TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 3, pp. 1201-1225, June 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

BLOW-UP OF A DEGENERATE NON-LINEAR HEAT EQUATION

Chi-Cheung Poon

Abstract. We study the blowup behavior of non-negative solutions of the following problem:

$$\begin{aligned} u_t &= u^p(\Delta u + u^q) && \text{in} \quad \Omega \times (0,T), \\ u(x,t) &= 0 && \text{whenever} \quad x \in \partial\Omega, \end{aligned}$$

with $p > 0$ and $q > 1$. We will show that it is possible to have solutions blowing up at only one point, and

$$\limsup_{t \to T^-} \Big((T-t)^{1/(p+q-1)} \max_{\Omega} u(x,t) \Big) = \infty.$$

1. Introduction

Here, we study the blowup behavior of positive solutions of the following problem:

$$\text{(1.1)} \qquad \begin{aligned} u_t &= u^p(\Delta u + u^q) && \text{in} \quad \Omega \times (0,T), \\ u(x,t) &= 0 && \text{whenever} \quad x \in \partial\Omega. \end{aligned}$$

We assume that Ω is a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^n$, and

$$q > 1, \quad \text{and} \quad p > 0.$$

We say a solution u blows up at a point $a \in \Omega$ at time $t = T$ if $u(x,t)$ is continuous in $\Omega \times (0,T)$ and there is a sequence $(x_k, t_k) \in \Omega \times (0,T)$ such that $x_k \to a$ and $t_k \to T$ as $k \to \infty$, and

$$\lim_{k \to \infty} u(x_k, t_k) = \infty.$$

It is easy to see that if u blows up at $t = T$, then there is a constant $C > 0$ such that

$$\max_{x \in \Omega} u(x,t) \geq C|T-t|^{-1/(p+q-1)}.$$

Received October 1, 2009, accepted December 29, 2009.
Communicated by J. S. Guo.
2000 *Mathematics Subject Classification*: 35K55.
Key words and phrases: Quasilinear parabolic equation, Blowup.

The question is: when can we have an inequality of the form

$$\max_{x\in\Omega} u(x,t) \le C|T-t|^{-1/(p+q-1)}? \tag{1.2}$$

When $p = 0$, equation (1.1) becomes

$$u_t = \Delta u + u^q.$$

When Ω is a bounded convex domain in $\mathbb{R}^n$, Friedman and McLeod [2], proved that for any $q > 1$, there is a constant $C > 0$ so that

$$\sup_x u(x,t) \le C|T-t|^{-1/(q-1)}, \tag{1.3}$$

provided that the initial data $u(x,0)$ satisfies the differential inequality

$$\Delta u(x,0) + u(x,0)^q \ge 0. \tag{1.4}$$

They also proved that, under the assumptions in the above, there are no boundary blowup points. Also, if Ω a ball centered at $x = 0$ and $u(x,t)$ is symmetric and depends on $|x|$ and t only, and if $u_r \le 0$, where $r = |x|$, then $x = 0$ the only blowup point.

In [4, 5], among other results, Giga and Kohn proved that, when $1 < q < (n+2)/(n-2)$, or $n \le 2$, Ω is a convex domain in $\mathbb{R}^n$, for any non-negative positive initial data, then there is no boundary blowup point and (1.3) is true. When Ω is a general bounded domain in $\mathbb{R}^n$, and $q \le (n+3)/(n+1)$, using a different method, Fila and Souplet [1], showed that (1.3) holds.

When $q = 1$ and $p > 0$, the equation (1.1) becomes

$$u_t = u^p(\Delta u + u).$$

Winkler, [7, 8], proved that

$$\max_{x\in\Omega} u(x,t) \le C|T-t|^{-1/p}, \qquad \text{when} \quad 0 < p < 2,$$

and

$$\limsup_{t\to T^-} \left((T-t)^{1/p} \max_{\Omega} u(x,t) \right) = \infty, \qquad \text{when} \quad p \ge 2.$$

In this paper, we always assume that the domain Ω is convex. The existence of solutions of (1.1) can be proved via many different methods. In the book [6], chapter VII, section 2, existence of solutions is obtained using Galerkin's method. Here, we follow the approach by Friedman and McLeod, [3]. From the construction, we can easily deduce some properties of the solution. For example, if the initial data satisfies the inequality (1.4), then $u_t(x,t) \ge 0$ whenever $u(x,t)$ is defined.

Our results are:

(i) Let $u(x,t)$ be a positive solution of (1.1) in $\Omega \times (0,T)$ with $0 < p < 2$ and $q > 1$. Suppose that the initial data $u_0(x) = u(x,0)$ satisfies the condition
$$\frac{1}{2}\int_\Omega |Du_0|^2(x)\ dx \le \frac{1}{q+1}\int_\Omega u_0^{q+1}(x)\ dx.$$
Then, $u(x,t)$ blows up in finite time.

(ii) If $q > 1$ and $0 < p < 2$, we prove that for any solution of (1.1) which blows up at time T, then, there is $C > 0$ so that
$$\left(\int_\Omega u^{2-p}(x,t)\ dx\right)^{1/(2-p)} \le C|T-t|^{-1/(q+p-1)}.$$

(iii) Suppose that Ω a ball centered at $x = 0$. If the solution $u(x,t)$ is symmetric and depends on $|x|$ and t only, and if $u_r \le 0$, where $r = |x|$, then $x = 0$ the only blowup point.

(iv) For non-symmetric solutions, if $p > 0$ and $q > 1$, the solution does not blow-up in a neighborhood of $\partial\Omega$.

(v) If $p > 0$ and $q > 1$, and the solution blows up at time T, then we show that
$$\max_{x\in\Omega} u(x,t) \le C|T-t|^{-1/(q-1)}.$$
This result is probably not optimal.

(vi) If $p \ge 2$ and $q > 1$, we then show that if u is a solution of (1.1) and is symmetric and is radial decreasing, and if u blows up at $t = T$, then
$$\limsup_{t\to T^-}\left((T-t)^{1/(p+q-1)}\max_\Omega u(x,t)\right) = \infty.$$

2. Existence of Solution

Let Ω be a $C^{2,\alpha}$, bounded, convex domain in $\mathbb{R}^n$. Let $u_0(x) \in C^{2,\alpha}(\Omega)\cap C^1(\bar\Omega)$ and $u_0(x) > 0$ for $x \in \Omega$, and satisfies the differential inequality

$$\Delta u_0 + u_0^q \ge 0 \tag{2.1}$$

in Ω. Let $p > 0$, $q > 1$. Following the method of Friedman and McLeod, [3], we let $g_\epsilon(u)$ be a smooth function defined for $u \in (0,\infty)$ so that $g_\epsilon(u) = \epsilon$ for $u \in (0,\epsilon/2)$ and $g_\epsilon(u) = u^p$ for $u \in [\epsilon,\infty)$.

For each $\epsilon > 0$, we consider the problem

$$\begin{aligned} &u_t = g_\epsilon(u)(\Delta u + u^q) && \text{in} \quad \Omega\times(0,T),\\ &u(x,t) = \epsilon && \text{whenever} \quad x\in\partial\Omega\\ &u(x,0) = u_0(x)+\epsilon && \text{in} \quad \Omega \end{aligned} \tag{2.2}$$

There is a $T_\epsilon > 0$ so that, for $t \in (0, T_\epsilon)$, there is a positive solution $u_\epsilon \in C^\infty(\Omega \times (0, T_\epsilon]$. We note that $w(x) = \epsilon$ is a sub-solution, i.e., $\Delta w + w^q > 0$ in Ω, and $w(x) = \epsilon$ for $x \in \partial\Omega$. Therefore, by the maximum principle, $u_\epsilon \geq \epsilon$ in $\Omega \times (0, T_\epsilon)$. Thus, in fact, u_ϵ satisfies the equation

$$u_t = u^p(\Delta u + u^q)$$

Moreover, by the maximum principle, if $\epsilon > \delta > 0$, then, we have $u_\epsilon(x,t) \geq u_\delta(x,t)$, whenever both $u_\epsilon(x,t)$ and $u_\delta(x,t)$ are defined. Suppose that for $\epsilon = 1$, $u_1(x,t)$ is defined for $t \in (0, T_1]$. Then, by the maximum principle, for any $0 < \epsilon < 1$, the function $u_\epsilon(x,t)$ is defined for $t \in (0, T_1]$.

Suppose that $x_0 \in \partial\Omega$. Since Ω is convex, after a translation and rotation, we may assume that $x_0 = 0$ and $\Omega \subset \{x = (x_1, x_2, ..., x_n) : x_1 > 0\}$. Let

$$0 < \gamma < \max\{1, \frac{2}{q}\} \tag{2.3}$$

be fixed and $A > 1$ be a number to be determined. We define the function

$$\phi_\epsilon(x_1) = A(x_1^\gamma + \epsilon), \quad \text{for} \quad x_1 \geq 0. \tag{2.4}$$

Let

$$C_0 = \max_{\Omega \times (0, T_1]} u_1(x,t).$$

Then, by the maximum principle, for all $\epsilon \in (0,1)$, we have

$$C_0 \geq \max_{\Omega \times (0, T_1]} u_\epsilon(x,t).$$

Let C_1 be a positive constant so that

$$u_0(x) \leq C_1 x_1 \qquad \text{for} \qquad x \in \Omega.$$

By (2.3), we may choose $0 < L < 1$ so that

$$L^{2-q\gamma} < \frac{\gamma(1-\gamma)}{2^q C_0^{q-1}}, \quad L^{2-\gamma} < \frac{\gamma(1-\gamma)}{2^q} \quad \text{and} \quad L^{\gamma-1} > C_1. \tag{2.5}$$

This implies that

$$\left(\frac{C_0}{L^\gamma}\right)^{q-1} < \frac{\gamma(1-\gamma)}{2^q L^{2-\gamma}} \quad \text{and} \quad 1 < \frac{\gamma(1-\gamma)}{2^q L^{2-\gamma}}.$$

Then, we choose $A > 1$ such that

$$\left(\frac{C_0}{L^\gamma}\right)^{q-1} < A^{q-1} < \frac{\gamma(1-\gamma)}{2^q L^{2-\gamma}}. \tag{2.6}$$

Note that, both A and L are independent of ϵ. Let $\Omega_L = \Omega \cap \{x : x_1 < L\}$. We claim that, for any $0 < \epsilon < 1$,

$$\phi_\epsilon(x) \geq u_\epsilon(x,t) \qquad \text{for} \quad x \in \Omega_L \quad t \in (0, T_1]. \tag{2.7}$$

If $x \in \Omega_L$, by (2.5) and (2.6), we have

$$\begin{aligned} \Delta\phi_\epsilon + \phi_\epsilon^q &= A\gamma(\gamma-1)x_1^{\gamma-2} + A^q\left(x_1^\gamma + \epsilon\right)^q \\ &< A\gamma(\gamma-1)x_1^{\gamma-2} + (2A)^q \\ &< Ax_1^{\gamma-2}\left(\gamma(\gamma-1) + 2^q A^{q-1} x_1^{2-\gamma}\right) \\ &\leq Ax_1^{\gamma-2}\left(\gamma(\gamma-1) + 2^q A^{q-1} L^{2-\gamma}\right) \\ &< 0. \end{aligned}$$

For $x \in \Omega_L$, since $A > 1$ and $\gamma < 1$, we have

$$\phi_\epsilon(x) - u_\epsilon(x,0) \geq Ax_1^\gamma - C_1x_1 = x_1\left(Ax_1^{\gamma-1} - C_1\right) \geq x_1\left(L^{\gamma-1} - C_1\right) \geq 0.$$

Also, for all $t \in (0, T_1]$, if $x \in \partial\Omega \cap \{x : x_1 > 0\}$, $u_\epsilon(x,t) = \epsilon \leq \phi_\epsilon$. If $x \in \Omega \cap \{x : x_1 = L\}$, by (2.6), $\phi_\epsilon(x) \geq C_0 \geq u_\epsilon(x,t)$. Hence, by the maximum principle, for $x \in \Omega \cap \{x : x_1 < L\}$, and $t \in (0, T_1)$, we have

$$\phi_\epsilon(x) \geq u_\epsilon(x,t).$$

This proves the claim (2.7).

As mentioned before, for $(x,t) \in \Omega \times (0, T_1]$, we have

$$u_\epsilon(x,t) \leq u_\delta(x,t) \quad \text{if} \quad 0 < \epsilon \leq \delta.$$

We may define

$$u(x,t) = \lim_{\epsilon \to 0} u_\epsilon(x,t).$$

By the claim (2.7), we have $u(x,t) = 0$ whenever $x \in \partial\Omega$.

Let $K \subset \Omega$ be an compact set. Suppose that $u(x,t) > 0$ for $(x,t) \in K \times [0, T_1]$. There is a constant $\kappa > 0$ so that $u(x,t) \geq \kappa$ in $K \times [0, T_1]$. Since $u_\epsilon(x,t) \geq u(x,t)$, we have $u_\epsilon(x,t) \geq \kappa$ in $K \times [0, T_1]$, for all $0 < \epsilon < 1$. By the parabolic regularity theory, the functions u_ϵ is uniformly bounded in $C^{2+\alpha, 1+\alpha/2}(K \times [0, T_1])$. Thus, by choosing a subsequence, we see that u_ϵ converges to u in $C^{2+\beta, 1+\beta/2}(K \times [0, T_1])$, with $0 < \beta < \alpha$. This implies that $u(x,t)$ is a smooth solution of the equation $u_t = u^p(\Delta u + u^q)$ in $K \times [0, T_1]$. Thus, we obtain a non-negative function $u(x,t)$, which satisfies the equation $u_t = u^p(\Delta u + u^q)$ in every open set where $u(x,t) > 0$ holds. Moreover, by repeating the process, either $u(x,t)$ is defined for all $t > 0$, or, there is $T > 0$ so that $\max_x u(x,t) \to \infty$ as $t \to T$.

Lemma 2.1. *If we further assume that the initial data* $u_0(x) = u(x, 0) > 0$ *and satisfies the differential inequality (2.1) in* Ω*, then* $u_t(x, t) \geq 0$ *whenever* $u(x, t)$ *is defined.*

Proof. If (2.1) holds in Ω, for any $0 < \epsilon < 1$, by the maximum principle, $u_{\epsilon t}(x, t) \geq 0$ whenever $u_\epsilon(x, t)$ is defined. Thus, for each $x \in \Omega$, $t \to u_\epsilon(x, t)$ is an increasing function. When letting $\epsilon \to 0$, for each $x \in \Omega$, $t \to u(x, t)$ is also an increasing function. Thus, $u_t(x, t) \geq 0$ whenever $u(x, t)$ is defined. ∎

From the construction, it is easy to see that, if $u_0 \geq 0$ in Ω, then $u(x, t) \geq 0$ for $x \in \Omega$ and $t \in (0, T_1]$. In general, even if $u(x, 0) > 0$ for $x \in \Omega$, we do not know whether $u(x, t) > 0$ for $x \in \Omega$ and $t > 0$. However, if $u_0(x) > 0$ in Ω, and if (2.1) is true, by Lemma 2.1, we always have $u(x, t) > 0$ whenever $x \in \Omega$ and $t \in (0, T_1)$. Furthermore, for any compact subset $K \subset \Omega$, u_ϵ converges to u in $C^{2+\beta, 1+\beta/2}(K \times [0, T_1])$, with $0 < \beta < \alpha$.

Let ψ_1 be the solution of the O.D.E.

$$\psi'' + \psi^q = 0, \qquad \psi'(0) = 0, \quad \psi(0) = 1.$$

For any $M > 0$, let

$$\psi_M(x) = M\psi_1\left(M^{(q-1)/2}x\right).$$

Then, ψ_M the solution of the O.D.E.

$$\psi'' + \psi^q = 0, \qquad \psi'(0) = 0, \quad \psi(0) = M.$$

Suppose that $x_0 \in \partial\Omega$. Since Ω is convex, after a translation and rotation, we may assume that $x_0 = 0$ and $\Omega \subset \{x = (x_1, x_2, ..., x_n) : x_1 > 0\}$. Let $M > 0$ be a constant to be determined. For each $\epsilon > 0$, let ψ^ϵ be a translation of ψ_M so that $\psi^\epsilon(-\epsilon) = 0$ and ψ^ϵ is increasing for $x \in (-\epsilon, M^{-(q-1)/2} - \epsilon)$. Let v_ϵ be a function defined on the region

$$\Omega_\epsilon = \{x = (x_1, x_2, ..., x_n) \in \Omega : x_1 \in (0, M^{-(q-1)/2} - \epsilon)\}.$$

The function v_ϵ is a function depending on x_1 only and $v_\epsilon(x) = \psi^\epsilon(x_1)$. Then v_ϵ satisfies the equation $\Delta v + v^q = 0$ on Ω_ϵ. Now, we choose M so that

$$M \geq \max\{u(x, t) : x \in \Omega, \quad t \in [0, T_1]\}$$

and $v_\epsilon \geq u_0$ in Ω_ϵ. By the maximum principle, we have $u(x, t) \leq v_\epsilon(x)$ for all $x \in \Omega_\epsilon$. Since it is true for all $\epsilon > 0$, we conclude that $u(x, t) \leq v_0(x)$. This implies that, there is a constant $A > 0$, probably depending on t, so that

$$0 \leq u(x, t) \leq A \ \mathrm{dist}(x, \partial\Omega). \tag{2.8}$$

When the domain is a ball,

$$\Omega = \{x \in \mathbb{R}^n : |x| < R\},$$

and u_0 depends on $r = |x|$ only, then, for any $0 < \epsilon < 1$, the solutions, u_ϵ, to the problem (2.2) are symmetric. If we further assume that $u_{0r}(x) \leq 0$ for all $x \in \Omega$, then by the reflection principle, we have $u_{\epsilon r}(x,t) \leq 0$ whenever $u_\epsilon(x,t)$ is defined. By letting $\epsilon \to 0$, we conclude that $u(x,t)$ is symmetric and $u_r(x,t) \leq 0$ whenever $u_(x,t)$ is defined.

3. The Case $0 < p < 2$

Let $u(x,t)$ be a positive solution of (1.1), i.e., $u(x,t) > 0$ for all $x \in \Omega$ and $t \in [0,T)$. Using the scheme in section 2, we can find $T_1 > 0$ and solutions u_ϵ of (2.2) so that for any $K \subset\subset \Omega$, u_ϵ converges to u in $C^{2+\beta,1+\beta/2}(K \times [0,T_1])$.

Given any $\eta > 0$, we choose

$$\Gamma = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq \eta\}.$$

Since u_ϵ converges to u in $C^{2+\beta,1+\beta/2}(\Gamma \times [0,T_1])$, if $\epsilon < \eta$ and is small enough,

$$\sup_{\Gamma\times[0,T_1]} |u_\epsilon - u| < \eta.$$

Thus, we have

$$\sup_{\Omega\times[0,T_1]} |u_\epsilon - u| \leq \sup_{\Gamma\times[0,T_1]} |u_\epsilon - u| + \sup_{(\Omega-\Gamma)\times[0,T_1]} (u_\epsilon + u) \leq \eta + A(\eta+\eta) + A\eta.$$

Hence, we conclude that u_ϵ converges to u uniformly on $\Omega \times [0,T_1]$.

From equation (2.2), for any $0 < \epsilon < 1$, we have

$$\int_\Omega \frac{u_{\epsilon t}^2}{u_\epsilon^p}\,dx = \int_\Omega u_{\epsilon t}(\Delta u_\epsilon + u_\epsilon^q)\,dx = -\frac{d}{dt}\int_\Omega \left(\frac{|Du_\epsilon|^2}{2} - \frac{u_\epsilon^{q+1}}{q+1}\right)\,dx$$

Thus, if $0 < s < T_1$,

$$\begin{aligned}
&\int_0^s \int_\Omega \frac{u_{\epsilon t}^2(x,t)}{u_\epsilon^p(x,t)}\,dx\,dt + \frac{1}{2}\int_\Omega |Du_\epsilon|^2(x,s)\,dx \\
=&\frac{1}{q+1}\int_\Omega u_\epsilon^{q+1}(x,s)\,dx + \int_\Omega \left(\frac{|Du_\epsilon|^2(x,0)}{2} - \frac{u_\epsilon^{q+1}(x,0)}{q+1}\right)\,dx \\
=&\frac{1}{q+1}\int_\Omega u_\epsilon^{q+1}(x,s)\,dx + \int_\Omega \left(\frac{|Du|^2(x,0)}{2} - \frac{(u(x,0)+\epsilon)^{q+1}}{q+1}\right)\,dx
\end{aligned}$$

As $\epsilon \to 0$, u_ϵ converges to u uniformly on $\Omega \times (0, T_1]$, and Du_ϵ, $u_{\epsilon t}$ converge to Du, u_t almost everywhere on $\Omega \times (0, T_1]$. By Fatou's Lemma, when $\epsilon \to 0$, we have

$$\begin{aligned}&\int_0^s \int_\Omega \frac{u_t^2(x,t)}{u^p(x,t)}\, dx\, dt + \frac{1}{2}\int_\Omega |Du|^2(x,s)\, dx \\ \le& \frac{1}{q+1}\int_\Omega u^{q+1}(x,s)\, dx + \int_\Omega \left(\frac{|Du|^2(x,0)}{2} - \frac{u^{q+1}(x,0)}{q+1}\right)\, dx\end{aligned} \tag{3.1}$$

Equation (3.1) implies that, for $t \in (0, T_1)$,

$$\int_\Omega \left(\frac{|Du|^2(x,t)}{2} - \frac{u^{q+1}(x,t)}{q+1}\right) dx \le \int_\Omega \left(\frac{|Du|^2(x,0)}{2} - \frac{u^{q+1}(x,0)}{q+1}\right)\, dx, \tag{3.2}$$

and

$$\int_0^t \int_\Omega \frac{u_s^2(x,s)}{u^p(x,s)}\, dx\, ds \le \int_\Omega \left(\frac{|Du|^2(x,0)}{2} - \frac{u^{q+1}(x,0)}{q+1}\right)\, dx. \tag{3.3}$$

By repeating the process, we see that (3.2) and (3.3) are true for all $t \in (0, T)$.

On the other hand, let

$$\Omega(\epsilon) = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \ge \epsilon\}.$$

When $0 < p < 2$, using integration by parts, we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2-p}\int_{\Omega(\epsilon)} u^{2-p}\, dx\right) = \int_{\Omega(\epsilon)} u^{1-p}u_t\, dx = \int_{\Omega(\epsilon)} u(\Delta u + u^q)\, dx \\ =& -\int_{\Omega(\epsilon)} (|Du|^2 - u^{q+1})\, dx + \int_{\partial\Omega(\epsilon)} u\frac{\partial u}{\partial \nu(\epsilon)}\, d\sigma(\epsilon),\end{aligned}$$

where $\nu(\epsilon)$ is the unit outward normal to $\partial\Omega(\epsilon)$ and $d\sigma(\epsilon)$ is the volume form on $\partial\Omega(\epsilon)$. For any $0 < s_1 < s_2 < T$, we have

$$\begin{aligned}&\frac{1}{2-p}\left(\int_{\Omega(\epsilon)} u^{2-p}(x,s_2)\, dx - \int_{\Omega(\epsilon)} u^{2-p}(x,s_2)\, dx\right) \\ =& -\int_{s_1}^{s_2}\int_{\Omega(\epsilon)} (|Du|^2 - u^{q+1})\, dx\, dt + \int_{s_1}^{s_2}\int_{\partial\Omega(\epsilon)} u\frac{\partial u}{\partial\nu(\epsilon)}\, d\sigma(\epsilon)\, dt,\end{aligned} \tag{3.4}$$

We claim that for any $0 < t < T$, there is $\epsilon_i \to 0$ as $i \to \infty$ such that

$$\left(\int_{\partial\Omega(\epsilon_i)} u^2\, d\sigma(\epsilon_i)\right)\left(\int_{\partial\Omega(\epsilon_i)} |Du|^2\, d\sigma(\epsilon_i)\right) \to 0. \tag{3.5}$$

In fact, if it is not true, there are $t \in (0, T)$ and a constant $c_0 > 0$ so that, for any $\epsilon > 0$,

$$\left(\int_{\partial\Omega(\epsilon)} u^2 \, d\sigma(\epsilon)\right)\left(\int_{\partial\Omega(\epsilon)} |Du|^2 \, d\sigma(\epsilon)\right) \geq c_0.$$

By (2.8), when ϵ is small enough, we have

$$\int_{\partial\Omega(\epsilon)} u^2 \, d\sigma(\epsilon) \leq C\epsilon^2.$$

Thus,

$$\int_{\partial\Omega(\epsilon)} |Du|^2 \, d\sigma(\epsilon) \geq C\epsilon^{-2}.$$

Let $\epsilon_0 > 0$ be small enough so that the function $\operatorname{dist}(x, \partial\Omega)$ is Lipschitz continuous for $x \in \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \epsilon_0\}$. By the coarea formula,

$$\int_\Omega |Du|^2 \, dx \geq C \int_0^{\epsilon_0} \int_{\partial\Omega(\epsilon)} |Du|^2 \, d\sigma(\epsilon) \, d\epsilon \geq C \int_0^{\epsilon_0} \epsilon^{-2} \, d\epsilon = \infty.$$

This contradicts (3.2). Therefore, (3.5) is true.

By (3.5) and Holder's inequality, we have

$$\int_{\partial\Omega(\epsilon_i)} u \frac{\partial u}{\partial\nu(\epsilon_i)} \, d\sigma(\epsilon_i) \, dt \to 0 \qquad \text{as} \quad i \to \infty.$$

Now, we may replace ϵ by ϵ_i in (3.4) and let $i \to \infty$. Then, we have

$$\begin{aligned} &\frac{1}{2-p}\left(\int_\Omega u^{2-p}(x, s_2) \, dx - \int_\Omega u^{2-p}(x, s_2) \, dx\right) \\ &= -\int_{s_1}^{s_2} \int_\Omega (|Du|^2 - u^{q+1}) \, dx \, dt. \end{aligned}$$

This implies that, for almost all $t \in (0, T)$, the function

$$\int_\Omega u^{2-p}(x, t) \, dx$$

is differentiable and

$$\frac{d}{dt}\left(\frac{1}{2-p}\int_\Omega u^{2-p}(x, t) \, dx\right) = -\int_\Omega (|Du|^2 - u^{q+1}) \, dx. \tag{3.6}$$

Theorem 3.2. *Let $u(x, t)$ be a positive solution of (1.1) with $0 < p < 2$ and $q > 1$. Suppose that the initial data $u_0(x) = u(x, 0)$ satisfies the condition*

$$\frac{1}{2}\int_\Omega |Du_0|^2(x) \, dx \leq \frac{1}{q+1}\int_\Omega u_0^{q+1}(x) \, dx. \tag{3.7}$$

Then, $u(x, t)$ blows up in finite time.

We note that given any $v \in C^1(\bar{\Omega})$, if $k > 0$ is chosen large enough, then the function $u_0(x) = kv(x)$ would satisfies (3.7).

Proof. From (3.2) and (3.7), for any $t > 0$, we have

$$\int_\Omega |Du|^2(x,t)\ dx \le \frac{2}{q+1}\int_\Omega u^{q+1}(x,t)\ dx.$$

Thus, when $0 < p < 2$, from (3.6),

$$\frac{d}{dt}\left(\frac{1}{2-p}\int_\Omega u^{2-p}(x,t)\ dx\right) \ge \frac{q-1}{q+1}\int_\Omega u^{q+1}(x,t)\ dx. \tag{3.8}$$

Also, by Holder's inequality, there is a constant $C_0 > 0$ so that

$$\int_\Omega u^{q+1}\ dx \ge C_0\left(\int_\Omega u^{2-p}\ dx\right)^{\frac{q+1}{2-p}}. \tag{3.9}$$

Let

$$I(t) = \int_\Omega u^{2-p}(x,t)\ dx.$$

From (3.8) and (3.9), we obtain

$$I'(t) \ge C_1 I^{\frac{q+1}{2-p}}.$$

Since $q > 1$ and $p > 0$, we have

$$\gamma = \frac{q+1}{2-p} > 1.$$

If the solution $u(x,t)$ exists in the time interval $(0,t)$, for certain constant $C_1 > 0$, we have

$$I^{1-\gamma}(t) \le I^{1-\gamma}(0) - (\gamma-1)C_1 t,$$

or

$$\int_\Omega u^{2-p}(x,t)\ dx \ge \left(\frac{1}{I^{1-\gamma}(0) - (\gamma-1)C_1 t}\right)^{\frac{1}{\gamma-1}}.$$

Therefore, the solution has to blow up in finite time. ■

Theorem 3.3. *Let $u(x,t)$ be a positive solution of (1.1) in $\Omega \times (0,T)$ with $0 < p < 2$ and $q > 1$. Then, there is $C > 0$ so that*

$$\left(\int_\Omega u^{2-p}(x,t)\ dx\right)^{\frac{1}{2-p}} \le C|T-t|^{-1/(q+p-1)}.$$

Proof. From (3.2), we have

$$\int_\Omega |Du|^2\, dx \le \frac{2}{q+1}\int_\Omega u^{q+1}(x,t)\, dx + B,$$

where

$$B = \int_\Omega \left(\frac{|Du|^2(x,0)}{2} - \frac{u^{q+1}(x,0)}{q+1} \right) dx.$$

Thus, when $0 < p < 2$, from (3.6),

$$\frac{d}{dt}\left(\frac{1}{2-p}\int_\Omega u^{2-p}(x,t)\, dx \right) \ge \frac{q-1}{q+1}\int_\Omega u^{q+1}\, dx - B. \tag{3.10}$$

As before, let

$$I(t) = \int_\Omega u^{2-p}(x,t)\, dx.$$

Combining (3.10) and (3.9), we see that there are constants $C_2 > 0$ and $C_3 > 0$ so that,

$$I' \ge -C_2 + C_3 I^{\frac{q+1}{2-p}}. \tag{3.11}$$

If

$$-2C_2 + C_3 I^{\frac{q+1}{2-p}} \le 0 \qquad \text{for} \qquad t \in (0,T) \tag{3.12}$$

then there is constant $C_4 > 0$ so that

$$I(t) = \int_\Omega u^{2-p}(x,t)\, dx \le C_4$$

for all $t \in (0,T)$, and the Theorem is true.

If (3.12) is not true, either there is $s_1 > 0$ such that

$$-C_2 + C_3 I^{\frac{q+1}{2-p}}(t) > 0,$$

for all $t \in (s_1, T)$, or, there is an interval (s_1, S) such that

$$I^{\frac{q+1}{2-p}}(t) > \frac{C_2}{C_3},$$

for all $t \in (s_1, S)$, and

$$I^{\frac{q+1}{2-p}}(S) = \frac{C_2}{C_3}.$$

The latter case implies that $I'(t) < 0$ for some $t \in (s_1, S)$. It contradicts the equation (3.11). Therefore, for all $t \in (s_1, T)$, $I'(t) \geq -C_2 + C_3 I^{\frac{q+1}{2-p}}(t) > 0$, and $I(t) \geq I(s_1)$. Thus, we can find $C_4 > 0$ so that

$$I'(t) \geq C_4 (I(t))^{\frac{q+1}{2-p}} \quad \text{for} \quad t > s_1. \tag{3.13}$$

Let S_n be a sequence so that $S_n > s_1$ for all n and $S_n \to T$ as $nto\infty$ and

$$\limsup_{n\to\infty} \int_\Omega u^{2-p}(x, S_n)\, dx = \infty.$$

After integrating equation (3.13), from t to S_n, with $t > s_1$, we have

$$\frac{q+p-1}{2-p}\left(I^{-\frac{q+p-1}{2-p}}(t) - I^{-\frac{q+p-1}{2-p}}(S_n)\right) \geq C_4(S_n - t).$$

By letting $n \to \infty$, we have

$$\left(\int_\Omega u^{2-p}(x,t)\, dx\right)^{\frac{1}{2-p}} \leq C_5 |T-t|^{-1/(q+p-1)}. \qquad \blacksquare$$

4. Symmetric Solutions

In this section, we let

$$\Omega = \{x \in \mathbb{R}^n : |x| < R\}.$$

Let $r = |x|$. We assume that $u(x,t)$ depends on r and t only. Then equation (1.1) becomes

$$\begin{aligned} & u_t = u^p\left(u_{rr} + \frac{n-1}{r}u_r + u^q\right) \qquad \text{for} \qquad r \in (0,R), \quad t \in (0,T), \\ & u_r(0,t) = 0, \quad u(R,t) = 0 \qquad \text{for} \qquad t \in (0,T), \\ & u(r,0) = u_0(r) \qquad \text{for} \qquad r \in (0,R). \end{aligned} \tag{4.1}$$

We assume that $u_0 \in C^{2,\alpha}(0,R) \cap C^2[0,R]$, and

$$\begin{aligned} & u_0(R) = 0 \quad u_0'(0) = 0, \quad u_0'(R) < 0 \quad \text{and} \quad u_0''(0) < 0 \\ & u_0(r) > 0, \quad u_0'(r) < 0, \quad u_0'' + \frac{n-1}{r}u_0' + u_0^q \geq 0 \qquad \text{for} \quad r \in (0,R). \end{aligned} \tag{4.2}$$

It follows that

$$u_t(r,t) \geq 0, \quad u_r(r,t) \leq 0 \qquad \text{for} \qquad r \in (0,R), \quad t \in (0,T). \tag{4.3}$$

Under these conditions, using the method of Friedman and McLeod, [2], we will show that $x = 0$ is the only blowup point.

Lemma 4.1. *Let $p > 0$, $q > 1$ and $\gamma \in (1, q)$. There is a constant $\epsilon > 0$ so that*

$$-u_r \geq \epsilon r^2 u^\gamma \qquad \text{for} \qquad r \in (0, R), \quad t \in (0, T).$$

Proof. We let $f(u) = u^q$ and $F(u) = u^\gamma$, with $1 < \gamma < q$. Let $c(r) = \epsilon r^{n+1}$ and

$$J = r^{n-1} u_r + c(r) F(u).$$

Then,

$$\begin{aligned} J_t &= r^{n-1} u_{tr} + c(r) F' u_t \\ &= p r^{n-1} u^{-1} u_t u_r + r^{n-1} u^p (u_{rrr} - \frac{n-1}{r^2} u_r + \frac{n-1}{r} u_{rr} + f' u_r), \\ J_r &= (n-1) r^{n-2} u_r + r^{n-1} u_{rr} + c, F + cF' u_r, \end{aligned}$$

and

$$\begin{aligned} J_{rr} = &(n-1)(n-2) r^{n-3} u_r + 2(n-1) r^{n-2} u_{rr} + r^{n-1} u_{rrr} \\ &+ c'' F + 2c' F' u_r + c F'' u_r^2 + c F' u_{rr}. \end{aligned}$$

Hence,

$$\begin{aligned} &J_t - u^p \left(J_{rr} - \frac{n-1}{r} J_r \right) \\ = &p r^{n-1} u^{-1} u_t u_r + r^{n-1} u^p f' u_r - u^p \left(c'' F + 2c' F' u_r + c F'' u_r^2 + c F' u_{rr} \right) \\ &+ \frac{n-1}{r} u^p (c' F + c F' u_r) + c F' u_t. \end{aligned}$$

Using equation (4.1), we obtain

$$\begin{aligned} &J_t - u^p \left(J_{rr} - \frac{n-1}{r} J_r \right) \\ = &p r^{n-1} u^{-1} u_t u_r + r^{n-1} u^p f' u_r - u^p \left(c'' F + 2c' F' u_r + c F'' u_r^2 \right) \\ &+ c F' u^p \left(\frac{n-1}{r} u_r + f \right) + \frac{n-1}{r} u^p (c' F + c F' u_r) \\ = &p r^{n-1} u^{-1} u_t - c F'' u^p u_r^2 + u^p \left(r^{n-1} f' - 2c' F' + \frac{2(n-1)}{r} c F' \right) u_r \\ &+ u^p \left(c F' f + \frac{n-1}{r} c' F - c'' F \right). \end{aligned}$$

Now, we use the fact that

$$u_r = \frac{1}{r^{n-1}} (J - cF)$$

and have

$$(4.4)\quad \begin{aligned} &J_t - u^p\left(J_{rr} - \frac{n-1}{r}J_r\right) \\ =&pr^{n-1}u^{-1}u_tu_r - cF''u^pu_r^2 + u^p\left(f' - \frac{2}{r^{n-1}}c'F' + \frac{2(n-1)}{r^n}cF'\right)J \\ &+ u^p\left(cF'f + \frac{n-1}{r}c'F - c''F - cFf' + \frac{2}{r^{n-1}}cc'F'F\right. \\ &\left.-\frac{2(n-1)}{r^n}c^2FF'\right). \end{aligned}$$

Since $f = u^q$, $F = u^\gamma$ with $1 < \gamma < q$ and $c = \epsilon r^{n+1}$, we have

$$\begin{aligned} &cF'f + \frac{n-1}{r}c'F - c''F - cFf' + \frac{2}{r^{n-1}}cc'F'F - \frac{2(n-1)}{r^n}c^2FF' \\ =&4\epsilon^2r^{n+2}u^{2\gamma-1} - (q-\gamma)\epsilon r^{n+1}u^{q+\gamma-1} - \epsilon(n+1)r^{n-1}u^\gamma \end{aligned}$$

We choose ϵ small enough so that $4\epsilon R \le q - \gamma$. Then, when $u \ge 1$, since $\gamma < q$, we have

$$4\epsilon^2r^{n+2}u^{2\gamma-1} - (q-\gamma)\epsilon r^{n+1}u^{q+\gamma-1} \le 4\epsilon^2Rr^{n+1}u^{2\gamma-1} - (q-\gamma)\epsilon r^{n+1}u^{q+\gamma-1} < 0.$$

Also, we choose ϵ small enough so that $4\epsilon R^3 < n+1$. When $u \le 1$, since $\gamma > 1$, we have

$$4\epsilon^2r^{n+2}u^{2\gamma-1} - \epsilon(n+1)r^{n-1}u^\gamma \le 4\epsilon^2R^3r^{n-1}u^{2\gamma-1} - \epsilon(n+1)r^{n-1}u^\gamma < 0.$$

Therefore, for any $r > 0$ and $u > 0$, if ϵ is chosen small enough, we have

$$cF'f + \frac{n-1}{r}c'F - c''F - cFf' + \frac{2}{r^{n-1}}cc'F'F - \frac{2(n-1)}{r^n}c^2FF' < 0.$$

From (4.4) and our assumptions (4.3), the function J satisfies an equation of the form

$$J_t = u^p\left(J_{rr} - AJ_r + BJ\right) \qquad \text{for} \qquad r \in (0, R) \quad \text{and} \quad t \in (0, T),$$

where

$$A = \frac{n-1}{r} \qquad \text{and} \qquad B = qu^{q-1} - 2\epsilon\gamma(n+1)ru^{\gamma-1} + 2\epsilon(n-1)ru^{\gamma-1}.$$

When $r = 0$, we have $J = 0$. When $r = R$, since $u_r(r,t) \le 0$, we have

$$\limsup_{r \to R} u_r(r,t) \le 0.$$

This implies that

$$\limsup_{r\to R} J(r,t) \le 0 \quad \text{for all} \quad t \in (0,T).$$

Also, from the fact that $u_0''(0) < 0$ and the mean value theorem, we can see that when r is small enough, for some small constant $C > 0$, we have $u_0'(r) \le -Cr$. Hence, if r is small, $J(r,0) \le -Cr^n + \epsilon r^{n+1}u_0^q(r)$, and $u_0(0) > 0$. We may choose ϵ small enough, so that $J(r,0) \le 0$ for all $r \in (0,R)$. Then, by the maximum principle, we have $J(r,t) \le 0$ for all $r \in (0,R)$ and $t \in (0,T)$. ∎

Theorem 4.2. *Let $u(x,t)$ be a solution of (1.1) in $B_R(0) \times (0,T)$ with $q > 1$ and $p > 0$. We assume that $u(x,t)$ depends on $r = |x|$ and t only. If the initial data $u_0(r)$ satisfies assumptions (4.2), then the point $x = 0$ is the only blow-up point.*

Proof. By Lemma 4.1, for some $\gamma > 1$, we have $-u_r \ge r^2 u^\gamma$. For any $0 < r < R$ and $t \in (0,T)$, we have

$$-\int_0^r \frac{u_r(s,t)}{u^\gamma(s,t)}\, ds \ge \epsilon \int_0^r s^2\, ds.$$

It follows that

$$u^{1-\gamma}(r,t) \ge \frac{\epsilon r^3}{3}.$$

Thus, for any $r > 0$, we have

$$\limsup_{t\to T} u(r,t) < \infty. \qquad ∎$$

5. Non-symmetric Solutions

In this section, we will show that if $u(x,t)$ is a non-negative solution of (1.1) in $\Omega \times (0,T)$ with $q > 1$ and $p > 0$, then there is a constant $C > 0$ such that

$$u(x,t) \le C(T-t)^{-1/(q-1)}.$$

Again, we follow the method of Friedman and McLeod, [2].

Theorem 5.1. *Let Ω be a bounded convex $C^{2,\alpha}$ domain in $\mathbb{R}^n$ and $u(x,t)$ be a non-negative solution of (1.1) in $\Omega \times (0,T)$ with $q > 1$ and $p > 0$. Let $u_0(x) = u(x,0)$ be the initial data of u. We assume that $u_0 \in C^{2,\alpha}(\Omega) \cap C^2(\bar{\Omega})$,*

$$u_0 = 0 \quad \text{and} \quad \frac{\partial u_0}{\partial \nu} < 0 \quad \text{on} \quad \partial\Omega \tag{5.1}$$

and

(5.2) $$u_0 > 0 \qquad \text{and} \qquad \Delta u_0 + u_0^q \geq 0 \qquad \text{in} \quad \Omega.$$

Then, there are constants $\alpha > 0$ and $M > 0$ such that $u(x,t) \leq M$ whenever dist$(x, \partial\Omega) < \alpha/2$.

Proof. Take any $\tilde{x} \in \Omega$. After a translation and a rotation, we may assume that $\tilde{x} = 0$, $\Omega \subset \{x : x_1 < 0\}$, and that the hyperplane $x_1 = 0$ is tangent to $\partial\Omega$ at $\tilde{x}$. Given $\alpha > 0$, we define

$$\Omega_\alpha = \{x \in \Omega : -\alpha < x_1 < 0\}.$$

By (5.1), there is $\alpha > 0$ such that

$$\frac{\partial u_0}{\partial x_1} < 0 \qquad \text{for} \quad (x,t) \in \Omega_\alpha \times (0,T).$$

Moreover, the choice of α depends only on Ω and the initial data u_0. Then, by the reflection principle, we have

$$\frac{\partial u}{\partial x_1} \leq 0 \qquad \text{for} \quad (x,t) \in \Omega_\alpha \times (0,T).$$

Let $f(u) = u^q$ and $F(u) = u^\gamma$ with $1 < \gamma < q$. We introduce the function

$$J = u_{x_1} + \epsilon(x_1 + \alpha)^2 F(u),$$

where $\epsilon > 0$ is to be determined. Using (1.1), we compute that

$$J_t = u_{tx_1} + \epsilon(x_1+\alpha)^2 F' u_t = p u^{-1} u_{x_1} u_t + u^p(\Delta u_{x_1} + f' u_{x_1}) + \epsilon(x_1+\alpha)^2 F' u_t,$$

and

$$\Delta J = \Delta u_{x_1} + 4\epsilon(x_1+\alpha) F' u_{x_1} + \epsilon(x_1+\alpha)^2 F'' |Du|^2 + \epsilon(x_1+\alpha)^2 F' \Delta u + 2\epsilon F.$$

Thus, we have

$$\begin{aligned}
&J_t - u^p \Delta J \\
=& p u^{-1} u_{x_1} u_t - \epsilon(x_1+\alpha)^2 F'' |Du|^2 u^p + u^p f' u_{x_1} - 2\epsilon F u^p \\
&- 4\epsilon(x_1+\alpha) F' u^p u_{x_1} + \epsilon(x_1+\alpha)^2 F' f u^p \\
=& p u^{-1} u_{x_1} u_t - \epsilon(x_1+\alpha)^2 F'' |Du|^2 u^p + u^p f'(J - \epsilon(x_1+\alpha)^2 F) - 2\epsilon F u^p \\
&- 4\epsilon(x_1+\alpha) F' u^p (J - \epsilon(x_1+\alpha)^2 F) + \epsilon(x_1+\alpha)^2 F' f u^p
\end{aligned}$$

In $\Omega_\alpha \times (0,T)$, since $u_{x_1} \leq 0$ and $u_t \geq 0$, we obtain,

$$\begin{aligned}
&J_t - u^p \Delta J \\
\leq &(f' - 4\epsilon(x_1+\alpha)F')u^p J - \epsilon(x_1+\alpha)^2 u^p (f'F - fF') \\
&+ 4\epsilon^2(x_1+\alpha)^3 u^p FF' - 2\epsilon u^p F
\end{aligned}$$

From the definitions of f and F, we have $f'F - fF' = (q-\gamma)u^{q+\gamma-1} > 0$ and

$$\begin{aligned}
&\epsilon(x_1+\alpha)^2 u^p (f'F - fF') - 4\epsilon^2(x_1+\alpha)^3 u^p FF' \\
= &\epsilon(q-\gamma)(x_1+\alpha)^2 u^{p+q+\gamma-1} - 4\epsilon^2\gamma(x_1+\alpha)^3 u^{p+2\gamma-1}.
\end{aligned}$$

Since $-\alpha < x_1 < 0$, if $u \geq 1$, and $\epsilon > 0$ is chosen small enough, since $q > \gamma$, we have

$$\begin{aligned}
&\epsilon(x_1+\alpha)^2 u^p (f'F - fF') - 4\epsilon^2(x_1+\alpha)^3 u^p FF' \\
\geq &\epsilon(q-\gamma)(x_1+\alpha)^2 u^{p+q+\gamma-1} - 4\epsilon^2\alpha\gamma(x_1+\alpha)^2 u^{p+2\gamma-1} \\
\geq &0.
\end{aligned}$$

If $0 \leq u \leq 1$, and $\epsilon > 0$ is chosen small enough, since $\gamma > 1$, we have

$$\begin{aligned}
&4\epsilon^2(x_1+\alpha)^3 u^p FF' - 2\epsilon u^p F \\
= &4\epsilon^2\gamma(x_1+\alpha)^3 u^{p+2\gamma-1} - 2\epsilon u^{p+\gamma} \\
\leq &4\epsilon^2\gamma\alpha^3 u^{p+2\gamma-1} - 2\epsilon u^{p+\gamma} \\
\leq &0.
\end{aligned}$$

Hence, when ϵ is chosen small enough, the function J satisfies an equation of the form

$$J_t \leq u^p(\Delta J + EJ)$$

in $\Omega_\alpha \times (0,T)$, with $E = (f' - 4\epsilon(x_1+\alpha)F')u^p$. We also choose ϵ small enough such that $J(x,0) \leq 0$ for $x \in \Omega_\alpha$. It is easy to check that $J(x,t) \leq 0$ for all $x \in \partial\Omega_\alpha$. By the maximum principle, we have $J(x,t) \leq 0$ in $\Omega_\alpha \times (0,T)$. Then, for $(x,t) \in \Omega_{\alpha/2} \times (0,T)$, we have $u_{x_1} \leq -\epsilon(x_1+\alpha)^2 u^\gamma$. Fix $t \in (0,T)$. We let $w(s) = u(s, 0', t)$, where $0' = (0,0,...0) \in \mathbb{R}^{n-1}$. Then, $w' \leq -\epsilon(s+\alpha)^2 w^\gamma$. For all $s \in (-\alpha, 0)$, we have

$$-\frac{1}{\gamma-1}\left(w^{-(\gamma-1)}(s) - w^{-(\gamma-1)}(-\alpha)\right) \leq -\frac{\epsilon(s+\alpha)^3}{3}.$$

Then, when $s \in (-\alpha/2, 0)$, we have

$$-\frac{1}{\gamma-1}w^{-(\gamma-1)}(s) \leq -\frac{\epsilon\alpha^3}{24}.$$

We note that $\gamma > 1$, therefore,

$$w^{\gamma-1}(s) \leq \frac{24}{(\gamma-1)\epsilon\alpha^3},$$

and the Theorem follows. ∎

Theorem 5.2. *Let $u(x,t)$ be a non-negative solution of (1.1) in $\Omega \times (0,T)$ with $q > 1$ and $p > 0$. We assume that $u_0(x) = u(x,0)$ is of $C^{2,\alpha}$ and satisfies (5.1) and (5.2). Then there is a constant $C > 0$ such that*

$$u(x,t) \le C(T-t)^{-1/(q-1)}.$$

Proof. By Lemma 2.1, we have

$$u_t(x,t) \ge 0 \qquad \text{for} \quad \text{all} \quad (x,t) \in \Omega \times (0,T).$$

For any $x \in \Omega$, $u(x,t) \ge u(x,0) > 0$. Let $\alpha > 0$ be the constant in Theorem 5.1, and

$$\Omega' = \{x \in \Omega : \alpha/4 < \text{dist}(x, \partial\Omega) < \alpha/2\}.$$

Let $c > 0$ be a constant, so that $u_0(x) \ge c$ for $x \in \Omega'$. From Theorem 5.1, there are positive constants α and M such that $u(x,t) \le M$ whenever dist$(x, \partial\Omega) < \alpha/2$. Thus, $c \le u(x,t) \le M$ for $x \in \Omega'$. By the parabolic regularity theory, there is a constant $C_1 > 0$ such that

$$u_t(x,t) \le C_1 \qquad \text{for} \qquad (x,t) \in \Omega' \times (0,T).$$

The function $w(x,t) = u_t(x,t)$ satisfies the equation

$$w_t = u^p \Delta w + (q u^{p+q-1} + p u^{-1} w) w$$

in $\Omega' \times (0,T)$. It follows that there is $C_2 > 0$ such that

$$0 < C_2 \le w(x,t) \le C_1 \qquad \text{for} \qquad (x,t) \in \Omega'' \times (\alpha, T),$$

where $\Omega'' = \{x \in \Omega : 5\alpha/16 < \text{dist}(x, \partial\Omega) < 7\alpha/16\}$. Let

$$\tilde{\Omega} = \{x \in \Omega : \text{dist}(x, \partial\Omega) > 3\alpha/8\}.$$

There is a constant $0 < \delta \le 1$ such that

$$u_t(x,t) - \delta u^q(x,t) \ge 0 \qquad \text{for} \qquad (x,t) \in \partial\tilde{\Omega} \times (\alpha, T). \tag{5.3}$$

Moreover, by the maximum principle, we have $w(x,t) > 0$ for $(x,t) \in \Omega \times (0,T)$. Therefore, we can choose δ small enough so that

$$u_t(x,\alpha) - \delta u^q(x,\alpha) \ge 0 \qquad \text{for} \qquad x \in \tilde{\Omega}. \tag{5.4}$$

Let $\gamma > 1$ and δ be the constant in (5.3). Let

$$J = u_t - \delta u^\gamma = u^p(\Delta u + u^q) - \delta u^\gamma.$$

By direct computations, we have

$$J_t = pu^{-1}u_t^2 + u^p(\Delta u_t + qu^{q-1}u_t) - \delta\gamma u^{\gamma-1}u_t,$$

and

$$\Delta J = \Delta u_t - \delta\gamma u^{\gamma-1}\Delta u - \delta\gamma(\gamma-1)u^{\gamma-2}|Du|^2.$$

Thus,

$$\begin{aligned} &J_t - u^p\Delta J \\ =& pu^{-1}u_t^2 + qu^{p+q-1}u_t - \delta\gamma u^{\gamma-1}u_t + \delta\gamma u^{p+\gamma-1}\Delta u + \delta\gamma(\gamma-1)u^{p+\gamma-2}|Du|^2. \end{aligned}$$

Using equation (1.1) and the fact that $u_t = J + \delta u^\gamma$, we have

$$\begin{aligned} &J_t - u^p\Delta J \\ =& pu^{-1}u_t^2 + qu^{p+q-1}u_t - \delta\gamma u^{p+q+\gamma-1} + \delta\gamma(\gamma-1)u^{p+\gamma-2}|Du|^2 \\ =& pu^{-1}u_t^2 + \delta\gamma(\gamma-1)u^{p+\gamma-2}|Du|^2 + qu^{p+q-1}(J+\delta u^\gamma) - \delta\gamma u^{p+q+\gamma-1} \end{aligned}$$

Thus, we conclude that

$$\begin{aligned} &J_t - u^p\Delta J \\ =& pu^{-1}u_t^2 + \delta\gamma(\gamma-1)u^{p+\gamma-2}|Du|^2 + qu^{p+q-1}J + \delta(q-\gamma)u^{p+q+\gamma-1}. \end{aligned} \tag{5.5}$$

Then, in equation (5.5), we choose $\gamma = q$. Then, the function satisfies an inequality of the form

$$J_t \geq u^p\Delta J + BJ \quad \text{in} \quad \tilde{\Omega} \times (\alpha, T)$$

with $B = qu^{p+q-1}$. By (5.3), we have $J(x,t) \geq 0$ whenever $x \in \partial\tilde{\Omega}$. Also, when $t = \alpha$, by (5.4), $J = u_t(x,\alpha) - \delta u^q(x,\alpha) \geq 0$. Then, by the maximum principle, $J = u_t - \delta u^q \geq 0$ in $\tilde{\Omega} \times (\alpha, T)$. For any $\alpha < t < s < T$, we have

$$\frac{1}{q-1}\Big(u^{-(q-1)}(x,t) - u^{-(q-1)}(x,s)\Big) \geq \delta(s-t).$$

Hence,

$$\frac{1}{q-1}u^{-(q-1)}(x,t) \geq \delta(s-t).$$

When letting $s \to T$, we have

$$\frac{1}{q-1}u^{-(q-1)}(x,t) \geq \delta(T-t),$$

and the Theorem follows. ■

6. The Case $p \geq 2$

Lemma 6.1. *Let* $0 < 2R < L$, $\lambda_1 > 0$, $0 < \lambda_2 < 1$, $p \geq 2$, $q > 1$. *Let* $w(x) \in C^2(2R, L) \cap C[2R, L]$ *be a solution of the ODE*

$$w'' - \lambda_1 w^{-p} w' x - \lambda_2 w^{1-p} + w^q = 0 \qquad \text{on} \qquad (2R, L) \tag{6.1}$$

which is decreasing in x *and satisfies the boundary conditions:* $w(2R) = w_0 > 0$, *and* $w(L) = 0$. *Let*

$$\epsilon = \frac{\lambda_2}{2\lambda_1 L} > 0.$$

Then, there is constant $\delta > 0$ *so that* $w'(x) + \epsilon w(x) \leq 0$ *when* $x \in (L - \delta, L)$.

Proof. Let $w(x)$ be a solution as described in the lemma. Suppose that there is a point $a \in (2R, L)$ such that $w'(a) + \epsilon w(a) > 0$. Since w is decreasing in $(2R, L)$ and $w(L) = 0$, we have

$$\limsup_{x \to L^-} (w'(x) + \epsilon w(x)) \leq 0.$$

Thus, we can find an interval $(a, b) \subset (2R, L)$ such that $w'(x) + \epsilon w(x) > 0$ for $x \in (a, b)$ and $w'(b) + \epsilon w(b) = 0$. Then, for $x \in (a, b)$, we have

$$\begin{aligned} w''(x) + \epsilon w'(x) &= \lambda_1 w^{-p} w' x + \lambda_2 w^{1-p} - w^q + \epsilon w' \\ &\geq -\lambda_1 \epsilon w^{1-p} x + \lambda_2 w^{1-p} - w^q - \epsilon^2 w \\ &\geq -\lambda_1 \epsilon L w^{1-p} + \lambda_2 w^{1-p} - w^q - \epsilon^2 w \\ &\geq \frac{\lambda_2}{2} w^{1-p} - w^q - \epsilon^2 w. \end{aligned}$$

We let $\eta > 0$ so that if $0 < w < \eta$, then

$$\frac{1}{2}\lambda_2 w^{1-p} - w^q - \epsilon^2 w \geq 0.$$

Since w is decreasing and $w(L) = 0$, there is $\delta > 0$ such that $0 < w(x) < \eta$ for $x \in (L - \delta, L)$. If $a > L - \delta$, then $0 < w(x) < \eta$ for $x \in (a, b)$. This implies that

$$w''(x) + \epsilon w'(x) \geq 0 \qquad \text{in} \qquad (a, b).$$

Hence, $w'(x) + \epsilon w(x)$ is an increasing function in (a, b). However, $w'(x) + \epsilon w(x) > 0$ in (a, b) and $w'(b) + \epsilon w(b) = 0$, and we have a contradiction. ∎

We let

$$F(w) = \frac{\lambda_2}{p-2} w^{2-p} + \frac{1}{q+1} w^{q+1} \quad \text{when} \quad p > 2, \tag{6.2}$$

and

$$F(w) = -\lambda_2 \log w + \frac{1}{q+1} w^{q+1} \quad \text{when} \quad p = 2. \tag{6.3}$$

In both cases, we have

$$\lim_{x \to 0^+} F(w) = \lim_{x \to \infty} F(w) = \infty,$$

and the function $F(w)$ has a unique minimum at $w = \lambda_2^{1/(p+q-1)}$, for $w \in (0, \infty)$. It is easy to check that, since $0 < \lambda_2 < 1$, we have

$$F\left(\lambda_2^{1/(p+q-1)}\right) > 0.$$

Lemma 6.2. *Let $R > 0$, $\lambda_1 > 0$, $0 < \lambda_2 \leq 1$, $p \geq 2$, $q > 1$. Let $w(x)$ be a solution of the ODE (6.1) with initial data $w(2R) = w_0 > \lambda_2^{1/(p+q-1)} > 0$, and $w'(2R) = 0$. Then, either $w(x)$ can be extended as a positive, decreasing function defined on $(2R, \infty)$ and*

$$F(m) \geq F(w_0), \qquad \text{with} \qquad m = \lim_{x \to \infty} w(x) < \lambda_2^{1/(p+q-1)}, \tag{6.4}$$

or, there is $K > 2R$ such that $w(x)$ is decreasing in $(2R, K)$, $w'(K) = 0$, and

$$F(\eta) \geq F(w_0), \qquad \text{with} \qquad \eta = w(K) < \lambda_2^{1/(p+q-1)}. \tag{6.5}$$

Proof. By our assumption on $w(2R)$, we have $w''(2R) < 0$. Thus w is a decreasing function near $x = 2R$.

Let $K > 2R$ be the first point where $w'(K) = 0$ and $\eta = w(K) > 0$. We first assume that $p > 2$. From the equation (6.1), when $p > 2$, we have

$$\frac{d}{dx}\left(\frac{1}{2}w'^2 + \frac{\lambda_2}{p-2}w^{2-p} + \frac{1}{q+1}w^{q+1}\right) = \lambda_1 w^{-p} w'^2 x \geq 0. \tag{6.6}$$

Thus, if $F(w)$ is the function in (6.2), we have $F(\eta) \geq F(w_0)$.

Suppose that the point K in the above does not exist. Then, either $w(x)$ is defined for all $x \in (2R, \infty)$ and is a decreasing function, or there is $L > 2R$ so that $w(x)$ is a decreasing function in $(2R, L)$ and $w(L) = 0$. In the first case, let $w(x) \to m \geq 0$ as $x \to \infty$. Then, there is an increasing sequence x_n such that $x_n \to \infty$ as $n \to \infty$ and $w'(x_n) \to 0$ as $n \to \infty$. When $p > 2$, from (6.6), we have

$$\frac{1}{2}w'^2(x_n) + F(w(x_n)) \geq F(w_0).$$

When $n \to \infty$, we see that $F(m) \geq F(w_0)$.

Suppose that there is $L > 2R$ so that $w(x)$ is a decreasing function in $(2R, L)$ and $w(L) = 0$. By Lemma 6.1, there are $\delta > 0$ and $\epsilon > 0$ such that $w'(x) + \epsilon w(x) \geq 0$ in $(L - \delta, L)$. Then, we have

$$\begin{aligned} w'' - \frac{\lambda_2}{2} w^{1-p} + w^q &= w'' + \epsilon\lambda_1 L w^{1-p} - \lambda_2 w^{1-p} + w^q \\ &\geq w'' - \lambda_1 w^{-p} w' x - \lambda_2 w^{1-p} + w^q \\ &= 0. \end{aligned}$$

Since $w' \leq 0$, when $p > 2$, we have

$$\frac{d}{dx}\left(\frac{1}{2}w'^2 + \frac{\lambda_2}{2(p-2)}w^{2-p} + \frac{1}{q+1}w^{q+1}\right) \leq 0.$$

Thus, for $x \in (L - \delta, L)$,

$$\begin{aligned} &\frac{1}{2}w'^2(x) + \frac{\lambda_2}{2(p-2)}w^{2-p}(x) + \frac{1}{q+1}w^{q+1}(x) \\ \leq &\frac{1}{2}w'^2(L-\delta) + \frac{\lambda_2}{2(p-2)}w^{2-p}(L-\delta) + \frac{1}{q+1}w^{q+1}(L-\delta) \end{aligned}$$

and is bounded from above. However, since $w(x) \to 0$ as $x \to L$, this is impossible.

When $p = 2$, we let $F(w)$ be the function in (6.3). Using the same arguments, we obtain the same result. ∎

Theorem 6.3. *Let $\Omega = \{x : |x| < R_0\}$ and let $u(x,t)$ be a positive solution of (1.1) in $\Omega \times (0,T)$ with $p \geq 2$ and $q > 1$. Suppose that u is symmetric, and is radially decreasing, and blows up at $t = T$, then*

$$\limsup_{t \to T^-}\left((T-t)^{1/(p+q-1)} \max_{\Omega} u(x,t)\right) = \infty.$$

Proof. Let $u(x,t)$ be a positive solution of (1.1) in $\Omega \times (0,T)$ with $p \geq 2$ and $q > 1$. We assume that u depends on r and t only, where $r = |x|$, and $u_r(x,t) \leq 0$ for all $(x,t) \in \Omega \times (0,T)$. Note that

$$u(0,t) = \max_{x \in \Omega} u(x,t).$$

If the Theorem is not true, then there is a constant $M > 0$ such that

$$\limsup_{t \to T}(T-t)^{1/(p+q-1)}u(0,t) = M < \infty. \tag{6.7}$$

Let $a = (-a_1, 0, ..., 0) \in \Omega$ with $a_1 > 0$. We let $w(y,s)$ be the rescaled function of u at a, i.e.,

$$w(y,s) = (T-t)^{1/(p+q-1)} u\left(a + y(T-t)^{(q-1)/2(p+q-1)}, t\right) \text{ with } s = -\log(T-t).$$

Then, $w(y,s)$ satisfies the equation

$$w_s = w^p\left(\Delta w - \frac{q-1}{2(p+q-1)}w^{-p}Dw\cdot y - \frac{1}{p+q-1}w^{1-p} + w^q\right) \tag{6.8}$$

on the set

$$\Gamma_a\left\{(y,s) : s > -\log T,\ a + y(T-t)^{(q-1)/2(p+q-1)} \in \Omega\right\}.$$

Let

$$\Gamma_a(s) = \left\{y :\ a + y(T-t)^{(q-1)/2(p+q-1)} \in \Omega\right\} \quad \text{with} \quad s = -\log(T-t).$$

We note that, for each $s > 0$, the set $\Gamma_a(s)$ is a ball centered at $(T-t)^{-(q-1)/2(p+q-1)}$ $(a_1, 0, ..., 0)$ with radius $(T-t)^{-(q-1)/2(p+q-1)}R_0$, and $s = \ln(T-t)$. When $y \in \partial\Gamma_a(s)$, we have $w(y,s) = 0$. For $y \notin \Gamma_a(s)$, we let $w(y,s) = 0$. Then, $w(y,s)$ is defined for all $y \in \mathbb{R}^n$ and $s > -\log T$. From our assumptions, we have

$$\frac{\partial w}{\partial y_1}(y,t) \le 0 \quad \text{when} \quad y = (y_1, 0, ..., 0), \quad y_1 > (T-t)^{-(q-1)/2(p+q-1)}a_1. \tag{6.9}$$

Moreover, if $y_1 \in (T-t)^{-(q-1)/2(p+q-1)}(a_1, a_1+R_0)$, and $(y_1, y') \in \Gamma_a(s)$, then we have $w(y_1, y'; t) \le w(y_1, 0'; t)$. Here $y' = (y_2, ..., y_n)$ and $0' = (0, ..., 0) \in \mathbb{R}^{n-1}$. By (6.7), there is a sequence t_k such that $t_k \to \infty$ as $k \to \infty$ and

$$\lim_{n\to\infty}(T-t_k)^{1/(p+q-1)}u(0,t_k) = M. \tag{6.10}$$

Let

$$R = T^{-(q-1)/2(p+q-1)}R_0.$$

Let $\phi(z)$ be a solution of the ODE (6.1), with

$$\lambda_1 = \frac{q-1}{2(p+q-1)} \qquad \text{and} \qquad \lambda_2 = \frac{1}{p+q-1},$$

and $\phi(2R) = \alpha > 0$ and $\phi'(2R) = 0$, where

$$\alpha = 2\max\left(M, \lambda_2^{1/(p+q-1)}\right).$$

By Lemma 6.2, either ϕ can be extended as as decreasing function for $z \in (2R, \infty)$, or $\phi(z)$ is defined on $(2R, K)$, $\phi'(z) \le 0$ in $(2R, K)$ and $\phi'(K) = 0$. By equation (6.4) and (6.5), we choose $\alpha = \phi(2R)$ large enough so that

$$m = \lim_{z\to\infty}\phi(z) < M/2 \tag{6.11}$$

or

$$\phi(K) < M/2. \tag{6.12}$$

We first assume that ϕ is a decreasing function and is defined on $[2R, \infty)$. We let $\phi(z) = \phi(2R)$ for $z \in [0, 2R)$, and define the function $\varphi(y)$ to be a function depending on y_1 only, and $\varphi(y) = \phi(y_1)$. Then, we have $\varphi(y) > w(y, -\ln T)$. Let $a = (a_1, 0')$, and

$$s_k = -\log(T - t_k) \qquad \text{and} \qquad y_k = a(T - t_k)^{-(q-1)/2(p+q-1)},$$

where t_k is the sequence in (6.10). Note that

$$|y_k| \to \infty \quad \text{as} \quad k \to \infty,$$

and

$$\lim_{k\to\infty} w(y_k, s_k) = M.$$

Hence, by (6.11), when k is large, we have $w(y_k, s_k) > \varphi(y_k)$. Thus, there is $s_0 > -\ln T$ such that $w(y, s) < \varphi(y)$ for all $y \in \mathbb{R}^n$ and $-\log T < s < s_0$, and, for certain $y_0 \in \mathbb{R}^n$, $w(y_0, s_0) = \varphi(y_0)$. By our assumption, we must have $y_0 = (y_{01}, 0')$, and $y_{01} > 2R$. Then, in a neighborhood of y_0, the function $\varphi(y)$ is also a solution of the equation (1.1). Also, we have $w(y, s) \le \phi(y)$ for all y and $s < s_0$, but $w(y_0, s_0) = \varphi(y_0)$. By the maximum principle, it is impossible.

Next, we assume that ϕ is a decreasing function for $x \in (R, K)$, $\phi'(2R) = \phi'(K) = 0$. By (6.5), we choose $\alpha = \phi(2R)$ large enough so that $\phi(K) < M/2$. Then, $\phi''(K) > 0$ and we may extend ϕ to be function on the interval $(2R, \bar{K})$, for some $\bar{K} > K$ so that on $(K, \bar{K})$, the function ϕ is strictly increasing. When $z \in (0, 2R)$, we let $\phi(z) = \phi(2R)$. When $z > \bar{K}$, we let $\phi(z) = \phi(\bar{K})$. We then define the function $\varphi(y)$ to be a function depending on y_1 only, and $\varphi(y) = \phi(y_1)$. Then, we have $\varphi(y) > w(y, -\log T)$. As in the above, let $a = (a_0, 0')$, and

$$s_k = -\log(T - t_k) \qquad \text{and} \qquad y_k = a(T - t_k)^{-(q-1)/2(p+q-1)}.$$

Then, we have $|y_k| \to \infty \quad \text{as} \quad k \to \infty$, and $\lim_{k\to\infty} w(y_k, s_k) = M$. Hence, by (6.12), when k is large, we have $w(y_k, s_k) > \varphi(y_k)$. Thus, there is $s_0 > -\ln T$ such that $w(y, s) < \varphi(y)$ for all $y \in \mathbb{R}^n$ and $-\log T < s < s_0$, and, for certain $y_0 \in \mathbb{R}^n$, $w(y_0, s_0) = \varphi(y_0)$. Let $y_0 = (y_{01}, y_0')$. We claim that $y_{01} \in (2R, K]$. By the choice of $\phi(2R)$, it is clear that $y_{01} > 2R$. If $y_{01} > K$, let $\tilde{y} = (K, 0')$. Since $w(y, s_0) \le \varphi(y)$ for all y, we have $w(\tilde{y}, s_0) \le \varphi(\tilde{y}) < \varphi(y_0) = w(y_0, s_0)$. It contradicts (6.9). Hence, $y_{01} \in (2R, K]$. In a neighborhood of y_0, $\varphi(y)$ is also a solution of the equation (1.1). Also, we have $w(y, s) \le \phi(y)$ for all y and $s < s_0$, but $w(y_0, s_0) = \varphi(y_0)$. By the maximum principle, it is also impossible. ∎

References

1. M. Fila and P. Souplet, The blow-up rate for semilinear parabolic problems on general domains, *Nonlinear Differ. Equ. Appl.*, **8** (2001), 473-480.

2. A. Friedman and B. McLeod, Blowup of positive solutions of semilinear heat equations, *Indiana Univ. Math. Journal*, **34** (1985), 425-447.

3. A. Friedman and B. McLeod, Blow-up of solutions of nonlinear degenerate parabolic equations, *Arch. Rational Mech. Anal.*, **96** (1986), 55-80.

4. Y. Giga and R. V. Kohn, Characterizing blowup using similarity variables, *Indiana Univ. Math. Journal*, **36(1)** (1987), 1-40.

5. Y. Giga and R. V. Kohn, Nondegeneracy of blowup for semilinear heat equation, *Comm. Pure Appl. Math.*, **42** (1989), 845-884.

6. A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhailov, *Blow-up in quasilinear parabolic equations*, Walter de Gryyter, Berlin, New York, 1995.

7. M. Winkler, Blow-up of solution to a degenerate parabolic equation not in divergence form, *J. Differential Equation*, **192** (2003), 445-474.

8. M. Winkler, Blow-up in a degenerate parabolic equation, *Indiana University Math. Journal*, **53(5)** (2004), 1415-1442.

Chi-Cheung Poon
Department of Mathematics
National Chung Cheng University
Minghsiung, Chiayi 621, Taiwan
E-mail: ccpoon@math.ccu.edu.tw