

3. Convergence Theorem by the Normal Hybrid Method

In this section, we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the normal hybrid method.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $T : C \to C$ be a relatively nonexpansive mapping such that $F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and*

$$\begin{cases} u_n &= J^{-1}\big(\alpha_n Jx_0 + (1-\alpha_n)JTJ_{r_n}x_n\big), \\ C_n &= \{z \in C : \phi(z, u_n) \le \phi(z, x_n) + \alpha_n\big(\|x_0\|^2 + 2\langle z, Jx_n - Jx_0\rangle\big)\}, \\ Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0 \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap A^{-1}0}x_0$, where $\Pi_{F(T)\cap A^{-1}0}$ is the generalized projection of E onto $F(T) \cap A^{-1}0$.

Proof. We first show that C_n and Q_n are closed and convex for each $n \ge 0$. From the definitions of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \ge 0$. Next, we prove that C_n is convex.
Since

$$\phi(z, u_n) \le \phi(z, x_n) + \alpha_n\big(\|x_0\|^2 + 2\langle z, Jx_n - Jx_0\rangle\big)$$

is equivalent to

$$0 \le \|x_n\|^2 - \|u_n\|^2 - 2\langle z, Jx_n - Ju_n\rangle + \alpha_n\big(\|x_0\|^2 + 2\langle z, Jx_n - Jx_0\rangle\big),$$

which is affine in z, and hence C_n is convex. So, $C_n \cap Q_n$ is a closed and convex subset of E for all $n \ge 0$. Let $u \in F(T) \cap A^{-1}0$. Put $y_n = J_{r_n}x_n$ for all $n \ge 0$. Since T and J_{r_n} are relatively nonexpansive mappings, we have

$$\begin{aligned} &\phi(u, u_n) \\ &= \phi\big(u, J^{-1}\big(\alpha_n Jx_0 + (1-\alpha_n)JTy_n\big)\big) \\ &= \|u\|^2 - 2\langle u, \alpha_n Jx_0 + (1-\alpha_n)JTy_n\rangle + \|\alpha_n Jx_0 + (1-\alpha_n)JTy_n\|^2 \\ &\le \|u\|^2 - 2\alpha_n\langle u, Jx_0\rangle - 2(1-\alpha_n)\langle u, JTy_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|Ty_n\|^2 \\ &= \alpha_n\big(\|u\|^2 - 2\langle u, Jx_0\rangle + \|x_0\|^2\big) + (1-\alpha_n)\big(\|u\|^2 - 2\langle u, JTy_n\rangle + \|Ty_n\|^2\big) \\ &= \alpha_n\phi(u, x_0) + (1-\alpha_n)\phi(u, Ty_n) \\ &\le \alpha_n\phi(u, x_0) + (1-\alpha_n)\phi(u, y_n) \end{aligned} \tag{3.1}$$

$$
\begin{aligned}
&= \alpha_n\phi(u,x_0)+(1-\alpha_n)\phi(u,J_{r_n}x_n)\\
&\le \alpha_n\phi(u,x_0)+(1-\alpha_n)\phi(u,x_n)\\
&= \phi(u,x_n)+\alpha_n\big(\phi(u,x_0)-\phi(u,x_n)\big)\\
&= \phi(u,x_n)+\alpha_n\big(\|u\|^2-2\langle u,Jx_0\rangle+\|x_0\|^2-\|u\|^2+2\langle u,Jx_n\rangle-\|x_n\|^2\big)\\
&\le \phi(u,x_n)+\alpha_n\big(\|x_0\|^2+2\langle u,Jx_n-Jx_0\rangle\big).
\end{aligned}
$$

So, $u \in C_n$ for all $n \ge 0$, which implies that $F(T)\cap A^{-1}0 \subset C_n$. Next, we show by induction that $F(T)\cap A^{-1}0 \subset Q_n$ for all $n \ge 0$. For $k = 0$, we have $F(T)\cap A^{-1}0 \subset C = Q_0$. Assume that $F(T)\cap A^{-1}0 \subset Q_k$ for $k \ge 0$. Because x_{k+1} is the projection of x_0 onto $C_k \cap Q_k$, by Lemma 2.5 we have

$$\langle x_{k+1}-z, Jx_0-Jx_{k+1}\rangle \ge 0, \qquad \forall z \in C_k\cap Q_k.$$

Since $F(T)\cap A^{-1}0 \subset C_k\cap Q_k$, we have

$$\langle x_{k+1}-z, Jx_0-Jx_{k+1}\rangle \ge 0, \qquad \forall z \in F(T)\cap A^{-1}0.$$

This together with definition of Q_{n+1} implies that $F(T)\cap A^{-1}0 \subset Q_{k+1}$ and hence $F(T)\cap A^{-1}0 \subset Q_n$ for all $n \ge 0$. So, we have that $F(T)\cap A^{-1}0 \subset C_n\cap Q_n$ for all $n \ge 0$. This implies that $\{x_n\}$ is well defined. From the definition of Q_n, we have that $x_n = \Pi_{Q_n}x_0$. So, from $x_{n+1} = \Pi_{C_n\cap Q_n}x_0 \in C_n\cap Q_n \subset Q_n$, we have

$$\phi(x_n,x_0) \le \phi(x_{n+1},x_0), \qquad \forall n \ge 0.$$

Therefore, $\{\phi(x_n,x_0)\}$ is nondecreasing. It follows from Lemma 2.6 and $x_n = \Pi_{Q_n}x_0$ that

$$\phi(x_n,x_0) = \phi(\Pi_{Q_n}x_0,x_0) \le \phi(u,x_0) - \phi(u,\Pi_{Q_n}x_0) \le \phi(u,x_0)$$

for all $u \in F(T)\cap A^{-1}0 \subset Q_n$. Therefore, $\{\phi(x_n,x_0)\}$ is bounded. So, the limit of $\{\phi(x_n,x_0)\}$ exists. Moreover, by the definition of ϕ, we know that $\{x_n\}$ and $\{J_{r_n}x_n\} = \{y_n\}$ are bounded. From $x_n = \Pi_{Q_n}x_0$, we also have

$$
\begin{aligned}
\phi(x_{n+1},x_n) &= \phi(x_{n+1},\Pi_{Q_n}x_0)\\
&\le \phi(x_{n+1},x_0) - \phi(\Pi_{Q_n}x_0,x_0) = \phi(x_{n+1},x_0) - \phi(x_n,x_0)
\end{aligned}
$$

for all $n \ge 0$. This implies that $\lim_{n\to\infty}\phi(x_{n+1},x_n) = 0$. From $x_{n+1} = \Pi_{C_n\cap Q_n}x_0 \in C_n$, we have

$$\phi(x_{n+1},u_n) \le \phi(x_{n+1},x_n) + \alpha_n\big(\|x_0\|^2 + 2\langle x_{n+1}, Jx_n - Jx_0\rangle\big).$$

By $\lim_{n\to\infty}\alpha_n = 0$, we obtain that $\lim_{n\to\infty}\phi(x_{n+1},u_n) = 0$.
Since $\lim_{n\to\infty}\phi(x_{n+1},x_n) = \lim_{n\to\infty}\phi(x_{n+1},u_n) = 0$ and E is uniformly convex and smooth, we have from Lemma 2.3 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\|x_{n+1}-u_n\| = 0.$$

So, we have $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Ju_n\| = \lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.2}$$

On the other hand, we have

$$\begin{aligned} \|Jx_{n+1} - Ju_n\| &= \|Jx_{n+1} - \alpha_n Jx_0 - (1-\alpha_n)JTy_n\| \\ &= \|\alpha_n(Jx_{n+1} - Jx_0) + (1-\alpha_n)(Jx_{n+1} - JTy_n)\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JTy_n) - \alpha_n(Jx_0 - Jx_{n+1})\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JTy_n\| - \alpha_n\|Jx_0 - Jx_{n+1}\|. \end{aligned}$$

This follows that

$$\|Jx_{n+1} - JTy_n\| \leq \frac{1}{1-\alpha_n}\big(\|Jx_{n+1} - Ju_n\| + \alpha_n\|Jx_0 - Jx_{n+1}\|\big).$$

From (3.2) and $\lim_{n\to\infty} \alpha_n = 0$, we obtain that $\lim_{n\to\infty} \|Jx_{n+1} - JTy_n\| = 0$.
Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - Ty_n\| = 0.$$

From

$$\|x_n - Ty_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Ty_n\|,$$

we have

$$\lim_{n\to\infty} \|x_n - Ty_n\| = 0.$$

From (3.1), we have

$$\phi(u, y_n) \geq \frac{1}{1-\alpha_n}\big(\phi(u, u_n) - \alpha_n\phi(u, x_0)\big).$$

Using $y_n = J_{r_n}x_n$ and Lemma 2.8, we have

$$\phi(y_n, x_n) = \phi(J_{r_n}x_n, x_n) \leq \phi(u, x_n) - \phi(u, J_{r_n}x_n) = \phi(u, x_n) - \phi(u, y_n).$$

It follows that

$$\begin{aligned} &\phi(y_n, x_n) \\ &\leq \phi(u, x_n) - \phi(u, y_n) \\ &\leq \phi(u, x_n) - \frac{1}{1-\alpha_n}\big(\phi(u, u_n) - \alpha_n\phi(u, x_0)\big) \\ &= \frac{1}{1-\alpha_n}\big((1-\alpha_n)\phi(u, x_n) - \phi(u, u_n) + \alpha_n\phi(u, x_0)\big) \\ &= \frac{1}{1-\alpha_n}\big(\phi(u, x_n) - \phi(u, u_n) + \alpha_n(\phi(u, x_0) - \phi(u, x_n))\big) \\ &\leq \frac{1}{1-\alpha_n}\big(\phi(u, x_n) - \phi(u, u_n) + \alpha_n\phi(u, x_0)\big) \\ &= \frac{1}{1-\alpha_n}\big(\|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle + \alpha_n\phi(u, x_0)\big) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{1-\alpha_n}\big(|\|x_n\|^2-\|u_n\|^2|+2|\langle u, Jx_n-Ju_n\rangle|+\alpha_n\phi(u,x_0)\big)\\
&\le \frac{1}{1-\alpha_n}\big(|\|x_n\|-\|u_n\||(\|x_n\|+\|u_n\|)+2\|u\|\|Jx_n-Ju_n\|+\alpha_n\phi(u,x_0)\big)\\
&\le \frac{1}{1-\alpha_n}\big(|\|x_n-u_n\|(\|x_n\|+\|u_n\|)+2\|u\|\|Jx_n-Ju_n\|+\alpha_n\phi(u,x_0)\big).
\end{aligned}
$$

From (3.2), $\lim_{n\to\infty}\|x_n-u_n\|=0$ and $\lim_{n\to\infty}\alpha_n=0$, we have $\lim_{n\to\infty}\phi(y_n,x_n)=0$. Since E is uniformly convex and smooth, we have from Lemma 2.3 that

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.3}$$

From $\lim_{n\to\infty}\|x_n-Ty_n\|=0$, we have

$$\lim_{n\to\infty}\|y_n-Ty_n\|=0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup v$. From $\lim_{n\to\infty}\|x_n-y_n\|=0$, we have $y_{n_k}\rightharpoonup v$. Since T is relatively nonexpansive, we have that $v\in\widehat{F}(T)=F(T)$. Next, we show $v\in A^{-1}0$. Since J is uniformly norm-to-norm continuous on bounded sets, from (3.3) we have

$$\lim_{n\to\infty}\|Jx_n-Jy_n\|=0.$$

From $r_n\ge a$, we have

$$\lim_{n\to\infty}\frac{1}{r_n}\|Jx_n-Jy_n\|=0.$$

Therefore, we have

$$\lim_{n\to\infty}\|A_{r_n}x_n\|=\lim_{n\to\infty}\frac{1}{r_n}\|Jx_n-Jy_n\|=0.$$

For $(p,p^*)\in A$, from the monotonicity of A, we have $\langle p-y_n, p^*-A_{r_n}x_n\rangle\ge 0$ for all $n\ge 0$. Replacing n by n_k and letting $k\to\infty$, we get $\langle p-v,p^*\rangle\ge 0$. From the maximallity of A, we have $v\in A^{-1}0$. Let $w=\Pi_{F(T)\cap A^{-1}0}x_0$. From $x_{n+1}=\Pi_{C_n\cap Q_n}x_0$ and $w\in F(T)\cap A^{-1}0\subset C_n\cap Q_n$, we obtain that

$$\phi(x_{n+1},x_0)\le\phi(w,x_0).$$

Since the norm is weakly lower semicontinuous, we have

$$
\begin{aligned}
\phi(v,x_0) &= \|v\|^2-2\langle v,Jx_0\rangle+\|x_0\|^2\\
&\le \liminf_{k\to\infty}\big(\|x_{n_k}\|^2-2\langle x_{n_k},Jx_0\rangle+\|x_0\|^2\big)\\
&= \liminf_{k\to\infty}\phi(x_{n_k},x_0)\le\limsup_{k\to\infty}\phi(x_{n_k},x_0)\le\phi(w,x_0).
\end{aligned}
$$

From the definition of $\Pi_{F(T)\cap A^{-1}0}$, we obtain $v = w$. This means that

$$\lim_{k\to\infty} \phi(x_{n_k}, x_0) = \phi(w, x_0).$$

Therefore we have

$$\begin{aligned} 0 &= \lim_{k\to\infty} \big(\phi(x_{n_k}, x_0) - \phi(w, x_0)\big) \\ &= \lim_{k\to\infty} \big(\|x_{n_k}\|^2 - \|w\|^2 - 2\langle x_{n_k} - w, Jx_0\rangle\big) \\ &= \lim_{k\to\infty} \big(\|x_{n_k}\|^2 - \|w\|^2\big). \end{aligned}$$

Since E has the Kadec-Klee property, we obtain that $x_{n_k} \to w = \Pi_{F(T)\cap A^{-1}0}x_0$. Therefore, $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap A^{-1}0}x_0$. This completes the proof. ■

As a direct consequence of Theorem 3.1, we can obtain the following result.

Corollary 3.2. (Inoue, Takahashi and Zembayashi [4]). *Let E be a uniformly convex and uniformly smooth Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in E$ and*

$$\begin{cases} u_n = J_{r_n}x_n, \\ C_n = \{z \in E : \phi(z, u_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n\cap Q_n}x_0 \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}x_0$, where $\Pi_{A^{-1}0}$ is the generalized projection of E onto $A^{-1}0$.

Proof. Putting $T = I$, $C = E$ and $\alpha_n = 0$ in Theorem 3.1, we obtain Corollary 3.2. ■

Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Define the subdifferential of f as follows:

$$\partial f(x) = \{x^* \in E : f(y) \ge \langle y - x, x^*\rangle + f(x),\ \forall y \in E\}$$

for each $x \in E$. Then, we know that ∂f is a maximal monotone operator; see [20] for more details. From Theorem 3.1, we also have the following result.

Corollary 3.3. (Qin and Su [15]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E and let*

T be a relatively nonexpansive mapping from C into itself such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and

$$\begin{cases} u_n &= J^{-1}\big(\alpha_n Jx_0 + (1-\alpha_n)JTx_n\big), \\ C_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n) + \alpha_n\big(\|x_0\|^2 + 2\langle z, Jx_n - Jx_0\rangle\big)\}, \\ Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0 \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$. If $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection of E onto $F(T)$.

Proof. Set $A = \partial i_C$ in Theorem 3.1, where i_C is the indicator function, that is,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & otherwise. \end{cases}$$

Then, we have that A is a maximal monotone operator and $J_r = \Pi_C$ for $r > 0$. In fact, we have from Lemma 2.5 that for any $x \in E$ and $r > 0$,

$$\begin{aligned} z = J_r x &\Leftrightarrow Jz + r\partial i_C(z) \ni Jx \\ &\Leftrightarrow Jx - Jz \in r\partial i_C(z) \\ &\Leftrightarrow i_C(y) \geq \langle y - z, \frac{Jx - Jz}{r}\rangle + i_C(z),\ \forall y \in E \\ &\Leftrightarrow 0 \geq \langle y - z, Jx - Jz\rangle,\ \forall y \in C \\ &\Leftrightarrow z = \arg\min_{y \in C} \phi(y, x) \\ &\Leftrightarrow z = \Pi_C x. \end{aligned}$$

So, from Theorem 3.1, we obtain Corollary 3.3. ∎

4. Convergence Theorem by the Shrinking Projection Method

In this section, we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the shrinking projection method.

Theorem 4.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$.*

Let $T : C \to C$ be a relatively nonexpansive mapping such that $F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$ and

$$\begin{cases} u_n &= J^{-1}\big(\alpha_n Jx_0 + (1-\alpha_n)JTJ_{r_n}x_n\big), \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \alpha_n\big(\|x_0\|^2 + 2\langle z, Jx_n - Jx_0\rangle\big)\}, \\ x_{n+1} &= \Pi_{C_{n+1}}x_0 \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [a,\infty)$ for some $a > 0$. If $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap A^{-1}0}x_0$, where $\Pi_{F(T)\cap A^{-1}0}$ is the generalized projection of E onto $F(T) \cap A^{-1}0$.

Proof. We first show that C_n is closed and convex for each $n \geq 0$. From the definition of C_n, it is obvious that C_n is closed. Since

$$\phi(z, u_n) \leq \phi(z, x_n) + \alpha_n\big(\|x_0\|^2 + 2\langle z, Jx_n - Jx_0\rangle\big)$$

is equivalent to

$$0 \leq \|x_n\|^2 - \|u_n\|^2 - 2\langle z, Jx_n - Ju_n\rangle + \alpha_n\big(\|x_0\|^2 + 2\langle z, Jx_n - Jx_0\rangle\big),$$

which is affine in z, and hence C_n is convex. So, C_n is a closed and convex subset of E for all $n \geq 0$. Next, we show by induction that $F(T) \cap A^{-1}0 \subset C_n$ for all $n \geq 0$. For $k = 0$, we have $F(T) \cap A^{-1}0 \subset C = C_0$. Suppose that $F(T) \cap A^{-1}0 \subset C_k$ for $k \geq 0$. Let $u \in F(T) \cap A^{-1}0$. Put $y_n = J_{r_n}x_n$ for all $n \geq 0$. Since T and J_{r_n} are relatively nonexpansive mappings, we have

$$\begin{aligned} &\phi(u, u_n) \\ &= \phi(u, J^{-1}\big(\alpha_n Jx_0 + (1-\alpha_n)JTy_n\big)) \\ &= \|u\|^2 - 2\langle u, \alpha_n Jx_0 + (1-\alpha_n)JTy_n\rangle + \|\alpha_n Jx_0 + (1-\alpha_n)JTy_n\|^2 \\ &\leq \|u\|^2 - 2\alpha_n\langle u, Jx_0\rangle - 2(1-\alpha_n)\langle u, JTy_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|Ty_n\|^2 \\ &= \alpha_n\big(\|u\|^2 - 2\langle u, Jx_0\rangle + \|x_0\|^2\big) + (1-\alpha_n)\big(\|u\|^2 - 2\langle u, JTy_n\rangle + \|Ty_n\|^2\big) \\ &= \alpha_n\phi(u, x_0) + (1-\alpha_n)\phi(u, Ty_n) \\ &\leq \alpha_n\phi(u, x_0) + (1-\alpha_n)\phi(u, y_n) \\ &= \alpha_n\phi(u, x_0) + (1-\alpha_n)\phi(u, J_{r_n}x_n) \\ &\leq \alpha_n\phi(u, x_0) + (1-\alpha_n)\phi(u, x_n) \\ &= \phi(u, x_n) + \alpha_n\big(\phi(u, x_0) - \phi(u, x_n)\big) \\ &\leq \phi(u, x_n) + \alpha_n\big(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0\rangle\big). \end{aligned} \tag{4.1}$$

So, we have $u \in C_{k+1}$ and hence $F(T) \cap A^{-1}0 \subset C_n$ for all $n \geq 0$. This implies that $\{x_n\}$ is well defined. From $C_{n+1} \subset C_n$ and $x_n = \Pi_{C_n}x_0$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \qquad \forall n \geq 0.$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows from Lemma 2.6 and $x_n = \Pi_{C_n} x_0$ that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{C_n} x_0) \leq \phi(u, x_0)$$

for all $u \in F(T) \cap A^{-1}0 \subset Q_n$. So, the limit of $\{\phi(x_n, x_0)\}$ exists. Therefore, $\{\phi(x_n, x_0)\}$ is bounded. Moreover, by the definition of ϕ, we know that $\{x_n\}$ and $\{J_{r_n} x_n\} = \{y_n\}$ are bounded. Since $x_n = \Pi_{C_n} x_0$, we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{Q_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) = \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \end{aligned}$$

for all $n \geq 0$. This implies that $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. From $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \alpha_n \big(\|x_0\|^2 + 2\langle x_{n+1}, Jx_n - Jx_0\rangle\big).$$

By $\lim_{n\to\infty} \alpha_n = 0$, we obtain that $\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0$.
Since $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = \lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0$ and E is uniformly convex and smooth, we have from Lemma 2.3 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.$$

So, we have $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Ju_n\| = \lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{4.2}$$

On the other hand, we have

$$\begin{aligned} \|Jx_{n+1} - Ju_n\| &= \|Jx_{n+1} - \alpha_n Jx_0 - (1-\alpha_n) JTy_n\| \\ &= \|\alpha_n (Jx_{n+1} - Jx_0) + (1-\alpha_n)(Jx_{n+1} - JTy_n)\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JTy_n) - \alpha_n (Jx_0 - Jx_{n+1})\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JTy_n\| - \alpha_n \|Jx_0 - Jx_{n+1}\|. \end{aligned}$$

This follows that

$$\|Jx_{n+1} - JTy_n\| \leq \frac{1}{1-\alpha_n}\big(\|Jx_{n+1} - Ju_n\| + \alpha_n \|Jx_0 - Jx_{n+1}\|\big).$$

From (4.2) and $\lim_{n\to\infty} \alpha_n = 0$, we obtain that $\lim_{n\to\infty} \|Jx_{n+1} - JTy_n\| = 0$.
Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - Ty_n\| = 0.$$

From

$$\|x_n - Ty_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - Ty_n\|,$$

we have

$$\lim_{n\to\infty} \|x_n - Ty_n\| = 0.$$

From (4.1), we have

$$\phi(u, y_n) \ge \frac{1}{1-\alpha_n}\big(\phi(u, u_n) - \alpha_n \phi(u, x_0)\big).$$

Using $y_n = J_{r_n} x_n$ and Lemma 2.8, we have

$$\phi(y_n, x_n) = \phi(J_{r_n} x_n, x_n) \le \phi(u, x_n) - \phi(u, J_{r_n} x_n) = \phi(u, x_n) - \phi(u, y_n).$$

It follows that

$$\begin{aligned}
&\phi(y_n, x_n)\\
&\le \phi(u, x_n) - \phi(u, y_n)\\
&\le \phi(u, x_n) - \frac{1}{1-\alpha_n}\big(\phi(u, u_n) - \alpha_n \phi(u, x_0)\big)\\
&= \frac{1}{1-\alpha_n}\big((1-\alpha_n)\phi(u, x_n) - \phi(u, u_n) + \alpha_n \phi(u, x_0)\big)\\
&= \frac{1}{1-\alpha_n}\big(\phi(u, x_n) - \phi(u, u_n) + \alpha_n(\phi(u, x_0) - \phi(u, x_n))\big)\\
&\le \frac{1}{1-\alpha_n}\big(\phi(u, x_n) - \phi(u, u_n) + \alpha_n \phi(u, x_0)\big)\\
&= \frac{1}{1-\alpha_n}\big(\|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle + \alpha_n \phi(u, x_0)\big)\\
&\le \frac{1}{1-\alpha_n}\big(|\|x_n\|^2 - \|u_n\|^2| + 2|\langle u, Jx_n - Ju_n\rangle| + \alpha_n \phi(u, x_0)\big)\\
&\le \frac{1}{1-\alpha_n}\big(|\|x_n\| - \|u_n\||(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\| + \alpha_n \phi(u, x_0)\big)\\
&\le \frac{1}{1-\alpha_n}\big(|\|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\| + \alpha_n \phi(u, x_0)\big).
\end{aligned}$$

From (4.2), $\lim_{n\to\infty} \|x_n - u_n\| = 0$ and $\lim_{n\to\infty} \alpha_n = 0$, we have $\lim_{n\to\infty} \phi(y_n, x_n) = 0$. Since E is uniformly convex and smooth, we have from Lemma 2.3 that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{4.3}$$

From $\lim_{n\to\infty} \|x_n - Ty_n\| = 0$, we have

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v$. From $\lim_{n\to\infty} \|x_n - y_n\| = 0$, we have $y_{n_k} \rightharpoonup v$. Since T is relatively nonexpansive, we have that $v \in \widehat{F}(T) = F(T)$. Next, we show $v \in A^{-1}0$. Since J is uniformly norm-to-norm continuous on bounded sets, from (4.3) we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0.$$

From $r_n \geq a$, we have

$$\lim_{n\to\infty} \frac{1}{r_n} \|Jx_n - Jy_n\| = 0.$$

Therefore, we have

$$\lim_{n\to\infty} \|A_{r_n} x_n\| = \lim_{n\to\infty} \frac{1}{r_n} \|Jx_n - Jy_n\| = 0.$$

For $(p, p^*) \in A$, from the monotonicity of A, we have $\langle p - y_n, p^* - A_{r_n} x_n \rangle \geq 0$ for all $n \geq 0$. Replacing n by n_k and letting $k \to \infty$, we get $\langle p - v, p^* \rangle \geq 0$. From the maximallity of A, we have $v \in A^{-1}0$. Let $w = \Pi_{F(T)\cap A^{-1}0} x_0$. From $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ and $w \in F(T) \cap A^{-1}0 \subset C_n \cap Q_n$, we obtain that

$$\phi(x_{n+1}, x_0) \leq \phi(w, x_0).$$

Since the norm is weakly lower semicontinuous, we have

$$\begin{aligned}
\phi(v, x_0) &= \|v\|^2 - 2\langle v, Jx_0 \rangle + \|x_0\|^2 \\
&\leq \liminf_{k\to\infty} \left(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0 \rangle + \|x_0\|^2 \right) \\
&= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \leq \phi(w, x_0).
\end{aligned}$$

From the definition of $\Pi_{F(T)\cap A^{-1}0}$, we obtain $v = w$. This means that

$$\lim_{k\to\infty} \phi(x_{n_k}, x_0) = \phi(w, x_0).$$

Therefore we have

$$\begin{aligned}
0 &= \lim_{k\to\infty} \left(\phi(x_{n_k}, x_0) - \phi(w, x_0) \right) \\
&= \lim_{k\to\infty} \left(\|x_{n_k}\|^2 - \|w\|^2 - 2\langle x_{n_k} - w, Jx_0 \rangle \right) \\
&= \lim_{k\to\infty} \left(\|x_{n_k}\|^2 - \|w\|^2 \right).
\end{aligned}$$

Since E has the Kadec-Klee property, we obtain that $x_{n_k} \to w = \Pi_{F(T)\cap A^{-1}0} x_0$. Therefore, $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap A^{-1}0} x_0$. This completes the proof. ■

As direct consequences of Theorem 4.1, we can obtain the following corollaries.

Corollary 4.2. (Inoue, Takahashi and Zembayashi [4]). *Let E be a uniformly convex and uniformly smooth Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in E$ and*

$$\begin{cases} u_n &= J_{r_n}x_n, \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}}x_0 \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}x_0$, where $\Pi_{A^{-1}0}$ is the generalized projection of E onto $A^{-1}0$.

Proof. Putting $T = I$, $C = C_0 = E$ and $\alpha_n = 0$ in Theorem 4.1, we obtain Corollary 4.2. ■

Corollary 4.3. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E and let T be a relatively nonexpansive mapping from C into itself such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$ and*

$$\begin{cases} u_n &= J^{-1}\big(\alpha_n Jx_0 + (1 - \alpha_n)JTx_n\big), \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \alpha_n\big(\|x_0\|^2 + 2\langle z, Jx_n - Jx_0\rangle\big)\}, \\ x_{n+1} &= \Pi_{C_{n+1}}x_0 \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0, 1]$. If $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection of E onto $F(T)$.

Proof. Set $A = \partial i_C$ in Theorem 4.1, where i_C is the indicator function. So, from Theorem 4.1, we obtain Corollary 4.3. ■

5. Applications

In this section, using Theorem 3.1 and Theorem 4.1, we obtain the following results in a Hilbert space.

Theorem 5.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A \subset H \times H$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (I + rA)^{-1}$ for all $r > 0$. Let $T : C \to C$ be a nonexpansive mapping* *such that* $F(T)\cap A^{-1}0\neq\emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0=x\in C$ *and*

$$\begin{cases} u_n &= \alpha_n x_0+(1-\alpha_n)TJ_{r_n}x_n,\\ C_n &= \{z\in C: \|z-u_n\|^2\leq \|z-x_n\|^2+\alpha_n\big(\|x_0\|^2+2\langle z, x_n-x_0\rangle\big)\},\\ Q_n &= \{z\in C: \langle x_n-z, x_0-x_n\rangle\geq 0\},\\ x_{n+1} &= P_{C_n\cap Q_n}x_0 \end{cases}$$

for all $n\in\mathbb{N}\cup\{0\}$, *where* $\{\alpha_n\}\subset[0,1]$ *and* $\{r_n\}\subset[a,\infty)$ *for some* $a>0$. *If* $\lim_{n\to\infty}\alpha_n=0$, *then* $\{x_n\}$ *converges strongly to* $P_{F(T)\cap A^{-1}0}x_0$, *where* $P_{F(T)\cap A^{-1}0}$ *is the metric projection of* H *onto* $F(T)\cap A^{-1}0$.

Proof. In a Hilbert space setting we know that every nonexpansive mapping is relatively nonexpansive, therefore T and J_r are relatively nonexpansive and we also know that $\phi(x,y)=\|x-y\|^2$ for all $x,y\in H$. By using Theorem 3.1, we are easily able to obtain the desired conclusion by putting $J=I$. This completes the proof. ■

Theorem 5.5. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* $A\subset H\times H$ *be a maximal monotone operator satisfying* $D(A)\subset C$ *and let* $J_r=(I+rA)^{-1}$ *for all* $r>0$. *Let* $T:C\to C$ *be a nonexpansive mapping such that* $F(T)\cap A^{-1}0\neq\emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0=x\in C$ *and*

$$\begin{cases} u_n &= \alpha_n x_0+(1-\alpha_n)TJ_{r_n}x_n,\\ C_{n+1} &= \{z\in C_n: \|z-u_n\|^2\leq \|z-x_n\|^2+\alpha_n\big(\|x_0\|^2+2\langle z, x_n-x_0\rangle\big)\},\\ x_{n+1} &= P_{C_{n+1}}x_0 \end{cases}$$

for all $n\in\mathbb{N}\cup\{0\}$, *where* $\{\alpha_n\}\subset[0,1]$ *and* $\{r_n\}\subset[a,\infty)$ *for some* $a>0$. *If* $\lim_{n\to\infty}\alpha_n=0$, *then* $\{x_n\}$ *converges strongly to* $P_{F(T)\cap A^{-1}0}x_0$, *where* $P_{F(T)\cap A^{-1}0}$ *is the metric projection of* H *onto* $F(T)\cap A^{-1}0$.

Proof. In a Hilbert space, it is known that T and J_r are relatively nonexpansive. By putting $J=I$ in Theorem 4.1, we obtain the desired conclusion. ■

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