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MULTIPLE SOLUTIONS FOR A NONLINEAR ELLIPTIC SYSTEM SUBJECT TO NONAUTONOMOUS PERTURBATIONS

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Abstract. In this paper we consider the following Neumann problem

$$\begin{cases} -\Delta u = \alpha(x)(F_u(u,v)-u)+\lambda G_u(x,u,v) \text{ in } \Omega \\ -\Delta v = \alpha(x)(F_v(u,v)-v)+\lambda G_v(x,u,v) \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases}$$

In particular, by means of a multiplicity theorem obtained by Ricceri, we establish that if the set of all global minima of the function $\mathbb{R}^2 \ni y \longmapsto \frac{|y|^2}{2} - F(y)$ (where $F \in C^1(\mathbb{R}^2)$ and it satisfies the condition $F(0,0)=0$) has at least $k \geq 2$ connected components, then the above Neumann problem admits at least $k+1$ weak solutions, k of which are lying in a given set.

1. Introduction

In this paper we study the existence and the multiplicity of the solutions for the following Neumann problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \alpha(x)(F_u(u,v)-u)+\lambda G_u(x,u,v) \text{ in } \Omega \\ -\Delta v = \alpha(x)(F_v(u,v)-v)+\lambda G_v(x,u,v) \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases}$$

(ν being the outer unit normal to $\partial\Omega$) by using the information about the number of the connected components of the set of all global minima of a given function, as the Ricceri's result [11] asserts. The analogous Neumann problem for one equation

$$(Q_\lambda) \qquad \begin{cases} -\Delta u = \alpha(x)(f(u)-u)+\lambda g(x,u) \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases}$$

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where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is an open, bounded and connected set, having boundary of class C^2, has been widely studied by Ricceri in [10]. In fact by assuming that the set of all global minima of the function $\mathbb{R} \ni \xi \longmapsto \frac{\xi^2}{2} - \int_0^\xi f(t)dt$ (here $f \in C^0(\mathbb{R})$ and satisfies the condition $\lim_{|\xi|\to+\infty} \frac{f(\xi)}{\xi} = 0$) has at least $k \geq 2$ connected components, the author finds that, for every $\alpha \in L^\infty(\Omega)$ and for every Carathéodory function $g : \Omega \times \mathbb{R} \longmapsto \mathbb{R}$ such that $\sup_{|\xi|\leq s} |g(\cdot, \xi)| \in L^p(\Omega)$ for some $p > n$ and for all $s > 0$, the problem (Q_λ) admits at least $k + 1$ strong solutions, even in $W^{2,p}(\Omega)$. There is a wide literature dealing with multiple solutions for nonlinear elliptic problems. For instance, Neumann either non perturbed or perturbed problems, including the p-Laplacian but also the $p(x)$-Laplacian, have been studied, by using the variational principle of Ricceri [12], in recent years from different authors (see, for instance [1-9], [13]). In the present paper we intend to prove a multiplicity result of this type: for each integer $k > 1$, there is $\lambda^* > 0$ such that problem (P_λ) has at least $k + 1$ solutions for all $\lambda \in]0, \lambda^*[$. Our approach is based exactly on a general multiplicity theorem, which for the convenience of the reader, we state as follows:

Theorem A. ([11] Theorem 8). *Let X be a reflexive and separable real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and continuously Gâteaux-differentiable functionals, with Ψ also coercive. Let us assume that the functional $\Psi + \lambda\Phi$ satisfies the Palais-Smale condition for every $\lambda > 0$ small enough and that the set of all global minima of Ψ has at least k connected components in the weak topology, with $k \geq 2$.*

Then, for each $\rho > \inf_X \Psi$, there exists $\lambda^* > 0$ such that, for every $\lambda \in]0, \lambda^*[$, the functional $\Psi + \lambda\Phi$ has at least $k+1$ critical points, k of which are lying in $\Psi^{-1}(]-\infty, \rho[)$.

2. Results

From now on, $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ will be an open, bounded and connected set, with boundary of class C^1. We shall consider the $W^{1,2}(\Omega)$ Sobolev space with the norm $\| u \| = (\int_\Omega (|\nabla u(x)|^2 + |u(x)|^2)dx)^{\frac{1}{2}}$ and the $[W^{1,2}(\Omega)]^2$ product space with the norm $\| (u, v) \| = \sqrt{\| u \|^2 + \| v \|^2}$. As usual, a weak solution of the problem (P_λ) is any $(u, v) \in [W^{1,2}(\Omega)]^2$ such that

$$\int_\Omega [\nabla u(x) \cdot \nabla \omega(x) + \nabla v(x) \cdot \nabla \phi(x) + \alpha(x)(u(x)\omega(x) + v(x)\phi(x))]dx$$
$$- \int_\Omega \alpha(x)[F_u(u(x), v(x))\omega(x) + F_v(u(x), v(x))\phi(x)]dx$$
$$-\lambda \int_\Omega [G_u(x, u(x), v(x))\omega(x) + G_v(x, u(x), v(x))\phi(x)]dx = 0$$

for all $(\omega, \phi) \in [W^{1,2}(\Omega)]^2$.

Theorem 2.1. *Let us assume that*

(i_1) $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is a continuously differentiable function such that* $F(0,0) = 0$; *moreover, the following condition holds*

(1) $$\lim_{|y|\to+\infty} \frac{|\nabla F(y)|}{|y|} = 0$$

(i_2) $G : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is a measurable function in* $x \in \Omega$ *for each* $y \in \mathbb{R}^2$, *such that* $G(x, 0, 0) = 0$, *for every* $x \in \Omega$, *and continuously differentiable in* y *for a.e.* $x \in \Omega$; *furthermore* G_u *and* G_v *are bounded functions in* $\mathbb{R}^2$ *for every* $x \in \Omega$ *with* $\sup_{y\in\mathbb{R}^2} |G_u(\cdot, y)|$ *and* $\sup_{y\in\mathbb{R}^2} |G_v(\cdot, y)|$ *functions of* $L^2(\Omega)$.

(i_3) *the set of all global minima of the function* $\mathbb{R}^2 \ni y \longmapsto \frac{|y|^2}{2} - F(y)$, *has at least* k *connected components with* $k \geq 2$.

Then, for every $\alpha \in L^\infty(\Omega)$, *with* $\operatorname{ess\,inf}_\Omega \alpha > 0$ *and for every number* ρ *satisfying*

$$\rho > \| \alpha \|_{L^1(\Omega)} \inf_{y\in\mathbb{R}^2} \left(\frac{|y|^2}{2} - F(y) \right)$$

there exists $\lambda^* > 0$ *such that, for each* $\lambda \in]0, \lambda^*[$, *the Neumann problem* (P_λ) *admits at least* $k+1$ *weak solutions*, k *of which belong to the set*

$$\left\{ (u, v) \in [W^{1,2}(\Omega)]^2 : \frac{1}{2} \int_\Omega (|\nabla u(x)|^2 + |\nabla v(x)|^2) dx + \right.$$
$$\left. + \int_\Omega \alpha(x) \left(\frac{|u(x)|^2 + |v(x)|^2}{2} - F(u(x), v(x)) \right) dx < \rho \right\}.$$

Proof. For the convenience of the reader, in the proof procedure of a such theorem we'll check, step by step, the hypotheses of Theorem **A**. First of all, we define the functionals $I,\ J,\ H : [W^{1,2}(\Omega)]^2 \longrightarrow \mathbb{R}$ as

$$I(u, v) = \frac{1}{2} \int_\Omega [|\nabla u(x)|^2 + |\nabla v(x)|^2 + \alpha(x)(|u(x)|^2 + |v(x)|^2)] dx,$$
$$J(u, v) = -\int_\Omega \alpha(x) F(u(x), v(x) dx$$

and

$$H(u, v) = -\int_\Omega G(x, u(x), v(x)) dx.$$

We put $\Im = I + J$ and we choose as X the $[W^{1,2}(\Omega)]^2$ Banach space and as functionals Ψ and Φ, $\Im$ and H respectively which are both well defined. From (1) follows, indeed, that for every $\epsilon > 0$ there exists $r > 0$ such that if $y \in \mathbb{R}^2$ and $|y| > r$ then $|\nabla F(y)| < \epsilon|y|$. Thus, for every $\epsilon > 0$, there exists $M > 0$ such that $|\nabla F(y)| < \epsilon|y| + M$ for every $y \in \mathbb{R}^2$. Then, by the Mean Value Theorem, it follows

$$|F(y)| < \epsilon|y|^2 + M|y| \tag{2}$$

for every $y \in \mathbb{R}^2$. In virtue of Lebesgue's dominated convergence theorem, the (2) ensures that the functional J is sequentially weakly continuous. Analogously, the assumption (i_2) implies that H is sequentially weakly continuous too. Moreover, by known results, the functional I is sequentially weakly lower semi-continuous, being convex and continuous, whence $\Im$ is sequentially weakly lower semi-continuous, as well as H. Finally, (i_1) and (i_2) ensure that $\Im$ and H are continuously Gâteaux-differentiable functionals with compact derivatives. Since the expressions of $\Im'$ and H' at any $(u, v) \in X$ are given as

$$\begin{aligned}
&\Im'(u,v)(\omega,\phi) \\
= &\int_\Omega [\nabla u(x)\cdot\nabla\omega(x) + \nabla v(x)\cdot\nabla\phi(x) + \alpha(x)(u(x)\omega(x) + v(x)\phi(x))]dx \\
&-\int_\Omega \alpha(x)[F_u(u(x),v(x))\omega(x) + F_v(u(x),v(x))\phi(x)]dx
\end{aligned}$$

and

$$H'(u,v)(\omega,\phi) = -\int_\Omega [G_u(x,u(x),v(x))\omega(x) + G_v(x,u(x),v(x))\phi(x)]\,dx,$$

for every $(\omega, \phi) \in X$, it is easy to show that the critical points of the functionals $\Im$ and H are rightly the weak solutions of (P_λ). We now prove that $\Im$ is also coercive. For this purpose, we choose $\epsilon > 0$ as in (2) and we observe that

$$\begin{aligned}
\int_\Omega \alpha(x)F(u(x),v(x))dx &\le \epsilon\int_\Omega \alpha(x)(|u(x)|^2 + |v(x)|^2)dx \\
&+M\int_\Omega \alpha(x)(|u(x)| + |v(x)|)dx \\
&\le \epsilon \parallel \alpha \parallel_\infty \parallel (u,v) \parallel^2 + M_1 \parallel \alpha \parallel_\infty \parallel (u,v) \parallel
\end{aligned}$$

Therefore, putting $C_1 = \frac{1}{2}\min\{1,\ \operatorname{ess\,inf}_\Omega \alpha\}$ one obtaines

$$\Im(u,v) \ge \left(C_1 - \epsilon \parallel \alpha \parallel_\infty\right) \parallel (u,v) \parallel^2 - M_1 \parallel \alpha \parallel_\infty \parallel (u,v) \parallel$$

whence follows

$$\lim_{\|(u,v)\|\to+\infty} \Im(u,v) = +\infty$$

provided that $0 < \epsilon < C_1/\| \alpha \|_\infty$. We check that the functional $\Im + \lambda H$ satisfies Palais-Smale's property; at first we observe that

$$\begin{aligned}
&\int_\Omega G(x,u(x),v(x))dx \\
&= \int_\Omega \Big(\int_0^1 [G_u(x,tu(x),tv(x))u(x) + G_v(x,tu(x),tv(x))v(x)]dt \Big) dx \\
&\le \int_\Omega \sup_{y\in\mathbb{R}^2} |G_u(x,y)||u(x)|dx + \int_\Omega \sup_{y\in\mathbb{R}^2} |G_v(x,y))||v(x)|dx \\
&\le \Big(\int_\Omega \sup_{y\in\mathbb{R}^2} |G_u(x,y)|^2 dx \Big)^{\frac{1}{2}} \Big(\int_\Omega |u(x)|^2 dx \Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_\Omega \sup_{y\in\mathbb{R}^2} |G_v(x,y)|^2 dx \Big)^{\frac{1}{2}} \Big(\int_\Omega |v(x)|^2 dx \Big)^{\frac{1}{2}}
\end{aligned}$$

and so by putting

$$C_2 = \max\left[\Big(\int_\Omega \sup_{y\in\mathbb{R}^2} |G_u(x,y)|^2 dx \Big)^{\frac{1}{2}}, \Big(\int_\Omega \sup_{y\in\mathbb{R}^2} |G_v(x,y)|^2 dx \Big)^{\frac{1}{2}} \right]$$

one has

$$H(u,v) \ge -C_2(\| u \| + \| v \|).$$

Therefore, for each $\lambda \ge 0$, we clearly have

$$(\Im + \lambda H)(u,v) \ge (C_1 - \epsilon \| \alpha \|_\infty) \| (u,v) \|^2 - M_1 \| \alpha \|_\infty \| (u,v) \| \\ -\lambda C_2(\| u \| + \| v \|)$$

whence follows that

$$\lim_{\|(u,v)\|\to+\infty} \Im(u,v) + \lambda H(u,v) = +\infty.$$

Thus, $\Im + \lambda H$ satisfies Palais-Smale's condition, as it is the sum of I, whose derivative is a homeomorphism between $[W^{1,2}(\Omega)]^2$ and its dual (see [4] and references therein), and of a functional $J + \lambda H$ with compact derivative in virtue of the conditions imposed on F and G. Finally, we prove that the set of all global minima of $\Im$ has at least k weakly connected components, with $k \ge 2$.
At first, let us observe that the function $y \longmapsto \frac{|y|^2}{2} - F(y)$ is coercive as it comes immediately from (2), for a convenient choice of ϵ. Thus, the set

$$\left\{ (s,r) \in \mathbb{R}^2 : \frac{s^2+r^2}{2} - F(s,r) = \inf_{y\in\mathbb{R}^2} \left(\frac{|y|^2}{2} - F(y) \right) \right\}$$

that we denote by $\mathfrak{M}$ is not empty and, by assumption, has at least k connected components. For each $(u,v) \in [W^{1,2}(\Omega)]^2$, one clearly has

$$\Im(u,v) \geq \inf_{y\in\mathbb{R}^2}\left(\frac{|y|^2}{2} - F(y)\right) \parallel \alpha \parallel_{L^1(\Omega)}$$

and if $(s_0, r_0) \in \mathfrak{M}$, the functions, defined by putting $u_0(x) = s_0$ and $v_0(x) = r_0$, both belong to $[W^{1,2}(\Omega)]^2$ holding the equality

$$\Im(u_0,v_0) = \inf_{y\in\mathbb{R}^2}\left(\frac{|y|^2}{2} - F(y)\right) \parallel \alpha \parallel_{L^1(\Omega)} .$$

If $(u,v) \in [W^{1,2}(\Omega)]^2$, with u or v not constant, as Ω is connected, one has $|\nabla u| > 0$ in some set Ω_1 of positive measure or $|\nabla v| > 0$ in some set Ω_2 of positive measure. Consequently, the pair (u,v) can not be a global minimum of $\Im$. Let $\Upsilon : \mathbb{R}^2 \longmapsto [W^{1,2}(\Omega)]^2$ the mapping that at each $(b,c) \in \mathbb{R}^2$ associates the pair (u,v) of the equivalence classes with u and v everywhere equal in Ω to b and c respectively. Since Υ is a homeomorphism between $\mathbb{R}^2$ and $\Upsilon(\mathbb{R}^2)$, endowed with the relativization of the weak topology on $[W^{1,2}(\Omega)]^2$, $\Upsilon(\mathfrak{M})$ is the set of all global minima of $\Im$ and then it has at least k weakly connected components. The proof is now complete.

Remark 2.1. We observe that the same conclusion of the Theorem 2.1. holds by replacing in (1) a limit $\ell < C_1/\parallel \alpha \parallel_\infty$. In such a case, in order to prove the coerciveness of the functional $\Im$, we have to choose $0 < \epsilon < (C_1/\parallel \alpha \parallel_\infty) - \ell$.

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