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CLASSIFICATION OF RULED SURFACES WITH POINTWISE 1-TYPE GAUSS MAP IN MINKOWSKI 3-SPACE

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Abstract. We study the ruled surfaces in Minkowski 3-space with pointwise 1-type Gauss map. As a result, we introduce some new examples of the ruled surfaces with pointwise 1-type Gauss map in Minkowski 3-space.

1. Introduction

The notion of finite type immersions has played an important role in classifying and characterizing the submanifolds in Euclidean space or pseudo-Euclidean space since it was introduced by B.-Y. Chen in the late 1970s ([3, 4]). Also, we can apply it to the smooth maps in Euclidean space or pseudo-Euclidean space naturally ([1, 2, 6]). A smooth map ϕ of a submanifold M of Euclidean space or pseudo-Euclidean space is said to be of *finite type* if ϕ can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of M, that is, $\phi = \phi_0 + \sum_{i=1}^{k} \phi_i$, where ϕ_0 is a constant function, $\phi_i (i = 1, \cdots, k)$ non-constant functions satisfying $\Delta\phi_i = \lambda_i\phi_i$, $\lambda_i \in \mathbb{R}$.

In this regards, it is worth investigating the classification of the submanifolds in Euclidean space or pseudo-Euclidean space in terms of finite type Gauss map. In general, the Laplacian of 1-type Gauss map of a submanifold in Euclidean or pseudo-Euclidean space satisfies $\Delta G = \lambda(G+\mathbb{C})$ for some constant λ and a constant vector $\mathbb{C}$.

On the other hand, the Gauss map of some minimal surfaces such as a helicoid of the first kind, a helicoid of the second kind, a helicoid of the third kind and the conjugate of Enneper's surface of the second kind in Minkowski 3-space satisfies $\Delta G = fG$ for some smooth function f. It looks like an eigenvalue problem but the

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function f turns out to be non-constant for such surfaces. For this reason, the notion of pointwise 1-type Gauss map in Euclidean space or pseudo-Euclidean space was initiated: A submanifold M in Euclidean space $\mathbb{E}^m$ or pseudo-Euclidean space $\mathbb{E}^m_s$ of index s is said to have *pointwise 1-type Gauss map* if

$$\Delta G = f(G + \mathbb{C}) \tag{1.1}$$

for a nonzero smooth function f and some constant vector $\mathbb{C}$. In particular, if $\mathbb{C}$ is zero, it is said to be *of the first kind*. Otherwise, it is said to be *of the second kind* ([5, 7, 8, 9, 10, 11, 12, 14, 15]).

Recently, the present authors have introduced some new examples of the ruled surfaces with pointwise 1-type Gauss map of the second kind in $\mathbb{E}^3$ called a cylinder of an infinite type and a rotational ruled surface ([10]). Two of the present authors gave the classification of the ruled surfaces with pointwise 1-type Gauss map of the first kind in Minkowski 3-space $\mathbb{E}^3_1$ ([14]).

In the present paper, we mainly focus on a ruled surface in Minkowski 3-space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the second kind. In fact, the class of solution spaces of equation (1.1) could be very big because it could have infinitely many solutions associated with a function f and a constant vector $\mathbb{C}$.

As a consequence, by combining the results in [14], we give a complete classification of ruled surfaces with pointwise 1-type Gauss map in Minkowski 3-space $\mathbb{E}^3_1$:

Theorem A (Classification). *Let M be a ruled surface in Minkowski 3-space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map. Then, M is an open part of a Euclidean plane, a Minkowski plane, a hyperbolic cylinder, a Lorentz circular cylinder, a circular cylinder of index 1, a cylinder of an infinite type, a helicoid of the first kind, a helicoid of the second kind, a helicoid of the third kind, the conjugate of Enneper's surface of the second kind, a rotational ruled surface of type I or type II, a transcendental ruled surface, or a B-scroll.*

2. Preliminaries

Let $\mathbb{E}^3_1$ be Minkowski 3-space with the Lorentz metric $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) denotes the standard coordinate system in $\mathbb{E}^3_1$. Let M be a non-degenerate connected surface in $\mathbb{E}^3_1$. The map $G : M \to Q^2(\epsilon) \subset \mathbb{E}^3_1$ which maps each point of M to the unit normal vector to M at the point is called the *Gauss map* of M, where ϵ $(= \pm 1)$ denotes the sign of the vector field G and $Q^2(\epsilon)$ is a 2-dimensional space form with constant sectional curvature ϵ.

Now, we define a ruled surface M in Minkowski 3-space $\mathbb{E}^3_1$. Let I and J be some open intervals in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve in $\mathbb{E}^3_1$ defined on I

and $\beta = \beta(s)$ a transversal vector field with $\alpha'(s)$ along α. Then, a parametrization of a ruled surface M is given by

$$x(s,t) = \alpha(s) + t\beta(s), \quad s \in I, \quad t \in J.$$

The curve $\alpha = \alpha(s)$ is called a *base curve* and $\beta = \beta(s)$ a *director vector field*. In particular, if β is constant, M is said to be *cylindrical*. Otherwise, it is said to be *non-cylindrical*.

First, we consider a base curve α is space-like or time-like. Then, the base curve α can be chosen to be orthogonal to the director vector field β which can be normalized as $\langle \beta, \beta \rangle = \pm 1$. In this case, we have five different types according to the character of α and β as follows: According as the base curve α is space-like or time-like, the ruled surface M is said to be of type M_+ or M_-, respectively. Also, the ruled surface of type M_+ can be divided into three types. If β is space-like, it is said to be of type M_+^1 or M_+^2 if β' is non-null or null, respectively. When β is time-like, β' is space-like because of the causal vector of β, which is said to be of type M_+^3. On the other hand, when α is time-like, β is always space-like. Accordingly, it is also said to be of type M_-^1 or M_-^2 if β' is non-null or null, respectively. The ruled surface of type M_+^1 or M_+^2 (resp. M_+^3, M_-^1 or M_-^2) is clearly space-like (resp. time-like).

A curve in $\mathbb{E}_1^3$ is said to be *null or light-like* if its tangent vector field is null along it. If the base curve α is null and the director vector field β along α is null, then the ruled surface M is called a *null scroll*. It is evidently a time-like surface.

Other cases such as α is non-null and β is null, or α is null and β is non-null are reduced to one of the types $M_\pm^1$, $M_\pm^2$ and M_+^3, or a null scroll by an appropriate change of the base curve ([13]).

3. Cylindrical Ruled Surfaces

In this section, we examine the cylindrical ruled surfaces with pointwise 1-type Gauss map of the second kind in Minkowski 3-space.

Let M be a cylindrical ruled surface in Minkowski 3-space $\mathbb{E}_1^3$ of type M_+^1, M_-^1 or M_+^3. For a unit constant vector field β, M is parameterized by

$$x(s,t) = \alpha(s) + t\beta$$

such that $\langle \alpha', \alpha' \rangle = \epsilon_1 \ (= \pm 1)$, $\langle \alpha', \beta \rangle = 0$ and $\langle \beta, \beta \rangle = \epsilon_2 \ (= \pm 1)$.

We consider two cases separately.

Case 1. Let M be a cylindrical ruled surface of type M_+^1 or M_-^1, i.e., $\epsilon_2 = 1$. Without loss of generality, we may assume that $\alpha(s) = (\alpha_1(s), \alpha_2(s), 0)$ is a plane curve parameterized by the arc-length s and the constant vector field β is chosen as $\beta = (0, 0, 1)$. Then, the Gauss map G and its Laplacian ΔG of M are given by $G = \alpha' \times \beta = (-\alpha_2{}', -\alpha_1{}', 0)$ and $\Delta G = (\epsilon_1 \alpha_2''', \epsilon_1 \alpha_1''', 0)$, respectively, where the prime denotes the differentiation with respect to s.

Suppose that M has pointwise 1-type Gauss map of the second kind, that is, the Gauss map G satisfies equation (1.1). Then, the third component of the constant vector $\mathbb{C}$ is zero and we have a system of differential equations:

$$\begin{aligned} \epsilon_1 \alpha_2''' &= f(-\alpha_2' + c_1), \\ \epsilon_1 \alpha_1''' &= f(-\alpha_1' + c_2), \end{aligned} \tag{3.1}$$

where $\mathbb{C} = (c_1, c_2, 0)$.

First, we consider the case that M is of type M_+^1. Since $\langle \alpha', \alpha' \rangle = -\alpha_1'^2 + \alpha_2'^2 = 1$, we may put

$$\alpha_1'(s) = \sinh \theta, \quad \alpha_2'(s) = \cosh \theta$$

for a function $\theta = \theta(s)$. Therefore, equation (3.1) can be written as

$$(\theta')^2 \cosh \theta + \theta'' \sinh \theta = f(-\cosh \theta + c_1),$$

$$(\theta')^2 \sinh \theta + \theta'' \cosh \theta = f(-\sinh \theta + c_2).$$

It follows

$$(\theta')^2 = f(-1 + c_1 \cosh \theta - c_2 \sinh \theta) \tag{3.2}$$

and

$$\theta'' = f(-c_1 \sinh \theta + c_2 \cosh \theta). \tag{3.3}$$

Suppose $\theta' \equiv 0$. Then, obviously $\Delta G = 0$. Since f is non-zero, (3.1) implies $\alpha_1' = c_2$, $\alpha_2' = c_1$ and thus $G = -\mathbb{C}$. Therefore, M is an open part of a Euclidean plane. If the interior Int(U) of a closed subset U $= \{p \in M | \theta'(p) = 0\}$ is non-empty, U must be M by the above argument and connectedness of M. Otherwise, if θ' has zeros, the set of zeros of θ' has measure zero.

Now we suppose $\theta' \neq 0$. (3.1) shows that f depends only on the parameter s, i.e., $f(s, t) = f(s)$. Differentiating (3.2) with respect to s and using (3.2) and (3.3), we obtain

$$\theta' = c \sqrt[3]{f} \tag{3.4}$$

for some non-zero constant c. On the other hand, combining (3.2) and (3.3), we get the following differential equation

$$\left(\frac{(\theta')^2}{f} + 1 \right)^2 - \left(\frac{\theta''}{f} \right)^2 = c_1^2 - c_2^2. \tag{3.5}$$

By using (3.4), equation (3.5) gives

$$\left(c^2 f^{-\frac{1}{3}}+1\right)^2-\left(-\frac{c}{2}\left(f^{-\frac{2}{3}}\right)'\right)^2=c_1^2-c_2^2. \tag{3.6}$$

If we put $f^{-\frac{1}{3}}=y$, then equation (3.6) becomes

$$(c^2y+1)^2-(cyy')^2=c_1^2-c_2^2.$$

If $\mathbb{C}$ is null, then the solution of the differential equation $(c^2y+1)^2-(cyy')^2=0$ is given by

$$c^2y-\ln|c^2y+1|=\pm c^3(s+k),$$

or, equivalently,

$$c^2f^{-\frac{1}{3}}-\ln|c^2f^{-\frac{1}{3}}+1|=\pm c^3(s+k) \tag{3.7}$$

for some constant k.

If $\mathbb{C}$ is non-null, the solution of (3.6) is obtained as follows:

$$\begin{aligned}&\sqrt{\left(c^2f^{-\frac{1}{3}}+1\right)^2+(-c_1^2+c_2^2)}\\&-\ln\left(c^2f^{-\frac{1}{3}}+1+\sqrt{\left(c^2f^{-\frac{1}{3}}+1\right)^2+(-c_1^2+c_2^2)}\right)\\&+\ln\sqrt{|-c_1^2+c_2^2|}=\pm c^3(s+k)\end{aligned} \tag{3.8}$$

for some constant k.

We now consider the case that M is of type M_-^1. Since $-{\alpha_1'}^2+{\alpha_2'}^2=-1$, we may put

$$\alpha_1'(s)=\cosh\theta,\quad \alpha_2'(s)=\sinh\theta$$

for a function $\theta=\theta(s)$. As is the previous case, if $\theta'\equiv 0$, M is an open portion of a Minkowski plane. If θ' is non-zero, we get $\theta'=c\sqrt[3]{f}$ and

$$\left(c^2f^{-\frac{1}{3}}-1\right)^2-\left(-\frac{c}{2}\left(f^{-\frac{2}{3}}\right)'\right)^2=-c_1^2+c_2^2 \tag{3.9}$$

for some non-zero constant c. In this case, if $\mathbb{C}$ is null or non-null, then its solution is obtained as, respectively,

$$c^2f^{-\frac{1}{3}}+\ln|c^2f^{-\frac{1}{3}}-1|=\pm c^3(s+k) \tag{3.10}$$

or

$$\begin{aligned}&\sqrt{\left(c^2f^{-\frac{1}{3}}-1\right)^2-(-c_1^2+c_2^2)}\\ &+\ln\left(c^2f^{-\frac{1}{3}}-1+\sqrt{\left(c^2f^{-\frac{1}{3}}-1\right)^2+|-c_1^2+c_2^2|}\right)\\ &-\ln\sqrt{|-c_1^2+c_2^2|}=\pm c^3(s+k),\end{aligned}\tag{3.11}$$

where k is a constant.

Case 2. Let M be a cylindrical ruled surface of type M_+^3. Then we may assume that $\beta=(1,0,0)$ and $\alpha(s)=(0,\alpha_2(s),\alpha_3(s))$ without loss of generality. Hence, the Gauss map G and its Laplacian ΔG of M are obtained by $G=(0,{\alpha_3}',-{\alpha_2}')$ and $\Delta G=(0,-\alpha_3^{'''},\alpha_2^{'''})$, respectively.

Suppose that the Gauss map G of M is of pointwise 1-type of the second kind. Then, we have

$$-\alpha_3^{'''}=f(\alpha_3'+c_2),$$

$$\alpha_2^{'''}=f(-\alpha_2'+c_3),$$

where $\mathbb{C}=(0,c_2,c_3)$. Since $\alpha(s)$ is parameterized by the arc length, i.e., $\langle\alpha',\alpha'\rangle={\alpha_2'}^2+{\alpha_3'}^2=1$, we may put

$$\alpha_2'(s)=\cos\theta,\quad \alpha_3'(s)=\sin\theta$$

for a function $\theta=\theta(s)$. Like a similar discussion developed in Case 1, M is an open portion of a Minkowski plane when $\theta'\equiv 0$. Otherwise, we can have $\theta'=c\sqrt[3]{f}$ for some non-zero constant c. Moreover, the smooth function f and the constant vector $\mathbb{C}$ satisfy

$$\sqrt{c_2^2+c_3^2-\left(c^2f^{-\frac{1}{3}}-1\right)^2}-\sin^{-1}\left(\frac{c^2f^{-\frac{1}{3}}-1}{\sqrt{c_2^2+c_3^2}}\right)=\pm c^3(s+k),\tag{3.12}$$

where c is a non-zero constant and k a constant.

Definition 3.1. ([10]). A cylindrical ruled surface over an infinite type base curve in Minkowski space is called a *cylinder of an infinite type.*

Thus, we have

Proposition 3.1. *Let M be a cylindrical ruled surface over a non-null base curve in Minkowski 3-space $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the second kind. If M is not totally geodesic, then the non-zero smooth function f satisfies* *one of the equations* (3.7)*,* (3.8)*,* (3.10)*,* (3.11) *or* (3.12) *depending upon the types of the base curve.*

Combining the results above and [14], we have

Theorem 3.2. *Let M be a cylindrical ruled surface over a non-null base curve in Minkowski 3-space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the first kind. Then, the curvature of the base curve is a non-zero constant. In other words, M is an open part of a hyperbolic cylinder, a Lorentz circular cylinder or a circular cylinder of index* 1*.*

Theorem 3.3. (Classification). *Let M be a cylindrical ruled surface over a non-null base curve in Minkowski 3-space $\mathbb{E}^3_1$. Then, M has pointwise 1-type Gauss map of the second kind if and only if M is an open part of a Euclidean plane, a Minkowski plane or a cylinder of an infinite type satisfying* (3.7)*,* (3.8)*,* (3.10)*,* (3.11) *or* (3.12) *up to rigid motion.*

4. Non-cylindrical Ruled Surfaces

In this section, we classify the non-cylindrical ruled surfaces in Minkowski 3-space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map.

Let M be a non-cylindrical ruled surface of type M^1_+, M^3_+ or M^1_- whose Gauss map is of pointwise 1-type of the second kind. Then, M is parameterized by, up to rigid motion,

$$x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \alpha', \beta\rangle = 0,\ \langle \beta, \beta\rangle = \epsilon_2\ (= \pm 1)$ and $\langle \beta', \beta'\rangle = \epsilon_3\ (= \pm 1)$. For later use, we define the smooth functions $q,\ u,\ Q$ and R as follows:

$$q = ||x_s||^2 = \epsilon_4\langle x_s,\ x_s\rangle,\ u = \langle \alpha', \beta'\rangle,\ Q = \langle \alpha',\ \beta\times\beta'\rangle,\ R = \langle \beta'',\ \beta\times\beta'\rangle,$$

where $\epsilon_4\ (= \pm 1)$ is the sign of the coordinate vector field $x_s = \frac{\partial x}{\partial s}$. For an orthonormal frame $\{\beta,\ \beta',\ \beta\times\beta'\}$ along the base curve α, we have

$$\begin{aligned}
\alpha' &= \epsilon_3 u\beta' - \epsilon_2\epsilon_3 Q\beta\times\beta',\\
\beta'' &= -\epsilon_2\epsilon_3\beta - \epsilon_2\epsilon_3 R\beta\times\beta',\\
\alpha'\times\beta &= -\epsilon_3 u\beta\times\beta' + \epsilon_3 Q\beta',\\
\beta\times\beta'' &= -\epsilon_3 R\beta',
\end{aligned}\tag{4.1}$$

which imply the smooth function q given by

$$q = \epsilon_4(\epsilon_3 t^2 + 2ut + \epsilon_3 u^2 - \epsilon_2\epsilon_3 Q^2).$$

We note that t must be chosen so that q takes positive values.

Furthermore, the Gauss map G and the mean curvature H of M are straightforwardly obtained by, respectively,

$$G = q^{-\frac{1}{2}}(\epsilon_3 Q\beta' - (\epsilon_3 u + t)\beta \times \beta'),$$
$$H = \frac{1}{2}\epsilon_2 q^{-3/2}(Rt^2 + (2\epsilon_3 uR + Q')t + u^2 R + \epsilon_3 uQ' - \epsilon_3 u'Q - \epsilon_2 Q^2 R).$$

On the other hand, the Laplacian ΔG of the Gauss map G can be expressed as follows ([14]):

$$\Delta G = q^{-7/2}\epsilon_4(A_1\beta + A_2\beta' + A_3\beta \times \beta'), \tag{4.2}$$

where we have put

$$\begin{aligned}
A_1 = {} & \epsilon_2 Rt^5 + (2\epsilon_2 Q' + 5\epsilon_2\epsilon_3 uR)t^4 + (-3\epsilon_2\epsilon_3\epsilon_4 u'Q - 2Q^2R + 8\epsilon_2\epsilon_3 uQ' + 10\epsilon_2 u^2 R)t^3 \\
& + (-4Q^2Q' + 3\epsilon_4 Q^2 Q' - 9\epsilon_2\epsilon_4 uu'Q - 6\epsilon_3 uQ^2R + 12\epsilon_2 u^2 Q' + 10\epsilon_2\epsilon_3 u^3 R)t^2 \\
& + (3\epsilon_3\epsilon_4 u'Q^3 + \epsilon_2 Q^4 R - 8\epsilon_3 uQ^2Q' + 6\epsilon_3\epsilon_4 uQ^2Q' - 9\epsilon_2\epsilon_3\epsilon_4 u^2u'Q - 6u^2Q^2R + 8\epsilon_2\epsilon_3 u^3 Q' \\
& + 5\epsilon_2 u^4 R)t + 2\epsilon_2 Q^4 Q' - 3\epsilon_2\epsilon_4 Q^4 Q' + 3\epsilon_4 uu'Q^3 + \epsilon_2\epsilon_3 uQ^4 R - 4u^2Q^2Q' + 3\epsilon_4 u^2 Q^2 Q' \\
& - 3\epsilon_2\epsilon_4 u^3 u'Q - 2\epsilon_3 u^3 Q^2 R + 2\epsilon_2 u^4 Q' + \epsilon_2\epsilon_3 u^5 R, \\
A_2 = {} & -\epsilon_3 R't^5 + (u'R - \epsilon_2\epsilon_3 QR^2 - \epsilon_3 Q'' - 5uR')t^4 + (u''Q + 2\epsilon_2\epsilon_3 Q^2 R' + 3u'Q' - 3\epsilon_2\epsilon_3 QQ'R \\
& + 4\epsilon_3 uu'R - 4\epsilon_2 uQR^2 - 4uQ'' - 10\epsilon_3 u^2 R')t^3 + (-3\epsilon_3 u'^2 Q - 2Q^3 + \epsilon_2 u'Q^2 R + 2\epsilon_3 Q^3 R^2 \\
& - 4\epsilon_2\epsilon_3 QQ'^2 + \epsilon_2\epsilon_3 Q^2 Q'' + 3\epsilon_3 uu''Q + 6\epsilon_2 uQ^2 R' + 9\epsilon_3 uu'Q' - 9\epsilon_2 uQQ'R + 6u^2u'R \\
& - 6\epsilon_2\epsilon_3 u^2 QR^2 - 6\epsilon_3 u^2 Q'' - 10u^3 R')t^2 + (-\epsilon_2 u''Q^3 - \epsilon_3 Q^4 R' + 5\epsilon_2 u'Q^2 Q' + 3\epsilon_3 Q^3 Q'R \\
& - 6uu'^2 Q - 4\epsilon_3 uQ^3 + 2\epsilon_2\epsilon_3 uu'Q^2 R + 4uQ^3 R^2 - 8\epsilon_2 uQQ'^2 + 2\epsilon_2 uQ^2 Q'' + 3u^2 u''Q \\
& + 6\epsilon_2\epsilon_3 u^2 Q^2 R' + 9u^2 u'Q' - 9\epsilon_2\epsilon_3 u^2 QQ'R + 4\epsilon_3 u^3 u'R - 4\epsilon_2 u^3 QR^2 - 4u^3 Q'' - 5\epsilon_3 u^4 R')t \\
& - \epsilon_2\epsilon_3 u'^2 Q^3 + 2Q^5 - 2u'Q^4 R - \epsilon_3 Q^5 R^2 - \epsilon_2\epsilon_3 uu''Q^3 - uQ^4 R' + 5\epsilon_2\epsilon_3 uu'Q^2 Q' + 3uQ^3 Q'R \\
& - 3\epsilon_3 u^2 u'^2 Q - 2u^2 Q^3 + \epsilon_2 u^2 u'Q^2 R + 2\epsilon_3 u^2 Q^3 R^2 - 4\epsilon_2\epsilon_3 u^2 QQ'^2 + \epsilon_2\epsilon_3 u^2 Q^2 Q'' + \epsilon_3 u^3 u''Q \\
& + 2\epsilon_2 u^3 Q^2 R' + 3\epsilon_3 u^3 u'Q' - 3\epsilon_2 u^3 QQ'R + u^4 u'R - \epsilon_2\epsilon_3 u^4 QR^2 - \epsilon_3 u^4 Q'' - u^5 R', \\
A_3 = {} & \epsilon_2 R^2 t^5 + (\epsilon_2 QR' + 2\epsilon_2 Q'R + 5\epsilon_2\epsilon_3 uR^2)t^4 + (2\epsilon_3 Q^2 - 3\epsilon_2\epsilon_3 u'QR - 2Q^2R^2 + \epsilon_2 Q'^2 \\
& + \epsilon_2 QQ'' + 4\epsilon_2\epsilon_3 uQR' + 8\epsilon_2\epsilon_3 uQ'R + 10\epsilon_2 u^2 R^2)t^3 + (-\epsilon_2\epsilon_3 u''Q^2 - 2Q^3 R' - 5\epsilon_2\epsilon_3 u'QQ' \\
& - Q^2 Q'R + 6uQ^2 - 9\epsilon_2 uu'QR - 6\epsilon_3 uQ^2 R^2 + 3\epsilon_2\epsilon_3 uQ'^2 + 3\epsilon_2\epsilon_3 uQQ'' + 6\epsilon_2 u^2 QR' \\
& + 12\epsilon_2 u^2 Q'R + 10\epsilon_2\epsilon_3 u^3 R^2)t^2 + (4\epsilon_2 u'^2 Q^2 - 2\epsilon_2\epsilon_3 Q^4 + 3\epsilon_3 u'Q^3 R + \epsilon_2 Q^4 R^2 + 3Q^2 Q'^2 \\
& - Q^3 Q'' - 2\epsilon_2 uu''Q^2 - 4\epsilon_3 uQ^3 R' - 10\epsilon_2 uu'QQ' - 2\epsilon_3 uQ^2 Q'R + 6\epsilon_3 u^2 Q^2 - 9\epsilon_2\epsilon_3 u^2 u'QR \\
& - 6u^2 Q^2 R^2 + 3\epsilon_2 u^2 Q'^2 + 3\epsilon_2 u^2 QQ'' + 4\epsilon_2\epsilon_3 u^3 QR' + 8\epsilon_2\epsilon_3 u^3 Q'R + 5\epsilon_2 u^4 R^2)t + \epsilon_3 u''Q^4 \\
& + \epsilon_2 Q^5 R' - 3\epsilon_2 u'Q^3 Q' - \epsilon_2 Q^4 Q'R + 4\epsilon_2\epsilon_3 uu'^2 Q^2 - 2\epsilon_2 uQ^4 + 3uu'Q^3 R + \epsilon_2\epsilon_3 uQ^4 R^2 \\
& + 3\epsilon_3 uQ^2 Q'^2 - \epsilon_3 uQ^3 Q'' - \epsilon_2\epsilon_3 u^2 u''Q^2 - 2u^2 Q^3 R' - 5\epsilon_2\epsilon_3 u^2 u'QQ' - u^2 Q^2 Q'R + 2u^3 Q^2 \\
& - 3\epsilon_2 u^3 u'QR - 2\epsilon_3 u^3 Q^2 R^2 + \epsilon_2\epsilon_3 u^3 Q'^2 + \epsilon_2\epsilon_3 u^3 QQ'' + \epsilon_2 u^4 QR' + 2\epsilon_2 u^4 Q'R + \epsilon_2\epsilon_3 u^5 R^2.
\end{aligned}$$

Now, we suppose that M has pointwise 1-type Gauss map of the second kind. Then, equation (1.1) together with (4.2) gives

(4.3) $q^{-7/2}\epsilon_4(A_1\beta+A_2\beta'+A_3\beta\times\beta') = f\{q^{-1/2}(\epsilon_3 Q\beta'-(\epsilon_3 u+t)\beta\times\beta')+\mathbb{C}\}.$

If we respectively take the scalar product to equation (4.3) with $\beta,\ \beta'$ and $\beta\times\beta'$, then we have the following system of equations:

$$q^{-7/2}\epsilon_4 A_1 = fc_1, \tag{4.4}$$

$$q^{-7/2}\epsilon_4 A_2 = f(q^{-1/2}\epsilon_3 Q + c_2), \tag{4.5}$$

$$-q^{-7/2}\epsilon_4 A_3 = f(q^{-1/2}(\epsilon_3 u + t) - c_3), \tag{4.6}$$

where $c_1 = \epsilon_2\langle\mathbb{C},\beta\rangle$, $c_2 = \epsilon_3\langle\mathbb{C},\beta'\rangle$ and $c_3 = -\epsilon_2\epsilon_3\langle\mathbb{C},\beta\times\beta'\rangle$. Differentiating the functions $c_1,\ c_2$ and c_3 with respect to s, we get

$$c_1' = \epsilon_2\epsilon_3 c_2, \tag{4.7}$$

$$c_1 + c_2' - \epsilon_3 c_3 R = 0, \tag{4.8}$$

$$c_3' - \epsilon_2\epsilon_3 c_2 R = 0. \tag{4.9}$$

Combining equations (4.4), (4.5) and (4.6), we find

$$q(A_2c_1 - A_1c_2)^2 - Q^2A_1^2 = 0, \tag{4.10}$$

$$q(A_1c_3 - A_3c_1)^2 - A_1^2(\epsilon_3 u + t)^2 = 0, \tag{4.11}$$

$$q(A_2c_3 - A_3c_2)^2 - (A_2(\epsilon_3 u + t) + \epsilon_3 QA_3)^2 = 0. \tag{4.12}$$

The left hand sides of (4.10), (4.11) and (4.12) are polynomials in t with functions of s as the coefficients. Therefore, all of them as functions of s of polynomials in t must be zero.

First, from the leading coefficient in t of the left hand side of (4.10) with the help of (4.7), we get

$$c_1R = \text{a constant}. \tag{4.13}$$

Also, the leading coefficient in t of the left hand side of (4.11) gives

$$\epsilon_3\epsilon_4(c_3 - c_1R)^2R^2 = R^2. \tag{4.14}$$

If $\epsilon_3\epsilon_4 = -1$, then R is identically zero. In this case, the leading coefficient of the left hand side of (4.11) implies

$$(c_3^2 + 1)Q'^2 = 0,$$

from which, Q is a constant. If we consider the coefficient of t^8 of the left hand side of (4.11), we also get

$$\left((3\epsilon_2 c_3 u' - 2\epsilon_3 c_1 Q)^2 + 9u'^2\right) Q^2 = 0.$$

If $Q \neq 0$, then we have

$$(3\epsilon_2 c_3 u' - 2\epsilon_3 c_1 Q)^2 + 9u'^2 = 0,$$

which implies $u' = 0$. It follows the mean curvature H vanishes on M. It contradicts that the Gauss map of M is of pointwise 1-type of the second kind. Thus, $Q = 0$. In turn, the mean curvature is zero identically, which is a contradiction, too. Consequently, we have

$$\epsilon_3 \epsilon_4 = 1.$$

Case 1. R is not identically zero on M.

We now consider an open subset $\mathbf{U} = \{p \in M | R(p) \neq 0\}$. Suppose $\mathbf{U}$ is not empty. Then, (4.14) yields

$$(c_3 - c_1 R)^2 = 1 \tag{4.15}$$

on $\mathbf{U}$. Differentiating equation (4.15), we obtain c_3 is a constant on a connected component $\mathbf{U}_0$ of $\mathbf{U}$ because $c_1 R$ is a constant. Therefore, (4.9) implies $c_2 = 0$ on $\mathbf{U}_0$. In view of (4.7), c_1 is a constant on $\mathbf{U}_0$. So, equation (4.8) yields R is a constant on $\mathbf{U}_0$. By continuity of R and connectedness of M, R is a non-zero constant on M. Therefore, by (4.7), (4.8) and (4.13), c_1 and c_3 are constants and $c_2 = 0$ on M. Thus, equations (4.10), (4.11) and (4.12) can be rewritten as follows:

$$q c_3^2 R^2 A_2^2 - Q^2 A_1^2 = 0, \tag{4.16}$$

$$q c_3^2 (A_1 - \epsilon_3 R A_3)^2 - A_1^2 (\epsilon_3 u + t)^2 = 0, \tag{4.17}$$

$$q c_3^2 A_2^2 - (A_2(\epsilon_3 u + t) + \epsilon_3 Q A_3)^2 = 0. \tag{4.18}$$

Moreover, combining equations (4.16) and (4.18), we get

$$Q^2 A_1^2 - R^2 \left(A_2(\epsilon_3 u + t) + \epsilon_3 Q A_3\right)^2 = 0. \tag{4.19}$$

From the leading coefficient of (4.17), we have

$$c_3^2 (1 - \epsilon_3 R^2)^2 = 1. \tag{4.20}$$

If we examine the coefficients of t^{10} and t^9 of the left hand side of (4.17) with the help of (4.20), respectively, we get

$$2Q^2 - 5\epsilon_2 Q^2 + 4u'QR - 6\epsilon_3 u'QR + \epsilon_2\epsilon_3 Q^2R^2 - 2\epsilon_3 Q'^2 = 0 \tag{4.21}$$

and

$$2QQ' - u''QR - 3u'Q'R + \epsilon_2\epsilon_3 QQ'R^2 = 0. \tag{4.22}$$

Furthermore, considering the leading coefficient of the left hand side of (4.19), we obtain

$$(u'R - \epsilon_3 Q'')^2 = Q^2. \tag{4.23}$$

Without loss of generality, we may assume

$$u'R - \epsilon_3 Q'' = Q.$$

From the coefficients of t^8 and t^7 of the left hand side of (4.19), respectively, we have

$$Q^2(u'Q - \epsilon_3 u'^2 R - \epsilon_2\epsilon_3 Q'^2 R) = 0 \tag{4.24}$$

and

$$Q^2(2u'R - \epsilon_3 Q' + 2\epsilon_3 uR) = 0. \tag{4.25}$$

Suppose the open subset $\mathbf{O} = \{p \in M | Q(p) \neq 0\}$ is not empty. Then, (4.24) and (4.25) imply

$$u'Q - \epsilon_3 u'^2 R - \epsilon_2\epsilon_3 Q'^2 R = 0 \tag{4.26}$$

and

$$2u'R - \epsilon_3 Q' + 2\epsilon_3 uR = 0. \tag{4.27}$$

On the other hand, considering the coefficient of t^8 of the left hand side of (4.16) with the help of (4.21), (4.22) and (4.26), we obtain

$$\epsilon_2 Q^3 - Q^3 = 0. \tag{4.28}$$

If we think of the non-empty subset $\mathbf{O}$, ϵ_2 must be 1. Therefore, (4.21) implies

$$-3Q^2 + 4u'QR - 6\epsilon_3 u'QR + \epsilon_3 Q^2R^2 - 2\epsilon_3 Q'^2 = 0. \tag{4.29}$$

Differentiating equation (4.29) with respect to s, we obtain

$$\begin{aligned} &-3QQ' + 2u''QR + 2u'Q'R - 3\epsilon_3 u''QR \\ &-3\epsilon_3 u'Q'R + \epsilon_3 QQ'R^2 - 2\epsilon_3 Q'Q'' = 0. \end{aligned} \tag{4.30}$$

Suppose $\epsilon_3 = 1$. Combining equations (4.22) and (4.30), we get $QQ' = 0$. Hence, Q is a non-zero constant on a connected component $\mathbf{O}_0$ of $\mathbf{O}$. By connectedness of M and continuity of Q, Q is a non-zero constant on M. Therefore, equations (4.22) and (4.27) respectively give the following

$$u'' = 0 \quad \text{and} \quad u' + u = 0.$$

Thus, we have $u' = 0$. It implies that $Q = 0$ because of (4.23), a contradiction. So, $\mathbf{O}$ is empty and thus, $Q \equiv 0$ on M.
Let us now assume $\epsilon_3 = -1$. Then, (4.30) with the help of (4.22) implies

$$Q'(3Q - 4u'R - 2QR^2) = 0. \tag{4.31}$$

Consider an open subset $\mathbf{O}_1 = \{p \in \mathbf{O} | Q'(p) \neq 0\}$ and suppose $\mathbf{O}_1$ is not empty. Then (4.31) gives

$$3Q - 4u'R - 2QR^2 = 0. \tag{4.32}$$

Since $u'R + Q'' = Q$, we have the following differential equation from (4.32)

$$Q'' - k^2 Q = 0, \tag{4.33}$$

where $k^2 = \frac{1+2R^2}{4}$ $(k > 0)$, a constant. Thus,

$$u'R = (1 - k^2)Q, \tag{4.34}$$

or, Q is given by

$$Q = \tilde{K}_1 \cosh ks + \tilde{K}_2 \sinh ks$$

for some constants $\tilde{K}_1$ and $\tilde{K}_2$. Together (4.34) with (4.22), we have

$$2 - 4(1 - k^2) - R^2 = 0,$$

or, using $k^2 = \frac{1+2R^2}{4}$, we get

$$R^2 = 1$$

on $\mathbf{O}_1$. Putting this into (4.26) and using (4.34), we get

$$Q = 0,$$

which is a contradiction on $\mathbf{O}_1$. Therefore, the open subset $\mathbf{O}_1$ is empty and Q is a non-zero constant on a connected component $\mathbf{O}_0$ of $\mathbf{O}$. Again, connectedness of M and continuity of Q imply Q is a non-zero constant on M. Hence, we have $u'R = Q$. Together with (4.26), $u' = 0$ is induced and we get $Q = 0$, a contradiction. Consequently, the open subset **O** is empty and $Q \equiv 0$ on M. Hence, no matter what cases of ϵ_3 may be, we have Q is zero on M.

Therefore, u is a non-zero constant by virtue of (4.23) and the first equation of (4.1) with the help of the fact that α is non-null.

On the other hand, it follows from the second equation in (4.1) that

$$\beta''' + \epsilon_2\epsilon_3(1 - \epsilon_3 R^2)\beta' = 0.$$

Let us give the initial conditions $\beta(0) = (a_1, 0, a_2)$, $\beta'(0) = (b_1, b_2, 0)$ and $\beta''(0) = -\epsilon_2\epsilon_3(a_1 + a_2b_2R, a_2b_1R, a_2 + a_1b_2R)$ of the above differential equation, where a_1, a_2, b_1 and b_2 are some constants satisfying $-a_1^2 + a_2^2 = \epsilon_2$, $-b_1^2 + b_2^2 = \epsilon_3$ and $a_1b_1 = 0$ with $(a_1, b_1) \neq (0, 0)$.

Considering equation (4.20), we only have the cases: $\epsilon_2\epsilon_3(1 - \epsilon_3R^2) > 0$ or $\epsilon_2\epsilon_3(1 - \epsilon_3R^2) < 0$.

First, if $\epsilon_2\epsilon_3(1-\epsilon_3R^2) > 0$, then we have $\beta'''(s) + a^2\beta'(s) = 0$ and its solution $\beta(s)$ is

$$\begin{aligned}\beta(s) = \Big(&-\frac{\epsilon_2\epsilon_3}{a^2}(\epsilon_3a_1R^2 + a_2b_2R) + \frac{\epsilon_2\epsilon_3}{a^2}(a_1 + a_2b_2R)\cos as + \frac{b_1}{a}\sin as,\\ &-\frac{\epsilon_2\epsilon_3}{a^2}a_2b_1R + \frac{\epsilon_2\epsilon_3}{a^2}a_2b_1R\cos as + \frac{b_2}{a}\sin as,\\ &-\frac{\epsilon_2\epsilon_3}{a^2}(\epsilon_3a_2R^2 + a_1b_2R) + \frac{\epsilon_2\epsilon_3}{a^2}(a_2 + a_1b_2R)\cos as\Big),\end{aligned} \tag{4.35}$$

where $a = \sqrt{\epsilon_2\epsilon_3(1 - \epsilon_3R^2)}$.

If $\epsilon_2\epsilon_3(1 - \epsilon_3R^2) < 0$, then the solution of $\beta'''(s) - a^2\beta'(s) = 0$ is obtained as follows

$$\begin{aligned}\beta(s) = \Big(&\frac{\epsilon_2\epsilon_3}{a^2}(\epsilon_3a_1R^2 + a_2b_2R) - \frac{\epsilon_2\epsilon_3}{a^2}(a_1 + a_2b_2R)\cosh as + \frac{b_1}{a}\sinh as,\\ &\frac{\epsilon_2\epsilon_3}{a^2}a_2b_1R - \frac{\epsilon_2\epsilon_3}{a^2}a_2b_1R\cosh as + \frac{b_2}{a}\sinh as,\\ &\frac{\epsilon_2\epsilon_3}{a^2}(\epsilon_3a_2R^2 + a_1b_2R) - \frac{\epsilon_2\epsilon_3}{a^2}(a_2 + a_1b_2R)\cosh as\Big),\end{aligned} \tag{4.36}$$

where $a = \sqrt{\epsilon_2\epsilon_3(\epsilon_3R^2 - 1)}$.

By applying the first equation of (4.1), $\alpha' = \epsilon_3u\beta'$ for some non-zero constant u. Therefore, we can easily obtain the base curve $\alpha(s)$ by means of $\beta(s)$ of the form (4.35) or (4.36).

Definition 4.1. A non-cylindrical ruled surface M generated by a base curve $\alpha(s)$ and the director vector field $\beta(s)$ satisfying (4.35) or (4.36) is called a *rotational ruled surface of type I* or a *rotational ruled surface of type II*, respectively.

Case 2. $R \equiv 0$ on M.

From (4.9), we see that c_3 is a constant. Combining equations (4.7) and (4.8), we have $c_1 + \epsilon_2\epsilon_3 c_1'' = 0$. Thus, depending upon the sign of ϵ_2 and ϵ_3, we get

$$c_1 = K_1 \cosh s + K_2 \sinh s \quad \text{or} \quad c_1 = K_3 \sin(s + s_0) \tag{4.37}$$

for some constants K_i $(i = 1, 2, 3)$ and s_0.

If we think of the leading coefficient of the left hand side of (4.11) with c_1 as above, we have

$$(c_3^2 - 1)Q'^2 = 0.$$

Suppose $c_3^2 \neq 1$. Then $Q' = 0$, that is, Q is a constant. If $Q = 0$, the mean curvature H vanishes on M, which is a contradiction. Therefore, Q is a non-zero constant.

If we consider the leading coefficients of the left hand sides of (4.10) and (4.11), respectively, we get

$$c_1 u'' + 3\epsilon_2 c_2 u' = 0 \quad \text{and} \quad (3\epsilon_2 c_3 u' + 2\epsilon_3 c_1 Q)^2 = 9u'^2. \tag{4.38}$$

Since $c_1' = \epsilon_2\epsilon_3 c_2$, the first equation of (4.38) implies

$$c_1 u'' + 3\epsilon_3 c_1' u' = 0.$$

Suppose c_1 is non-trivial. Then, the solution of the above differential equation is given by

$$u' = k_1 c_1^{\pm 3}, \tag{4.39}$$

where k_1 is a constant. Putting (4.37) and (4.39) into the second equation of (4.38), we obtain $k_1 = 0$ and $K_i = 0$ $(i = 1, 2, 3)$, which is a contradiction. Thus, we have $c_1 = 0$. From (4.4), we get $A_1 = 0$. It implies that $u' = 0$, that is, u is a constant. Hence we see that the mean curvature H vanishes identically that is again a contradiction. Consequently, we get $c_3^2 = 1$.

Now, if we consider the leading coefficient of the left hand side of (4.10) with the help of (4.7), we have

$$c_1 Q'' + 2c_1' Q' = 0.$$

Suppose c_1 is non-zero. Then the solution of the above equation is given by

$$Q' = k_2 c_1^{-2}, \tag{4.40}$$

where k_2 is a constant. Since the coefficient of t^9 of the left hand side of (4.11) with $c_3^2 = 1$ is zero, we have

$$Q'(QQ'' + Q'^2 + 2\epsilon_2\epsilon_3 Q^2) = 0.$$

Consider an open set $\mathbf{V} = \{p \in M | Q'(p) \neq 0\}$. Suppose $\mathbf{V}$ is not empty. Then we get

$$QQ'' + Q'^2 + 2\epsilon_2\epsilon_3 Q^2 = 0. \tag{4.41}$$

Equations (4.37), (4.40) and (4.41) lead to $k_2 = 0$ and so $Q' = 0$, a contradiction. Therefore, $\mathbf{V}$ is empty and Q is a non-zero constant on M since M is not minimal. Thus, the fact that the leading coefficient of the left hand side of (4.10) is zero gives $c_1 u'' + 3\epsilon_3 c_1' u' = 0$ because of (4.7). If c_1 is non-trivial, we have a solution of the form (4.39). Since $c_3^2 = 1$, from the leading coefficient of (4.11), we also get

$$c_1 Q \pm 3\epsilon_2\epsilon_3 u' = 0. \tag{4.42}$$

Putting (4.37) and (4.39) into (4.42), we obtain $c_1 = 0$.

Similarly as before, (4.4) yields $u' = 0$. Therefore, the mean curvature H vanishes, which is a contradiction. As a consequence, the case of $R = 0$ can never occur.

Consequently, we have

Theorem 4.1. *Let M be a non-cylindrical ruled surface of type M_+^1, M_-^1 or M_+^3 in Minkowski 3-space $\mathbb{E}_1^3$. Suppose that M has pointwise 1-type Gauss map of the second kind. Then, M is an open part of a rotational ruled surfaces of type I or type II.*

Now we examine a non-cylindrical ruled surface of type M_+^2 or M_-^2 with pointwise 1-type Gauss map of the second kind.

Let M be a non-cylindrical ruled surface of type M_+^2 or M_-^2. Then, the parametrization for M is given by

$$x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \alpha', \alpha' \rangle = \epsilon_1 (= \pm 1)$, $\langle \beta, \beta \rangle = 1$, $\langle \alpha', \beta \rangle = 0$ and β' is null. Let us also put

$$q = ||x_s||^2 = \epsilon_4 \langle x_s, x_s \rangle, \quad u = \langle \alpha', \beta' \rangle.$$

On the other hand, it is easy to see that $\beta \times \beta'$ is null. Since the null vector fields β' and $\beta \times \beta'$ are orthogonal, we may take

$$\beta' = \beta \times \beta'. \tag{4.43}$$

Moreover, we may assume $\beta(0) = (0, 0, 1)$. Thus, $\beta(s)$ is given by

$$\beta(s) = (as, as, 1) \tag{4.44}$$

for a non-zero constant a. For an orthonormal frame $\{\alpha',\ \beta,\ \alpha' \times \beta\}$ along α, we have

$$\beta' = \epsilon_1 u(\alpha' - \alpha' \times \beta) \quad \text{and} \quad \alpha'' = -u\beta + \frac{u'}{u}\alpha' \times \beta. \tag{4.45}$$

Let $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$. Since $\langle \alpha', \beta' \rangle = u$ and $\langle \alpha', \beta \rangle = 0$, we obtain

$$\alpha_1' - \alpha_2' = -\frac{u}{a} \quad \text{and} \quad \alpha_3' = -us. \tag{4.46}$$

Equation (4.46) together with $\langle \alpha', \alpha' \rangle = \epsilon_1$ implies

$$\alpha_1' + \alpha_2' = \frac{a\epsilon_1}{u} - aus^2. \tag{4.47}$$

Combining equations (4.46) and (4.47), we get

$$\alpha'(s) = \left(\frac{1}{2}\left(\frac{a\epsilon_1}{u} - aus^2 - \frac{u}{a} \right), \frac{1}{2}\left(\frac{a\epsilon_1}{u} - aus^2 + \frac{u}{a} \right), -us \right). \tag{4.48}$$

On the other hand, the Gauss map G of M is given by

$$G = q^{-\frac{1}{2}}(A - t\beta'), \tag{4.49}$$

where we put $A = \alpha' \times \beta$. By a straightforward computation, the Laplacian ΔG of the Gauss map G can be expressed as ([14])

$$\Delta G = q^{-\frac{7}{2}} \left((-2u^2q + u''tq - 4\epsilon_4 u'^2 t^2)(A - t\beta') - \epsilon_4 u\beta' q^2 + 3u'tA'q - \epsilon_4 A'' q^2 \right).$$

Now we suppose that M has pointwise 1-type Gauss map of the second kind. Then, we obtain

$$\begin{aligned} &(-2u^2q + u''tq - 4\epsilon_4 u'^2 t^2)(A - t\beta') - \epsilon_4 u\beta' q^2 + 3u'tA'q - \epsilon_4 A'' q^2 \\ &= f(q^3(A - t\beta') + q^{\frac{7}{2}}\mathbb{C}) \end{aligned} \tag{4.50}$$

for some non-zero smooth function f and a constant vector $\mathbb{C}$.

If we take the scalar product to equation (4.50) with α', β and $\alpha' \times \beta$, respectively, then we obtain the following:

$$B_1 = f(-q^3tu + q^{\frac{7}{2}}\langle \mathbb{C}, \alpha' \rangle), \tag{4.51}$$

$$B_2 = fq^{\frac{7}{2}}\langle \mathbb{C}, \beta \rangle, \tag{4.52}$$

$$B_3 = f(-\epsilon_1 q^3 - uq^3t + q^{\frac{7}{2}}\langle \mathbb{C}, \alpha' \times \beta \rangle), \tag{4.53}$$

where

$$\begin{aligned} B_1 &= 2u^3qt - uu''t^2q + 4\epsilon_4 uu'^2t^3 + 3\epsilon_1\frac{u'^2}{u}tq - \epsilon_1\epsilon_4\frac{u''}{u}q^2 + \epsilon_1\epsilon_4\frac{u'^2}{u^2}q^2, \\ B_2 &= -3uu'tq + 2\epsilon_4 u'q^2, \\ B_3 &= 2\epsilon_1 u^2 q - \epsilon_1 u''tq + 4\epsilon_1\epsilon_4 u'^2t^2 + 2u^3qt - uu''t^2q + 4\epsilon_4 uu'^2t^3 + \epsilon_1\epsilon_4 q^2\frac{u'^2}{u^2}. \end{aligned}$$

If we put $\mathbb{C} = c_1\alpha' + c_2\beta + c_3\alpha' \times \beta$, (4.51)-(4.53) imply

$$(c_2B_1 - \epsilon_1 c_1 B_2)^2 q = u^2t^2B_2^2, \tag{4.54}$$

$$(c_2B_3 + \epsilon_1 c_3 B_2)^2 q = (\epsilon_1 + ut)^2 B_2^2, \tag{4.55}$$

$$(utB_3 - (\epsilon_1 + ut)B_1)^2 = q(c_1B_3 + c_3B_1)^2, \tag{4.56}$$

which are polynomials in t with functions of s as the coefficients. Hence, the leading coefficient of the left hand side of (4.54) must be zero, which means $c_2^2u^3(uu'' - 2u'^2)^2 = 0$. Because $u \neq 0$, we get

$$c_2^2(uu'' - 2u'^2)^2 = 0. \tag{4.57}$$

Consider an open subset $\mathbf{U} = \{p \in M | (uu'' - 2u'^2)(p) \neq 0\}$. Suppose $\mathbf{U}$ is not empty. Then, $c_2 = 0$ on $\mathbf{U}$. Therefore, equation (4.54) can be reduced to

$$B_2^2(c_1^2q - u^2t^2) = 0. \tag{4.58}$$

Since the leading coefficient of the left hand side of (4.58) must be zero, $u^6u'^2 = 0$ on $\mathbf{U}$, from which, $u' = 0$ on $\mathbf{U}$. It is a contradiction. Thus, $\mathbf{U}$ is empty and we have

$$uu'' - 2u'^2 = 0. \tag{4.59}$$

Suppose there is a point $s_0 \in$ domain (α) such that $u'(s_0) = 0$. Then, (4.54) implies $c_2 = 0$. Also, (4.59) gives $u''(s_0) = 0$. If we evaluate the left hand side of (4.56), it turns out to be zero at s_0 and thus

$$\epsilon_1 c_1 + (c_1 + c_3)u(s_0)t = 0.$$

It holds for each t and hence $c_1 = c_3 = 0$, that is, $\mathbb{C}$ is zero vector, which is a contradiction. Therefore, $u' \neq 0$ everywhere. From (4.59), we get

$$\frac{u''}{u'} - \frac{2u'}{u} = 0,$$

from which,

$$u(s) = \frac{1}{bs + c}$$

for some constants $b \neq 0$ and c. Thus, from (4.48), the base curve $\alpha(s)$ is given by

$$\alpha(s) = \frac{1}{2}\Bigg(a\epsilon_1(\frac{b}{2}s^2 + cs) - \frac{a}{2b}s^2 + \frac{ac}{b^2}s - (\frac{ac^2}{b^3} + \frac{1}{ab})\ln|bs + c| + d_1,$$
$$a\epsilon_1(\frac{b}{2}s^2 + cs) - \frac{a}{2b}s^2 + \frac{ac}{b^2}s - (\frac{ac^2}{b^3} - \frac{1}{ab})\ln|bs + c| + d_2, \tag{4.60}$$
$$-\frac{2}{b}s + \frac{2c}{b}\ln|bs + c| + d_3\Bigg),$$

where d_i $(i = 1, 2, 3)$ are some integration constants.

Definition 4.2. A ruled surface M generated by the base curve of the form (4.60) and the director vector field given by (4.44) is called a *transcendental ruled surface*.

Consequently, we have

Theorem 4.2. *Let M be a non-cylindrical ruled surface of type M^2_+ or M^2_- in Minkowski 3-space $\mathbb{E}^3_1$. Suppose that the Gauss map G of M is of pointwise 1-type of the second kind. Then, M is an open portion of a transcendental ruled surface.*

Combining Theorem 4.1, Theorem 4.2 and the results of [14], we have

Theorem 4.3. (Classification). *Let M be a non-cylindrical ruled surface over a non-null base curve in Minkowski 3-space $\mathbb{E}^3_1$. Then, M has pointwise 1-type Gauss map if and only if M is an open part of a helicoid of the first kind, a helicoid of the second kind, a helicoid of the third kind, the conjugate of Enneper's surface of the second kind, a rotational ruled surface of type I or II, or a transcendental ruled surface.*

5. Null Scrolls

In this section, we study a null scroll with pointwise 1-type Gauss map in Minkowski 3-space $\mathbb{E}^3_1$. We mainly focus to prove the following theorem.

Theorem 5.1. *Let M be a null scroll with pointwise 1-type Gauss map of the second kind in Minkowski 3-space $\mathbb{E}^3_1$. Then, M is an open part of a Minkowski plane.*

Proof. Let $\alpha = \alpha(s)$ be a null curve in $\mathbb{E}^3_1$ and $\beta = \beta(s)$ a null vector field satisfying $\langle \alpha', \beta \rangle = 1$ along α. For a null scroll M parameterized by

$$x = x(s,t) = \alpha(s) + t\beta(s),$$

we have the natural coordinate frame $\{x_s, x_t\}$ given by

$$x_s = \alpha' + t\beta' \quad \text{and} \quad x_t = \beta(s).$$

Furthermore, we may choose an appropriate parameter s in such a way that $u = \langle \alpha', \beta' \rangle = 0$, which is possible if the base curve α is chosen as a null geodesic of M. Again, we define the smooth functions q and v as follows:

$$q = \langle x_s, x_s \rangle \quad \text{and} \quad v = \langle \beta', \beta' \rangle.$$

On the other hand, the Gauss map G of M is determined by

$$G = x_s \times x_t = \alpha' \times \beta + t\beta' \times \beta \tag{5.1}$$

and the Laplacian of the Gauss map G is obtained by ([14])

(5.2) $$\Delta G = -2\beta'' \times \beta + 2vt\beta' \times \beta.$$

In terms of the pseudo-orthonormal frame $\{\alpha', \beta, \alpha' \times \beta\}$, we obtain

(5.3) $$\beta' = -Q\alpha' \times \beta, \quad \beta' \times \beta = Q\beta \quad \text{and} \quad \beta'' \times \beta = R\beta - v\alpha' \times \beta,$$

where $Q = \langle \alpha', \beta' \times \beta \rangle$ and $R = \langle \alpha', \beta'' \times \beta \rangle$.

We now suppose that M has pointwise 1-type Gauss map of the second kind. Then, with the help of (5.3), we have

(5.4) $$(2vtQ - 2R)\beta + 2v\alpha' \times \beta = f(\alpha' \times \beta + tQ\beta + \mathbb{C})$$

for some non-zero smooth function f and a constant vector $\mathbb{C}$.

If we take the scalar product to equation (5.4) with α', β and $\alpha' \times \beta$, respectively, then we have the following system of equations:

(5.5) $$2vtQ - 2R = f(Qt + c_2),$$

(5.6) $$c_1 f = 0,$$

(5.7) $$2v = f(1 + c_3),$$

where $c_1 = \langle \mathbb{C}, \beta \rangle$, $c_2 = \langle \mathbb{C}, \alpha' \rangle$ and $c_3 = \langle \mathbb{C}, \alpha' \times \beta \rangle$. Clearly, (5.6) gives $c_1 = 0$. From (5.7), the function f depends only on the parameter s. Therefore, from (5.5), we can obtain

$$(2v - f)Q = 0 \quad \text{and} \quad 2R + fc_2 = 0.$$

Consider an open subset $\mathbf{U} = \{p \in M | Q(p) \neq 0\}$. Suppose that $\mathbf{U}$ is not empty. Then $f = 2v$ on $\mathbf{U}$ which implies $c_3 f = 0$ by (5.7) and thus $c_3 = 0$. Therefore, the constant vector $\mathbb{C}$ can be written as $\mathbb{C} = c_2\beta$. Differentiating the constant vector $\mathbb{C}$ with respect to s, we have $0 = c_2'\beta(s) + c_2\beta'(s)$. Since β and β' are linearly independent for each s, c_2 vanishes, which is a contradiction because $\mathbb{C}$ is not zero vector. Therefore, the open subset $\mathbf{U}$ is empty, that is, $Q = 0$. Hence, (5.3) gives β is a constant vector. It follows that $R = 0$ and $v = 0$. Thus, $\Delta G = 0$. Since the Gauss map is of pointwise 1-type of the second kind, we may get $G = -\mathbb{C}$. Thus, the surface M is an open part of a Minkowski plane. Consequently, the proof is completed. ∎

Combining Theorem 5.1 and the results in [14], we have

Theorem 5.2. (Classification). *Let M be a null scroll with pointwise 1-type Gauss map in Minkowski 3-space $\mathbb{E}^3_1$. Then, M is an open part of a Minkowski plane or a B-scroll.*

Remark. Summing up all the cases, Theorem 3.2, Theorem 3.3, Theorem 4.3 and Theorem 5.2, we have a complete classification theorem of the ruled surfaces in Minkowski 3-space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map, which is described in Section 1.

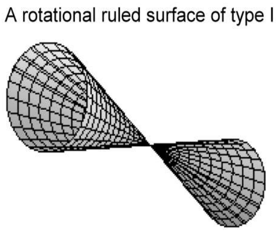

Fig. 1.

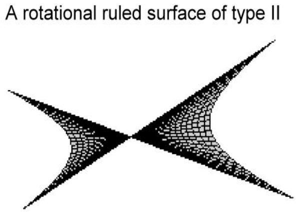

Fig. 2.

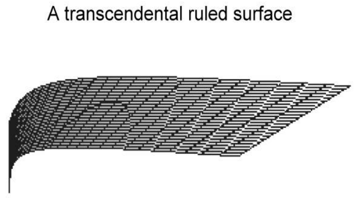

Fig. 3.

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