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L_p RADIAL MINKOWSKI HOMOMORPHISMS

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Abstract. Intersection bodies define a continuous and $GL(n)$ contravariant valuation which plays a crucial role in the solution of the Busemann-Petty problem. In this paper, we introduce the concept of L_p radial Minkowski homomorphisms and consider the Busemann-Petty type problem whether $\Phi_p K \subseteq \Phi_p L$ implies $V(K) \le V(L)$, where Φ_p is a homogeneous of degree $\left(\frac{n}{p} - 1\right)$, continuous operator on star bodies which is an $SO(n)$ equivariant valuation. Previous results by Schuster are generalized to a large class of L_p radial valuations.

1. Introduction

Let $vol_k(K)$ denote the k-dimensional Lebesgue measure of a compact convex set K. Instead of vol_n we usually write V. Let B denote the Euclidean unit ball and S^{n-1} the Euclidean unit sphere in $\mathbb{R}^n$. Let K be a body that is star-shaped with respect to the origin in $\mathbb{R}^n$. The radial function of K is given by

$$\rho_K(u) = max\{\lambda \ge 0 : \lambda u \in K\},\ u \in S^{n-1}. \tag{1.1}$$

We call K a star body if $\rho_K(\cdot)$ is continuous on S^{n-1} and K contains the origin in its interior. The radial distance of star bodies K and L is defined by $\delta(K, L) = \max_{u\in S^{n-1}} |\rho_K(u) - \rho_L(u)|$. A compact, convex set in $\mathbb{R}^n$ is said to be a convex body if it has non-empty interior.

The Busemann-Petty problem (see [5]) asks the following question: Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$vol_{n-1}(K \cap u^\perp) \le vol_{n-1}(L \cap u^\perp),\ \forall u \in S^{n-1}.$$

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Does it follow that

$$V(K) \le V(L)?$$

The Busemann-Petty problem has an affirmative answer if $n \le 4$ and a negative answer if $n \ge 5$. The solution appeared as the result of a sequence of papers: [20]$n \ge 12$, [3] $n \ge 10$, [10] and [4] $n \ge 7$, [29] and [6] $n \ge 5$, [7] $n = 3$, [37] and [9] $n = 4$. For a detailed account of the interesting history of the Busemann-Petty problem, see the books by Gardner [8] and Koldobsky [19].

The key to the complete solution of the Busemann-Petty problem in all dimensions, a connection between the problem and intersection bodies, was discovered by Lutwak [25] in 1988. The intersection body IK of a star body K is defined by

$$\rho(IK, u) = vol_{n-1}(K \cap u^{\perp}),\ u \in S^{n-1}.$$

From (1.1) and the fact that star bodies K and L satisfy $K \subset L$ if and only if $\rho(K, \cdot) \le \rho(L, \cdot)$, we see that the Busemann-Petty problem can be rephrased in the following way: Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. Is there the implication

$$IK \subset IL \Rightarrow V(K) \le V(L)? \tag{1.2}$$

If K is restricted to the class of intersection bodies, the Busemann-Petty problem has an affirmative answer. In addition, if L is a sufficiently smooth origin-symmetric star body with positive radial function which is not an intersection body, then there exists an origin-symmetric star body K such that $IK \subset IL$ but $V(K) > V(L)$ (see [25]). It is well known that the intersection body operator is a radial valuation.

A function Φ defined on the space $\mathcal{S}^n$ of star bodies in $\mathbb{R}^n$ and taking values in an abelian semigroup is called a radial valuation if

$$\Phi(K \cup L)\widetilde{+}\Phi(K \cap L) = \Phi K\widetilde{+}\Phi L, \tag{1.3}$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{S}^n$, respectively.

The theory of real valued valuations is at the center of convex geometry. Blaschke started a systematic investigation in the 1930s and then Hadwiger focused on classifying valuations on compact convex sets in $\mathbb{R}^n$ and obtained famous Hadwiger's Characterization Theorem. The survey [28] and the book[18] are an excellent source for the classical theory of valuations. For some of the more recent results, see [1, 2, 13-17, 20-24, 31-35].

First results on star body valued valuations were obtained by Klain [17] in 1996, where addition of star bodies is radial sum defined by $K\tilde{+}L = \{x\tilde{+}y :\ x \in K,\ y \in L\}$, where $x\tilde{+}y$ is defined to be the usual vector sum of the points x and y, if both of them are contained in a line through origin, and 0 otherwise. Moreover, he obtained a classification theorem for homogeneous valuations on star-shaped bodies which is a dual analogue of Hadwiger's Characterization Theorem of the elementary Minkowski mixed volumes.

A valuation Φ is called $SO(n)$ equivariant, if for all $\vartheta \in SO(n)$ and all $K \in \mathcal{S}^n$,

$$\Phi(\vartheta K) = \vartheta \Phi K. \tag{1.4}$$

A valuation Φ is called p-homogeneous, if for $K \in \mathcal{S}^n$ and $\lambda \geq 0$,

$$\Phi(\lambda K) = \lambda^p \Phi K. \tag{1.5}$$

A map $\Phi : \mathcal{S}^n \rightarrow \mathcal{S}^n$ is called an $(n-1)$-homogeneous radial Blaschke-Minkowski homomorphism if it is continuous, $SO(n)$ equivariant and satisfies $\Phi(K \widetilde{+}_{n-1} L) = \Phi K \widetilde{+} \Phi L$. Here $K \widetilde{+}_{n-1} L$ denotes the L_{n-1} radial sum of the star bodies K and L (see Section 2 for a precise definition). Obviously, a radial Blaschke-Minkowski homomorphism is a continuous radial valuation which is $SO(n)$ equivariant and $(n-1)$-homogeneous. Schuster introduced radial Blaschke-Minkowski homomorphisms and studied the Busemann-Petty problem type problem for them.

Theorem A. ([34]). *Let $\Phi : \mathcal{S}^n \rightarrow \mathcal{S}^n$ be a radial Blaschke-Minkowski homomorphism. If $K \in \Phi\mathcal{S}^n$ and $L \in \mathcal{S}^n$, then*

$$\Phi K \subseteq \Phi L \Rightarrow vol_n(K) \leq vol_n(L),$$

and $V(K) = V(L)$, if and only if $K = L$.

Theorem B. ([34]). *Let $\Phi : \mathcal{S}^n \rightarrow \mathcal{S}^n$ be a radial Blaschke-Minkowski homomorphism. If $\mathcal{S}^n(\Phi)$ does not coincide with $\mathcal{S}^n$, then there exist star bodies $K, L \in \mathcal{S}^n$, such that*

$$\Phi K \subseteq \Phi L,$$

but

$$V(K) > V(L).$$

Here $\mathcal{S}^n(\Phi)$ denotes the injectivity set of Φ (see Section 3 for a precise definition).

In recent years the investigations of convex body and star body valued valuations have received great attention from a series of articles by Ludwig[21-24], see also[14]. She started systematic studies and established complete classifications of convex and star body valued valuations with respect to L_p Minkowski sum and L_p radial which are compatible with the action of the group $GL(n)$. Based on these results, we study in this article the Busemann-Petty type problem for L_p radial Minkowski homomorphisms. We generalize the results of Schucher as follows:

Theorem 1.1. *Let $0 < p < n$ and let $\Phi_p : \mathcal{S}^n \to \mathcal{S}^n$ be an L_p radial Minkowski homomorphism. If $K \in \Phi_p\mathcal{S}^n$ and $L \in \mathcal{S}^n$, then*

$$\Phi_p K \subseteq \Phi_p L \Rightarrow V(K) \leq V(L), \tag{1.6}$$

and $V(K) = V(L)$, if and only if $K = L$.
If $p > n$, then

$$\Phi_p K \subseteq \Phi_p L \Rightarrow V(K) \geq V(L),$$

and $V(K) = V(L)$, if and only if $K = L$.

Theorem 1.2. *Let $0 < p < n$ and let $\Phi_p : \mathcal{S}^n \to \mathcal{S}^n$ be an L_p radial Minkowski homomorphism. If $\mathcal{S}^n(\Phi_p)$ does not coincide with $\mathcal{S}^n$, then there exist star bodies $K, L \in \mathcal{S}^n$, such that*

$$\Phi_p K \subseteq \Phi_p L, \tag{1.7}$$

but

$$V(K) > V(L). \tag{1.8}$$

If $p > n$, the inequality (1.8) is reverse.

2. Notation and Background Material

Let $\mathcal{S}^n$ be the space of star bodies in $\mathbb{R}^n$ and let $\mathcal{S}^n_e$ denote the subset of $\mathcal{S}^n$ that contains the origin-symmetric star bodies. We call a star body trivial if it contains only the origin. A star body $L \in \mathcal{S}^n$ is defined by the values of its radial function $\rho(L, \cdot)$ on S^{n-1}. From the definition of $\rho(L, \cdot)$, it follows immediately that for $\lambda > 0$ and $\vartheta \in SO(n)$,

$$\rho(\lambda L, u) = \lambda\rho(L, u) \; and \; \rho(\vartheta L, u) = \rho(L, \vartheta^{-1}u). \tag{2.1}$$

For $K, L \in S^n$, $p \in \mathbb{R}$ and $p \neq 0$, the L_p radial sum $K\widetilde{+}_p\varepsilon \cdot L$ is the star body defined by

$$\rho(K\widetilde{+}_p\varepsilon \cdot L, \cdot)^p = \rho(K, \cdot)^p + \varepsilon\rho(L, \cdot)^p, \tag{2.2}$$

where this addition and scalar multiplication are obviously dependent on p. The L_p dual mixed volume, $\widetilde{V}_p(K, L)$, of K and L is defined by (see [27])

$$\frac{n}{p}\widetilde{V}_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K\widetilde{+}_p\varepsilon \cdot L) - V(K)}{\varepsilon}.$$

The definition above and the polar coordinate formula for volume give the following integral representation of the dual mixed volume $\widetilde{V}_p(K,L)$

$$\widetilde{V}_p(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-p}(v)\rho_L^{p}(v)dS(v), \tag{2.3}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}. From the formula (2.3), it follows immediately that for each $K\in S^n$,

$$\widetilde{V}_p(K,K)=V(K). \tag{2.4}$$

From an application of the Hölder inequality, one can get the Minkowski inequality for the L_p dual mixed volume (see [12]).

Lemma 2.1. *For $K,L\in S^n$, if $0<p<n$, then*

$$\widetilde{V}_p(K,L)\le V(K)^{\frac{n-p}{n}}V(L)^{\frac{p}{n}}, \tag{2.5}$$

with equality if and only if K and L are dilates;
If $p<0$ or $p>n$, then

$$\widetilde{V}_p(K,L)\ge V(K)^{\frac{n-p}{n}}V(L)^{\frac{p}{n}}, \tag{2.6}$$

with equality if and only if K and L are dilates.

The quasi-L_p intersection body I_pK of a star body was introduced in [36]: Let K be a star body in $\mathbb{R}^n$, the quasi-L_p intersection body I_pK is defined by:

$$\rho(I_pK,u)^p=\int_{S^{n-1}\cap u^{\perp}}\rho(K,u)^{n-p}dS(u). \tag{2.7}$$

It is easy to check that $I_1K=(n-1)IK$ and $I_nK=((n-1)\omega_{n-1})^{\frac{1}{n}}B$.

Lemma 2.2. *The operator $I_p:\mathcal{S}^n\to\mathcal{S}^n$ has the following properties:*

(a) *I_p is continuous with respect to radial metric.*
(b) *$I_p(K\ \widetilde{+}_{n-p}\ L)=I_pK\ \widetilde{+}_p\ I_pL$ for all $K,L\in\mathcal{S}^n$.*
(c) *I_p is $SO(n)$ equivariant, i.e., $I_p(\vartheta K)=\vartheta I_pK$ for all $\vartheta\in SO(n)$.*

Proof. Since the p-th power of a continuous function is still continuous, (a) holds. From (2.7) and (2.2), we have

$$\begin{aligned}\rho(I_p(K\ \widetilde{+}_{n-p}\ L),u)^p &= \int_{S^{n-1}\cap u^{\perp}}\rho(K\ \widetilde{+}_{n-p}\ L,u)^{n-p}dS(u)\\ &= \int_{S^{n-1}\cap u^{\perp}}\rho(K,u)^{n-p}dS(u)+\int_{S^{n-1}\cap u^{\perp}}\rho(L,u)^{n-p}dS(u)\\ &= \rho(I_pK,u)^p+\rho(I_pL,u)^p\\ &= \rho(I_pK\widetilde{+}_pI_pL,u)^p.\end{aligned}$$

It remains to prove (b).

Using definition (2.7) and noting (2.1), for any $\vartheta \in SO(n)$ and $u \in S^{n-1}$, $\vartheta u \in S^{n-1}$, and $u \cdot v = 0$ if and only if $\vartheta^t u \cdot \vartheta^{-1} v = 0$, we have that

$$\begin{aligned}\rho(I_p\vartheta K, u)^p &= \int_{S^{n-1}\cap u^\perp} \rho(\vartheta K, u)^{n-p} dS(u)\\ &= \int_{S^{n-1}\cap u^\perp} \rho(K, \vartheta^t u)^{n-p} dS(u)\\ &= \int_{S^{n-1}\cap (\vartheta^t u)^\perp} \rho(K, u)^{n-p} dS(u)\\ &= \rho(I_pK, \vartheta^t u)^p = \rho(\vartheta^{-t} I_pK, u)^p\\ &= \rho(\vartheta I_pK, u)^p.\end{aligned}$$

This proves (c). ■

Some basic notions on spherical harmonics will be required. The background material on spherical harmonics is presented as in Schuster [34]. As usual, $SO(n)$ and S^{n-1} will be equipped with the invariant probability measures. Let $\mathcal{C}(SO(n))$, $\mathcal{C}(S^{n-1})$ be the spaces of continuous functions on $SO(n)$ and S^{n-1} with uniform topology and let $\mathcal{M}(SO(n)), \mathcal{M}(S^{n-1})$ denote their dual spaces of signed finite Borel measures with weak* topology. The group $SO(n)$ acts on these spaces by left translation, i.e., for $f \in \mathcal{C}(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1})$, we have $\vartheta f(u) = f(\vartheta^{-1}u), \vartheta \in SO(n)$, and $\vartheta\mu$ is the image measure of μ under the rotation ϑ. If $\mu, \sigma \in \mathcal{M}(SO(n))$, the convolution $\mu * \sigma$ is defined by[34]:

$$\int_{SO(n)} f(\vartheta) d(\mu * \sigma)(\vartheta) = \int_{SO(n)} \int_{SO(n)} f(\eta\tau) d\mu(\eta) d\sigma(\tau), \tag{2.8}$$

for every $f \in \mathcal{C}(SO(n))$. The sphere S^{n-1} is identified with the homogeneous space $SO(n)/SO(n-1)$, where $SO(n-1)$ denotes the subgroup of rotations leaving the pole $\widehat{e}$ of S^{n-1} fixed. The projection from $SO(n)$ onto S^{n-1} is $\vartheta \mapsto \widehat{\vartheta} := \vartheta\widehat{e}$. Right $SO(n-1)$-invariant functions on $SO(n)$ are defined by $\check{f}(\vartheta) = f(\widehat{\vartheta})$, for $f \in \mathcal{C}(S^{n-1})$. In fact, $\mathcal{C}(S^{n-1})$ is isomorphic to the subspace of right $SO(n-1)$-invariant functions in $\mathcal{C}(SO(n))$ and this correspondence carries over to an identification of the space $\mathcal{M}(S^{n-1})$ with right $SO(n-1)$-invariant measures in $\mathcal{M}(SO(n))$. It is easy to check that the Dirac measure $\delta_{\widehat{e}}$ is the unique rightneutral element for the convolution on $\mathcal{M}(S^{n-1})$.

The convolution $\mu * f \in \mathcal{C}(S^{n-1})$ of a measure $\mu \in \mathcal{M}(SO(n))$ and a function $f \in \mathcal{C}(S^{n-1})$ is defined by [34]

$$(\mu * f)(u) = \int_{SO(n)} \vartheta f(u) d\mu(\vartheta). \tag{2.9}$$

The canonical pairing of $f \in \mathcal{C}(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1})$ is defined by [34]

$$\langle \mu, f\rangle = \langle f, \mu\rangle = \int_{S^{n-1}} f(u)d\mu(u). \tag{2.10}$$

If $\mu, \nu \in \mathcal{M}(S^{n-1})$ and $f \in \mathcal{C}(S^{n-1})$, then

$$\langle \mu * \nu, f\rangle = \langle \mu, f * \nu\rangle. \tag{2.11}$$

A function $f \in \mathcal{C}(S^{n-1})$ is called zonal, if $\vartheta f = f$ for every $\vartheta \in SO(n-1)$. Zonal functions depend only on the value $u \cdot \widehat{e}$. The set of continuous zonal functions on S^{n-1} will be denoted by $\mathcal{C}(S^{n-1}, \widehat{e})$ and the definition of $\mathcal{M}(S^{n-1}, \widehat{e})$ is analogous. A map $\Lambda : \mathcal{C}[-1, 1] \to \mathcal{C}(S^{n-1}, \widehat{e})$ defined by

$$\Lambda f(u) = f(u \cdot \widehat{e}),\ u \in S^{n-1}. \tag{2.12}$$

The map Λ is also an isomorphism between functions on $[-1, 1]$ and zonal functions on S^{n-1}.
If $f \in \mathcal{C}(S^{n-1}), \mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ and $\eta \in SO(n)$, then

$$(f * \mu)(\widehat{\eta}) = \int_{S^{n-1}} f(\eta u)d\mu(u). \tag{2.13}$$

If $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$, for each $f \in \mathcal{C}(S^{n-1})$ and every $\vartheta \in SO(n)$, then

$$(\vartheta f) * \mu = \vartheta(f * \mu). \tag{2.14}$$

We use $\mathcal{H}_k^n$ to denote the finite dimensional vector space of spherical harmonics of dimension n and order k. Let $N(n, k)$ denote the dimension of $\mathcal{H}_k^n$. The space of all finite sums of spherical harmonics of dimension n is denoted by $\mathcal{H}^n$. The spaces $\mathcal{H}_k^n$ are pairwise orthogonal with respect to the usual inner product on $\mathcal{C}(S^{n-1})$. Clearly, $\mathcal{H}_k^n$ is invariant with respect to rotations.

Let $P_k^n \in \mathcal{C}[-1, 1]$ denote the Legendre polynomial of dimension n and order k. The zonal function ΛP_k^n is up to a multiplicative constant the unique zonal spherical harmonic in $\mathcal{H}_k^n$. In each space $\mathcal{H}_k^n$ we choose an orthonormal basis $H_{k1}, \cdots, H_{kN(n,k)}$. The collection $\{H_{k1}, \cdots, H_{kN(n,k)} :\ k \in \mathbb{N}\}$ forms a complete orthogonal system in $\mathcal{L}^2(S^{n-1})$. In particular, for every $f \in \mathcal{L}^2(S^{n-1})$, the series

$$f \sim \sum_{k=0}^{\infty} \pi_k f$$

converges to f in the $\mathcal{L}^2(S^{n-1})$-norm, where $\pi_k f \in \mathcal{H}_k^n$ is the orthogonal projection of f on the space $\mathcal{H}_k^n$. Using well-known properties of the Legendre polynomials, it is not hard to show that

$$\pi_k f = N(n, k)(f * \Lambda P_k^n). \tag{2.15}$$

This motivates the spherical expansion of a measure $\mu \in \mathcal{M}(S^{n-1})$,

$$\mu \sim \sum_{k=0}^{\infty} \pi_k \mu, \tag{2.16}$$

where $\pi_k \mu \in \mathcal{H}_k^n$ is defined by

$$\pi_k \mu = N(n,k)(\mu * \Lambda P_k^n). \tag{2.17}$$

From $P_0^n(t) = 1, N(n,0) = 1$ and $P_1^n(t) = t, N(n,1) = n$, we obtain, for $\mu \in \mathcal{M}(S^{n-1})$, the following special cases of (2.17):

$$\pi_0 \mu = \mu(S^{n-1}) \ and \ (\pi_1 \mu)(u) = n \int_{S^{n-1}} u \cdot v d\mu(v). \tag{2.18}$$

Let κ_n denote the volume of the Euclidean unit ball B. By definition (2.3) and (2.18), for every star body $K \in \mathcal{S}^n$, it follows that

$$\kappa_n \pi_0 \rho(K,\cdot)^p = \widetilde{V}_p(B,K) \ and \ \kappa_n \pi_0 \rho(K,\cdot)^{n-p} = \widetilde{V}_p(K,B). \tag{2.19}$$

A measure $\mu \in \mathcal{M}(S^{n-1})$ is uniquely determined by its series expansion (2.16). Using the fact that ΛP_k^n is (essentially) the unique zonal function in $\mathcal{H}_k^n$, a simple calculation shows that for $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ formula (2.17) becomes

$$\pi_k \mu = N(n,k) \langle \mu, \Lambda P_k^n \rangle \Lambda P_k^n. \tag{2.20}$$

Thus, a zonal measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ is defined by its so-called Legendre coefficients $\mu_k := \langle \mu, \Lambda P_k^n \rangle$. Using $\pi_k H = H$ for every $H \in \mathcal{H}_k^n$ and the fact that spherical convolution of zonal measures is commutative, we have the Funk-Hecke Theorem: If $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ and $H \in \mathcal{H}_k^n$, then $H * \mu = \mu_k H$.

A map $\Phi : \mathcal{D} \subseteq \mathcal{M}(S^{n-1}) \to \mathcal{M}(S^{n-1})$. is called a multiplier transformation[34] if there exist real numbers c_k, the multipliers of Φ, such that, for every $k \in \mathbb{N}$,

$$\pi_k \Phi \mu = c_k \pi_k \mu, \ \forall \mu \in \mathcal{D}. \tag{2.21}$$

3. L_p Radial Minkowski Homomorphisms and Convolutions

A map $\Phi_p : \mathcal{S}^n \to \mathcal{S}^n$ is called an L_p radial valuation[14]: if

$$\Phi(K \cup L) \widetilde{+}_p \Phi(K \cap L) = \Phi K \widetilde{+}_p \Phi L, \tag{3.1}$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{S}^n$.

Definition 3.1. A map $\Phi_p : \mathcal{S}^n \to \mathcal{S}^n$ satisfying properties (a), (b) and (c) from Lemma 2.2 is called an L_p radial Minkowski homomorphism.

It is easy to check that an L_p radial Minkowski homomorphism is an L_p radial valuation.

In order to prove our results, we need to quote some lemmas. We call a map $\Phi : \mathcal{C}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ monotone, if non-negative functions are mapped to non-negative ones.

Lemma 3.1. ([32]). *A map $\Phi : \mathcal{C}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ is a monotone, linear map that is $SO(n)$ equivariant if and only if there is a measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ such that*

$$\Phi f = f * \mu. \tag{3.2}$$

Lemma 3.2. ([11]). *Let $\mu_m, \mu \in \mathcal{M}(SO(n)), m = 1, 2, \cdots$ and $f \in \mathcal{C}(SO(n))$. If $\mu_m \to \mu$ weakly, then $f * \mu_m \to f * \mu$ and $\mu_m * f \to \mu * f$ uniformly.*

Theorem 3.3. *A map $\Phi_p : \mathcal{S}^n \to \mathcal{S}^n$ is an L_p radial Minkowski homomorphism if and only if there is a non-negative measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ such that*

$$\rho(\Phi_p K, \cdot)^p = \rho(K, \cdot)^{n-p} * \mu. \tag{3.3}$$

Proof. Suppose that a map $\Phi_p : \mathcal{S}^n \to \mathcal{S}^n$ satisfies $\rho(\Phi_p K, \cdot)^p = \rho(K, \cdot)^{n-p} * \mu$, where $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ is a nonnegative measure. The continuity of Φ_p follows from the fact that the radial function $\rho(K, \cdot)$ is continuous with respect to radial metric. From (2.1), for $\vartheta \in SO(n)$, we obtain

$$\rho(\Phi_p \vartheta K, \cdot)^p = \rho(\vartheta K, \cdot)^{n-p} * \mu = \rho(K, \vartheta^{-1} \cdot)^{n-p} * \mu = \rho(\Phi_p K, \vartheta^{-1} \cdot)^p = \rho(\vartheta \Phi_p K, \cdot)^p.$$

Taking $K = L$ in (3.3), we have

$$\rho(\Phi_p L, \cdot)^p = \rho(L, \cdot)^{n-p} * \mu. \tag{3.4}$$

Combining with (2.2) (3.3) and (3.4), we obtain

$$\begin{aligned} \rho(\Phi_p K \widetilde{+}_p \Phi_p L, \cdot)^p &= \rho(\Phi_p K, \cdot)^p + \rho(\Phi_p L, \cdot)^p \\ &= \rho(K, \cdot)^{n-p} * \mu + \rho(L, \cdot)^{n-p} * \mu \\ &= (\rho(K, \cdot)^{n-p} + \rho(L, \cdot)^{n-p}) * \mu \\ &= \rho(K \widetilde{+}_{n-p} L, \cdot)^{n-p} * \mu \\ &= \rho(\Phi_p (K \widetilde{+}_{n-p} L), \cdot)^p. \end{aligned} \tag{3.5}$$

Thus maps of the form of (3.3) are L_p radial Minkowski homomorphisms (satisfy the properties (a), (b)and (c) from Lemma 2.2). Thus, we have to show that for every such operator Φ_p, there is a measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ such that (3.3) holds.

Since every positive continuous function on S^{n-1} can be the radial function of some star body, the vector space $\{\rho(K,\cdot)^{n-p}-\rho(L,\cdot)^{n-p}:\ K,L\in\mathcal{S}^n\}$ coincides with $\mathcal{C}(S^{n-1})$. The operator $\bar\Phi:\mathcal{C}(S^{n-1})\to\mathcal{C}(S^{n-1})$ is defined by

$$\bar\Phi f=\rho(\Phi_pK_1,\cdot)^p-\rho(\Phi_pK_2,\cdot)^p, \tag{3.6}$$

where $f=\rho(K_1,\cdot)^{n-p}-\rho(K_2,\cdot)^{n-p}$.
The operator $\bar\Phi$ for $g=\rho(L_1,\cdot)^{n-p}-\rho(L_2,\cdot)^{n-p}$ immediately yields:

$$\bar\Phi g=\rho(\Phi_pL_1,\cdot)^p-\rho(\Phi_pL_2,\cdot)^p. \tag{3.7}$$

Combining with (3.6), (3.7), (2.2), and (3.5), we obtain

$$\begin{aligned}\bar\Phi f+\bar\Phi g&=\rho(\Phi_pK_1,\cdot)^p-\rho(\Phi_pK_2,\cdot)^p+\rho(\Phi_pL_1,\cdot)^p-\rho(\Phi_pL_2,\cdot)^p\\&=\rho(\Phi_pK_1\widetilde{+}_p\Phi_pL_1,\cdot)^p-\rho(\Phi_pK_2\widetilde{+}_p\Phi_pL_2,\cdot)^p\\&=\rho(\Phi_p(K_1\widetilde{+}_{n-p}L_1),\cdot)^p-\rho(\Phi_p(K_2\widetilde{+}_{n-p}L_2),\cdot)^p\\&=\bar\Phi(\rho(K_1\widetilde{+}_{n-p}L_1,\cdot)^{n-p}-\rho(K_2\widetilde{+}_{n-p}L_2,\cdot)^{n-p})\\&=\bar\Phi(\rho(K_1,\cdot)^{n-p}+\rho(L_1,\cdot)^{n-p}-\rho(K_2,\cdot)^{n-p}-\rho(L_2,\cdot)^{n-p})\\&=\bar\Phi(f+g)\end{aligned}.$$

So the operator $\bar\Phi$ is linear.
Noting that Φ_p is an L_p radial Minkowski homomorphism and $\vartheta f(u)=f(\vartheta^{-1}u)$, we obtain that the operator $\bar\Phi$ is $SO(n)$ equivariant.
Since the cone of radial functions is invariant under $\bar\Phi$, it is also monotone. Hence, by Lemma 3.1, there is a non-negative measure $\mu\in\mathcal{M}(S^{n-1},\widehat{e})$ such that $\bar\Phi f=f*\mu$. The statement now follows from $\bar\Phi\rho(K,\cdot)^{n-p}=\rho(\Phi_pK,\cdot)^p$. $\Box$

For example, the generating measure of the quasi-L_p intersection body operator $I_p:\mathcal{S}^n\to\mathcal{S}^n$ is the invariant measure $\mu_{S_0^{n-2}}$ concentrated on $S_0^{n-2}=S^{n-1}\cap\widehat{e}^{\perp}$ with total mass $(n-1)\omega_{n-1}$:

$$\rho(I_pK,\cdot)^p=\rho(K,\cdot)^{n-p}*\mu_{S_0^{n-2}}.$$

Theorem 3.4. *If $\Phi_p:\mathcal{S}^n\to\mathcal{S}^n$ is an L_p radial Minkowski homomorphism, then, for $K,L\in\mathcal{S}^n$,*

$$\widetilde{V}_p(K,\Phi_pL)=\widetilde{V}_p(L,\Phi_pK). \tag{3.8}$$

Proof. Let $\mu\in\mathcal{M}(S^{n-1},\widehat{e})$ be the generating measure of Φ_p. Applying definition (2.3), Theorem 3.3 and (2.11), it follows that

$$\begin{aligned}\widetilde{V}_p(K,\Phi_pL) &= \kappa_n\langle\rho(\Phi_pL,\cdot)^p,\rho(K,\cdot)^{n-p}\rangle\\ &= \kappa_n\langle\rho(L,\cdot)^{n-p}*\mu,\rho(K,\cdot)^{n-p}\rangle\\ &= \kappa_n\langle\rho(L,\cdot)^{n-p},\rho(K,\cdot)^{n-p}*\mu\rangle\\ &= \kappa_n\langle\rho(L,\cdot)^{n-p},\rho(\Phi_pK,\cdot)^{p}\rangle\\ &= \widetilde{V}_p(L,\Phi_pK).\end{aligned}$$ ■

Using Theorem 3.3 and the fact that spherical convolution operators are multiplier transformations, one obtains that

Lemma 3.5. *If Φ_p is an L_p radial Minkowski homomorphism which is generated by the zonal measure μ, then, for every star body $K\in\mathcal{S}^n$,*

$$\pi_k\rho(\Phi_pK,\cdot)^p=\mu_k\pi_k\rho(K,\cdot)^{n-p}, \tag{3.9}$$

where the numbers μ_k are the Legendre coefficients of μ.

Definition 3.2. If Φ_p is an L_p radial Minkowski homomorphism, generated by the zonal measure μ, then we call the subset $\mathcal{S}^n(\Phi_p)$ of $\mathcal{S}^n$, defined by

$$\mathcal{S}^n(\Phi_p)=\{K\in\mathcal{S}^n:\pi_k\rho(K,\cdot)^{n-p}=o\ if\ \mu_k=0\}, \tag{3.10}$$

the injectivity set of Φ_p. It is easy to verify that for every L_p radial Minkowski homomorphism, the set $\mathcal{S}^n(\Phi_p)$ is a non-empty rotation and dilatation invariant subset of which is closed under L_p radial sum. By Lemma 3.5, a star body $K\in\mathcal{S}^n(\Phi_p)$ is uniquely determined by its image Φ_pK.

Definition 3.3. A star body $K\in\mathcal{S}^n$ is called p-polynomial if $\rho(K,\cdot)^p\in\mathcal{H}^n$. Clearly, the set of p-polynomial star bodies is dense in $\mathcal{S}^n$ and the set of all origin-symmetric polynomial star bodies is dense in $\mathcal{S}^n_e$.

Theorem 3.6. *If $\Phi_p:\mathcal{S}^n\to\mathcal{S}^n$ is an L_p radial Minkowski homomorphism such that $\mathcal{S}^n_e\subseteq\mathcal{S}^n(\Phi_p)$, then for every p-polynomial star body $L\in\mathcal{S}^n_e$, there exist origin-symmetry star bodies $K_1,K_2\in\mathcal{S}^n_e$ such that*

$$L\widetilde{+}_p\Phi K_1=\Phi K_2. \tag{3.11}$$

Proof. Let $L\in\mathcal{S}^n$ be a p-polynomial star body. From definition (3.3) we have

$$\rho(L,\cdot)^p=\sum_{k=0}^{m}\pi_k\rho(L,\cdot)^p. \tag{3.12}$$

Since $L \in \mathcal{S}_e^n$ and by the properties of the orthogonal projection of f on the space $\mathcal{H}_k^n$, we have $\pi_k \rho(L, \cdot)^p = 0$ for all odd $k \in \mathbb{N}$.
Let $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ be the generating measure of Φ_p and let μ_k denote the Legendre coefficients of μ. From $\mathcal{S}_e^n \subseteq \mathcal{S}^n(\Phi_p)$ and definition (3.2), it follows that $\mu_k \neq 0$ for every even $k \in \mathbb{N}$. We define

$$f := \sum_{k=0}^{m} c_k \pi_k \rho(L, \cdot)^p, \tag{3.13}$$

where $c_k = 0$ for odd and $c_k = \mu_k^{-1}$ if k is even. Clearly, f is an even continuous function on S^{n-1} and since spherical convolution operators are multiplier transformations, one can obtain

$$f * \mu = \sum_{k=0}^{m} c_k \mu_k \pi_k \rho(L, \cdot)^p = \sum_{k=0}^{m} \pi_k \rho(L, \cdot)^p = \rho(L, \cdot)^p. \tag{3.14}$$

Denote by f^+ and f^- the positive and negative parts of f and let K_1 and K_2 be the star bodies such that $\rho(K_1, \cdot)^{n-p} = f^-$ and $\rho(K_2, \cdot)^{n-p} = f^+$. Hence, (3.14) can be rewritten as

$$\rho(K_2, \cdot)^{n-p} * \mu = \rho(K_1, \cdot)^{n-p} * \mu + \rho(L, \cdot)^p.$$

By Theorem 3.3, it follows that

$$L \widetilde{+}_p \Phi_p K_1 = \Phi_p K_2. \qquad \blacksquare$$

4. Main Results

Let $\Phi_p : \mathcal{S}^n \to \mathcal{S}^n$ be a non-trivial L_p radial Minkowski homomorphism, i.e., Φ_p is a continuous and $SO(n)$ equivariant map satisfying $\Phi_p(K \widetilde{+}_{n-p} L) = \Phi_p K \widetilde{+}_p \Phi_p L$ and Φ_p does not map every star body to the origin. In this section, we study the Busemann-Petty type problem for L_p radial Minkowski homomorphisms.

Problem 4.1. Let K and L be star bodies in $\mathcal{S}^n$, is there the implication:
If $0 < p < n$, then

$$\Phi_p K \subseteq \Phi_p L \;\Rightarrow\; V(K) \leq V(L)? \tag{4.1}$$

If $p > n$, then

$$\Phi_p K \subseteq \Phi_p L \;\Rightarrow\; V(K) \geq V(L)? \tag{4.2}$$

Proof of Theorem 1.1. For $K \in \Phi_p \mathcal{S}^n$, there exists a star body K_0 such that $K = \Phi_p K_0$. Using Lemma (3.3) and the fact that if $0 < p < n$, the L_p dual mixed volume $\widetilde{V}_p$ is monotone with respect to set inclusion, we can conclude

$$\widetilde{V}_p(L,K)=\widetilde{V}_p(L,\Phi_p K_0)=\widetilde{V}_p(K_0,\Phi_p L)\geq \widetilde{V}_p(K_0,\Phi_p K)=\widetilde{V}_p(K,\Phi_p K_0)=V(K).$$

Applying the Minkowski inequality (2.5), we obtain

$$V(K)\leq V(L).$$

Note that if $p>n$, we only need to consider $\widetilde{V}_p(K,L)$. And the same argument yields: if $p>n$,

$$\Phi_p K\subseteq \Phi_p L \ \Rightarrow\ V(K)\geq V(L).$$

Equality holds if and only if K and L are dilatations of each other. Clearly, star bodies of equal volume which are dilatations of each other must be equal. ■

Unfortunately, whether the set of L_p radial Minkowski homomorphisms coincides with the set of continuous radial valuations which are $SO(n)$ equivariant and $(\frac{n}{p}-1)$- homogeneous is not known.

Proof of Theorem 1.2. Let $\mu\in\mathcal{M}(S^{n-1},\widehat{e})$ be the generating measure of Φ_p and μ_k denote its Legendre coefficients. Since $\mathcal{S}^n(\Phi_p)\neq\mathcal{S}^n$ and Φ_p is non-trivial, by definition (3.2) there exists an integer $k\in\mathbb{N}$ and $k\geq 1$ such that $\mu_k=0$. We can choose $\alpha>0$ such that the function $f(u)=1+\alpha P_k^n(u\cdot\widehat{e}), u\in S^{n-1}$, is positive. Let $K\in\mathcal{S}^n$ be the star body with $\rho(K,\cdot)^{n-p}=f$. Since $\pi_k\rho(K,\cdot)^{n-p}=\pi_k(1+\alpha P_k^n(u\cdot\widehat{e}))\neq 0$, from definition (3.2) we have $K\notin\mathcal{S}^n(\Phi_p)$.
From (2.19) and the properties of the orthogonal projection on the space $\mathcal{H}_k^n$, we have

$$\widetilde{V}_p(K,B)=\kappa_n\pi_0\rho(K,\cdot)^{n-p}=\kappa_n=V(B). \tag{4.3}$$

Using the fact a star body $K\in\mathcal{S}^n(\Phi_p)$ is uniquely determined by its image $\Phi_p K$, we see that $\Phi_p B=\Phi_p K$.
If $0<p<n$, noting that K is just a perturbation of B, we use (4.3) and the Minkowski inequality (2.5) to get

$$V(B)=\widetilde{V}_p(K,B)<V(K)^{\frac{n-p}{n}}V(B)^{\frac{p}{n}}.$$

Hence

$$V(B)<V(K).$$

If $p>n$, the same argument yields:

$$V(B)>V(K).$$ ■

Theorem 4.1. *Suppose $\mathcal{S}_e^n\subseteq\mathcal{S}^n(\Phi_p)$ and $0<p<n$. If $L\in\mathcal{S}_e^n$ is a p-polynomial star body whose radial function is positive, then, if $L\notin\Phi_p\mathcal{S}^n$, there exists a star body $K\in\mathcal{S}_e^n$, such that*

$$\Phi_p K\subseteq\Phi_p L,$$

but

$$V(K) > V(L).$$

Proof. Let $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ be the generating measure of Φ_p. Since $L \in \mathcal{S}_e^n$ is p-polynomial, it follows from the proof Theorem 3.6 that there exists an even function $f \in \mathcal{H}^n$,such that

$$\rho(L, \cdot)^p = f * \mu. \tag{4.4}$$

The function must assume negative values, otherwise, by Lemma 3.2 we have $L = \Phi_p L_0$, where L_0 is the star body with $\rho(L_0, \cdot)^{n-p} = f$. Let $F \in \mathcal{C}(S^{n-1})$ be a non-constant even function, such that $F(u) \geq 0$ if $f(u) < 0$, and $F(u) = 0$ if $f(u) \geq 0$. By suitable approximation of the function F with spherical harmonics, we can find a non-negative, even function $G \in \mathcal{H}^n$ and an even function $H \in \mathcal{H}^n$ such that

$$\langle f, G \rangle < 0, \ \textit{and}\ G = H * \mu. \tag{4.5}$$

Since the radial function $\rho(L, \cdot)$ is positive, there exists a $\beta > 0$ and an origin-symmetric star body K such that

$$\rho(K, \cdot)^{n-p} = \rho(L, \cdot)^{n-p} - \beta H. \tag{4.6}$$

From (4.4) and Theorem 3.3, we see that $\rho(\Phi_p K, \cdot)^p = \rho(\Phi_p L, \cdot)^p - \beta G$. Since $G \geq 0$, it follows that

$$\rho(\Phi_p K, \cdot) \leq \rho(\Phi_p L, \cdot), \tag{4.7}$$

or equivalently

$$\Phi_p K \subseteq \Phi_p L.$$

On the other hand, applying (2.3) (4.4) (4.6) and (2.11), we obtain

$$\begin{aligned} V(L) - \widetilde{V}_p(K, L) &= \frac{1}{n} \int_{S^{n-1}} \rho(L, \cdot)^p (\rho(L, \cdot)^{n-p} - \rho(K, \cdot)^{n-p}) dS(u) \\ &= \kappa_n \beta \langle f * \mu, H \rangle \\ &= \kappa_n \beta \langle f, H * \mu \rangle \\ &= \kappa_n \beta \langle f, G \rangle \\ &< 0. \end{aligned} \tag{4.8}$$

To complete the proof, we can use (2.5) to conclude

$$V(K) > V(L). \qquad \blacksquare$$

Combining Theorems 1.1, 1.2 and 4.1, we obtain

Corollary 4.2. *For origin-symmetric star bodies in* $\mathcal{S}^n$, *when* $0 < p < n$, *Problem 4.1 has an affirmative answer if and only if every polynomial star body* $L \in \mathcal{S}_e^n$ *with positive radial function is contained in* $\Phi_p \mathcal{S}^n$.

If we restrict to origin-symmetric convex bodies and $p = 1$, Problem 4.1 is just the well-known Busemann-Petty problem.

Remark. If $p = 1$, Ludwig completely characterized the intersection body operator: A map $\Phi : \mathcal{S}^n \to \mathcal{S}^n$ is a continuous $GL(n)$ contravariant radial valuation if and only if there exists a constant $c \geq 0$ such that $\Phi = cI$. However, for radial valuations which are $SO(n)$ equivariant, the following conjecture is still open (see [34]): The set of radial Blaschke-Minkowski homomorphisms coincides with the set of continuous radial valuations which are $SO(n)$ equivariant and $(n-1)$-homogeneous.

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