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BOUNDS ON FEEDBACK NUMBERS OF DE BRUIJN GRAPHS

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Abstract. The feedback number of a graph G is the minimum number of vertices whose removal from G results in an acyclic subgraph. We use $f(d,n)$ to denote the feedback number of the de Bruijn graph $UB(d,n)$. R. Královic and P. Ruzicka [Minimum feedback vertex sets in shuffle-based interconnection networks. Information Processing Letters, 86 (4) (2003), 191-196] proved that $f(2,n) = \lceil \frac{2^n-2}{3} \rceil$. This paper gives the upper bound on $f(d,n)$ for $d \geq 3$, that is, $f(d,n) \leq d^n \left(1 - \left(\frac{d}{1+d}\right)^{d-1}\right) + \binom{n+d-2}{d-2}$.

1. Introduction

Let $G = (V,E)$ be a simple graph, i.e., loopless and without multiple edges, with vertex set $V(G)$ and edge set $E(G)$. It is well known that the cycle rank of a graph G is the minimum number of edges that must be removed in order to eliminate all cycles in the graph. That is, if G has υ vertices, ε edges, and ω components, then the minimum number of edges whose deletion from G leaves an acyclic graph equals the cycle rank (or Betti number) $\rho(G) = \varepsilon - \upsilon + \omega$ (see, for example Xu [26]). A corresponding problem is the removal of vertices. A subset $F \subset V(G)$ is called *a feedback vertex set* if the subgraph $G - F$ is acyclic, that is, if $G - F$ is a forest. The minimum cardinality of a feedback vertex set is called *the feedback number* (or *decycling number* proposed first by Beineke and Vandell [5]) of G. A feedback vertex set of this cardinality is called a minimum feedback vertex set.

Determining the feedback number of a graph G is equivalent to finding the greatest order of an induced forest of G proposed first by Erdös, Saks and Sós [8], since the sum of the two numbers equals the order of G. A review of recent results and open problems on the decycling number is provided by Bau and Beineke [4].

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Apart from its graph-theoretical interest, the minimum feedback vertex set problem has some important applications to several fields. For example, the problems are in operating systems to resource allocation mechanisms that prevent deadlocks [17], in artificial intelligence to the constraint satisfaction problem and Bayesian inference, in synchronous distributed systems to the study of monopolies and in optical networks to converters placement problem(see [7, 9]).

In fact, the problem of finding the feedback number is NP-hard for general graphs [13] (also see [11]). The best known approximation algorithm for this problem has approximation ratio 2 [1]. There are also polynomial time algorithms for a number of topologies, such as reducible graphs [22], cocomparability graphs [11], convex bipartite graphs [11], cyclically reducible graphs [23], and interval graphs [15].

Determining the feedback number is quite difficult even for some elementary graphs. We refer the reader to an original research paper [5] for some results. The lower and the upper bounds on the feedback numbers have been established for some graphs, such as regular graphs, cubic graphs, hypercubic graphs, meshes, toroids, butterflies, cube-connected cycles, hypercubes, star graphs and directed split-stars, Kautz digraphs (see [1-3, 7, 9, 10, 15, 16, 18-24, 27]).

The de Bruijn digraph has many attractive features superior to the hypercube, such as regular, Eulerian, Hamiltonian, small diameter, nearly optimal connectivity, simple recursive structure, and simple routing algorithm. It contains some other useful topologies as its subgraphs (see, for example, Section 3.3 in [25]). So it is thought of as a good candidate for the next generation of parallel system architectures, after the hypercube network [6].

For two given integers $d \geq 2$ and $n \geq 1$, the de Bruijn digraph, denoted by $B(d,n)$, is defined as follows. The vertex set of $B(d,n)$ is

$$V(d,n) = \{x_1x_2\ldots x_n |\ x_i \in \{0,1,2,\ldots,d-1\} \text{ for } i = 1,2,\ldots,n\},$$

and the edge set $E(d,n)$ consists of all edges from one vertex $x_1x_2\ldots x_n$ to d other vertices $x_2x_3\ldots x_n\alpha$, where $\alpha \in \{0,1,\ldots,d-1\}$.

The de Bruijn undirected graph, denoted by $UB(d,n)$, is obtained from $B(d,n)$ by deleting the orientation of all edges and omitting multiple edges and loops.

It is clear that $B(d,n)$ is d-regular, $|V(d,n)| = d^n$ and $|E(d,n)| = d^{n+1}$. Moreover, $B(d,n)$ has $\frac{1}{2}d(d-1)$ symmetric edges and d loops. Thus, $UB(d,n)$ has $d^n - \frac{1}{2}d(d-1) - d$ edges, the maximum degree $2d$ and the minimum degree $2d-2$.

We use $f(d,n)$ to denote the feedback number of $UB(d,n)$. Královic and Ruzicka [14] proved $f(2,n) = \lceil \frac{2^n-2}{3} \rceil$. In this paper, we establish the following bounds on $f(d,n)$ for any $d \geq 3$ and $n \geq 1$:

$$\left\lceil \frac{d^{n+1}-d-\frac{d(d-1)}{2}-d^n+1}{2d-1} \right\rceil \leq f(d,n) \leq d^n \left(1-\left(\frac{d}{1+d}\right)^{d-1}\right)+\binom{n+d-2}{d-2}.$$

The proof of the result is in Section 3. In Section 2, we construct a feedback vertex set of $UB(d,n)$ and gives several lemmas.

2. Feedback Vertex Sets and Lemmas

Throughout this paper, we follow Xu [26] for graph-theoretical terminology and notation not defined here. Let $G=(V,E)$ be a graph and $S \subset V(G)$. The symbol $N_G(S)$ denotes the set of neighbors of S, namely, $N_G(S) = \{x \in V(G-S) : xy \in E(G), y \in S\}$. The subgraph induced by S is denoted by $G[S]$. The set S is independent if no two of vertices in S are adjacent in G, and is cycle-free if $G[S]$ is acyclic, that is, $G[S]$ has no cycles.

For given positive integers k and d, we use $\mathsf{P}_{k,d}$ to denote the set of all non-negative integral solutions of the indefinite equation $z_1+z_2+\ldots+z_d=k$, that is, an ordered sequence $(n_1,n_2,\ldots,n_d) \in \mathsf{P}_{k,d}$ means $n_1+n_2+\ldots+n_d=k$. The following result is well known and contained in any textbook on combinatorics (see, for example, p.3 in [12]).

Lemma 2.1. *For any positive integers k and d,*

$$|\mathsf{P}_{k,d}| = \binom{k+d-1}{k}.$$

Let $\alpha^s\beta^t$ denote the sequence $\underbrace{\alpha\alpha\ldots\alpha}_{s}\underbrace{\beta\beta\ldots\beta}_{t}$ and let the set $I_d=\{0,1,2,\ldots,d-1\}$. We define $d+1$ subsets of $V(d,n)$ as follows.

$$S_0 = \{0x_2x_3\ldots x_n |\ x_i \in I_d\}.$$

For each $i=1,2,\ldots,d-1$, let

$$S_i = \left\{ \begin{array}{c} i^{n_i}(i-1)^{n_{i-1}}\ldots t^{n_t}\alpha x_{k+2}x_{k+3}\ldots x_n |\ x_j \in I_d, \\ \alpha \in I_d \setminus \{1,\ldots,t\},\ 1 \leq t \leq i, \\ (n_t,\ldots,n_i) \in \mathsf{P}_{k,i-t+1},\ n_i, n_t \neq 0, \\ 2 \leq k \leq n-1, k \equiv 0 \pmod 2 \end{array} \right\},$$

and

$$S_d = \left\{ \begin{array}{l} \{(d-1)^{n_{d-1}}(d-2)^{n_{d-2}}\ldots t^{n_t} |\ n_t \neq 0 \text{ and } n_t \equiv 0 \pmod 2, \\ \ (n_t,n_{t+1},\ldots,n_{d-1}) \in \mathsf{P}_{n,d-t}, 1 \leq t \leq d-1, n \equiv 0 \pmod 2\}; \\ \emptyset, \quad \text{for } n \equiv 1 \pmod 2. \end{array} \right.$$

It is easy to verify that $S_i \cap S_j = \emptyset$ for any $i, j \in I_d$, $i \neq j$. Let

(2.1) $$S = S_0 \cup S_1 \cup S_2 \cup \ldots \cup S_{d-1} \cup S_d.$$

For example, for $d = 3,\ n = 4$, we have

$$\begin{aligned} S_0 = \{&0000, 0001, 0002, 0010, 0011, 0012, 0020, 0021, 0022,\\ &0100, 0101, 0102, 0110, 0111, 0112, 0120, 0121, 0122,\\ &0200, 0201, 0202, 0210, 0211, 0212, 0220, 0221, 0222\};\\ S_1 = \{&1100, 1101, 1102, 1120, 1121, 1122\};\\ S_2 = \{&2100, 2101, 2102, 2120, 2121, 2122, 2200, 2201, 2202\};\\ S_3 = \{&1111, 2211, 2222\}. \end{aligned}$$

Let $G = UB(d, n)$ and, for $x = x_1x_2 \ldots x_n \in V(G)$, let

$$N_G^{(L)}(x) = \{\alpha x_1x_2 \ldots x_{n-1} \mid \alpha \in I_d\} \text{ and } N_G^{(R)}(x) = \{x_2x_3 \ldots x_n\beta \mid \beta \in I_d\}.$$

Then $N_G(x) = N_G^{(L)}(x) \cup N_G^{(R)}(x)$.

Lemma 2.2. *S_i is an independent set of $UB(d, n)$ for each $i = 1, 2, \ldots, d$.*

Proof. Let $G = UB(d, n)$ and $x = i^{n_i}(i-1)^{n_{i-1}} \ldots t^{n_t}\alpha x_{k+2}x_{k+3} \ldots x_n$ be any vertex in S_i $(1 \le i \le d-1)$. Then

(2.2) $$\begin{aligned} N_G^{(L)}(x) &= \{\gamma i^{n_i}(i-1)^{n_{i-1}} \ldots t^{n_t}\alpha x_{k+2}x_{k+3} \ldots x_{n-1} |\ \gamma \in I_d\},\\ N_G^{(R)}(x) &= \{i^{n_i-1}(i-1)^{n_{i-1}} \ldots t^{n_t}\alpha x_{k+2}x_{k+3} \ldots x_n\beta |\ \beta \in I_d\}. \end{aligned}$$

For any $y \in N_G^{(L)}(x)$, we have $y = \gamma i^{n_i}(i-1)^{n_{i-1}} \ldots t^{n_t}\alpha x_{k+2}x_{k+3} \ldots x_{n-1}$. If $\gamma \neq i$, then $y \notin S_i$ clearly. If $\gamma = i$, then $y = i^{n_i+1}(i-1)^{n_{i-1}} \ldots t^{n_t}\alpha x_{k+2}x_{k+3} \ldots x_{n-1}$. Since $k = n_t + n_{t+1} + \ldots + (n_i+1) \equiv 1 \pmod 2$, that is, $(n_t, n_{t+1}, \ldots, (n_i+1)) \notin \mathsf{P}_{k,i-t+1}$, $y \notin S_i$. This implies $N_G^{(L)}(x) \cap S_i = \emptyset$.

For any $y \in N_G^{(R)}(x)$, $y = i^{n_i-1}(i-1)^{n_{i-1}}(i-2)^{n_{i-2}} \ldots t^{n_t}\alpha x_{k+2}x_{k+3} \ldots x_n\beta$.

If $n_i > 1$, then $n_i - 1 \neq 0$. But $k = n_t + n_{t+1} + \ldots + (n_i - 1) \equiv 1 \pmod 2$, that is, $(n_t, n_{t+1}, \ldots, (n_i - 1)) \notin \mathsf{P}_{k,i-t+1}$, and so $y \notin S_i$.

If $n_i = 1$, then $y = (i-1)^{n_{i-1}}(i-2)^{n_{i-2}} \ldots t^{n_t}\alpha x_{k+2}x_{k+3} \ldots x_n\beta$. Since $k = n_t + n_{t+1} + \ldots + n_{i-1} \equiv 1 \pmod 2$, that is, $(n_t, n_{t+1}, \ldots, n_{i-1}) \notin \mathcal{P}_{k,i-t}$, and so $y \notin S_i$. This implies $N_G^{(R)}(x) \cap S_i = \emptyset$.

So, $N_G(x) \cap S_i = \emptyset$ for any $x \in S_i$, which implies that no two vertices of S_i are adjacent. Thus, S_i is an independent set for each $i = 1, 2, \ldots, d-1$.

Similarly, we can prove that S_d is also an independent set. The lemma follows. ∎

Lemma 2.3. *For each* $k = 0, 1, \ldots, d$*, let* $G_k = G[S_0 \cup S_1 \cup \ldots \cup S_k]$. *Then* G_k *is acyclic.*

Proof. The proof proceeds by induction on $k \geq 0$. Suppose to the contrary that G_0 contains a cycle C. Then C contains a vertex $x = x_1x_2 \ldots x_n$ different from $u = 00 \ldots 0$. Assume the i-th position $x_i \neq 0$ in x. Then the vertex in C with the first position x_i is not in S_0, a contradiction. Thus, the conclusion holds when $k = 0$

Assume the conclusion is true for each ℓ with $0 \leq \ell < i$ and $i < d$. To prove that G_i is acyclic, we only need to show that any vertex $x \in S_i$ has at most one neighbor in G_{i-1} since G_{i-1} is acyclic by the induction hypothesis.

Choose any $x \in S_i$, that is,

$$x = i^{n_i}(i-1)^{n_{i-1}} \ldots t^{n_t} \alpha x_{k+2} x_{k+3} \ldots x_n,$$

where $k = n_t + \ldots + n_i$, $k \equiv 0 \pmod 2$ and $n_i, n_t \neq 0$. Then $N_G^{(L)}(x)$ and $N_G^{(R)}(x)$ are expressed as that in (2.2). It is clear that $N_G^{(L)}(x)$ can be expressed as

$$\begin{aligned} N_G^{(L)}(x) = & \{0 i^{n_i}(i-1)^{n_{i-1}} \ldots t^{n_t} \alpha x_{k+2} x_{k+3} \ldots x_{n-1}\} \\ & \cup \{i^{n_i+1}(i-1)^{n_{i-1}} \ldots t^{n_t} \alpha x_{k+2} x_{k+3} \ldots x_{n-1}\} \\ & \cup \{\gamma\, i^{n_i}(i-1)^{n_{i-1}} \ldots t^{n_t} \alpha x_{k+2} \ldots x_{n-1} |\ \gamma \in I_d, \gamma \neq 0, i\}. \end{aligned}$$

Since $n_t + \ldots + n_i \equiv 0 \pmod 2$, $n_t + \ldots + (n_i + 1) \equiv 1 \pmod 2$, and so $i^{n_i+1}(i-1)^{n_{i-1}} \ldots t^{n_t} \alpha x_{k+2} x_{k+3} \ldots x_{n-1}$ is not in neither S_i nor S_j for each $j = 0, 1, \ldots, i-1$. Since $\gamma \notin \{0, i\}$, the vertex $\gamma\, i^{n_i}(i-1)^{n_{i-1}} \ldots t^{n_t} \alpha x_{k+2} \ldots x_{n-1}$ is not in S_j for each $j = 0, 1, \ldots, i-1$. Thus,

$$N_G^{(L)}(x) \cap V(G_{i-1}) = \{0 i^{n_i}(i-1)^{n_{i-1}} \ldots t^{n_t} \alpha x_{k+2} \ldots x_{n-1}\} \subset S_0.$$

For $N_G^{(R)}(x)$, if $n_i > 1$ then $n_t + n_{t+1} + \ldots + n_i - 1 \equiv 1 \pmod 2$; if $n_i = 1$ then $n_t + n_{t+1} + \ldots + n_{i-1} \equiv 1 \pmod 2$, and so $N_G^{(R)}(x) \cap V(G_{i-1}) = \emptyset$. Thus,

$$N_{G_{i-1}}(x) = N_G(x) \cap V(G_{i-1}) = N_G^{(L)}(x) \cap V(G_{i-1}) = \{y\},$$

where $y = 0 i^{n_i}(i-1)^{n_{i-1}} \ldots t^{n_t} \alpha x_{k+2} \ldots x_{n-1} \in S_0$.

Because S_i is an independent set and G_{i-1} is acyclic, the induced subgraph G_i is acyclic for each $i = 0, 1, \ldots, d-1$.

Similarly, we can prove that G_d is acyclic. The lemma follows. ■

For each $k = 0, 1, 2, \ldots, n$, let

$$T_0 = S_0,$$
$$T_k = \left\{\begin{array}{l} (d-1)^{n_{d-1}} \ldots t^{n_t} \alpha x_{k+2} x_{k+3} \ldots x_n |\ x_j \in I_d, \\ \alpha \in I_d \setminus \{1, 2, \ldots, t\}, (n_t, \ldots, n_{d-1}) \in \mathsf{P}_{\ k,d-t}, \\ n_t \neq 0, 1 \leq t \leq d-1 \end{array}\right\},\ 1 \leq k \leq n-1,$$
$$T_n = \{(d-1)^{n_{d-1}} \ldots t^{n_t} |\ (n_t, \ldots, n_{d-1}) \in \mathsf{P}_{\ n,d-t}, n_t \neq 0, 1 \leq t \leq d-1\}.$$

It is clear that

$$V(d,n) = \bigcup_{j=0}^{n} T_j \quad \text{and } T_i \cap T_j = \emptyset, \quad 1 \leq i \neq j \leq n. \tag{2.3}$$

For example, for $d = 3,\ n = 4$, we have

$$\begin{aligned} T_0 = S_0 = \ & \{0000, 0001, 0002, 0010, 0011, 0012, 0020, 0021, 0022, \\ & 0100, 0101, 0102, 0110, 0111, 0112, 0120, 0121, 0122, \\ & 0200, 0201, 0202, 0210, 0211, 0212, 0220, 0221, 0222\}; \\ T_1 = \ & \{1000, 1001, 1002, 1010, 1011, 1012, 1020, 1021, 1022, \\ & 1200, 1201, 1202, 1210, 1211, 1212, 1220, 1221, 1222, \\ & 2000, 2001, 2002, 2010, 2011, 2012, 2020, 2021, 2022\}; \\ T_2 = \ & \{1100, 1101, 1102, 1120, 1121, 1122, 2120, 2121, \\ & 2122, 2200, 2201, 2202, 2100, 2101, 2102, \}; \\ T_3 = \ & \{1110, 1112, 2112, 2212, 2220, 2110, 2210\}; \\ T_4 = \ & \{1111, 2111, 2211, 2221, 2222\}. \end{aligned}$$

and $V(3,4) = T_0 \cup T_1 \cup T_2 \cup T_3 \cup T_4$.

Theorem 2.1. *Let $\overline{S} = V(d,n) \backslash S$, where S is defined in (2.1). Then $\overline{S}$ is a feedback vertex set of $UB(d,n)$. Moreover,*

$$|\overline{S}| = \sum_{j=1, j \equiv 1 \,(\mathrm{mod}\, 2)}^{n-1} |T_j| + |T_n - S_d|. \tag{2.4}$$

Proof. By Lemma 2.3, $\overline{S}$ is a feedback vertex set of $UB(d,n)$ immediately. We prove (2.4) below. We first show that

$$\bigcup_{i=0}^{d-1} S_i = \bigcup_{j=1, j \equiv 0 \,(\mathrm{mod}\, 2)}^{n-1} T_j \quad \text{and } S_d \subseteq T_n. \tag{2.5}$$

We only need to consider the case of $i, j \neq 0$ since $S_0 = T_0$. Arbitrarily choose i $(1 \leq i \leq d-1)$ and

$$x = i^{n_i}(i-1)^{n_{i-1}} \dots t^{n_t} \alpha x_{k+2} x_{k+3} \dots x_n \in S_i,$$

where α, n_i, n_t, k are fixed and $\alpha \in I_d \setminus \{1, \dots, t\}$, $1 \leq t \leq i, (n_t, \dots, n_i) \in \mathsf{P}_{k,i-t+1}$, and $n_i, n_t \neq 0, 2 \leq k \leq n-1, k \equiv 0 \pmod 2$. It is clear that $(n_t, \dots, n_i) \in \mathsf{P}_{k,i-t+1}$ means $(n_t, \dots, n_i, 0, \dots, 0) \in \mathsf{P}_{k,d-t}$, and so $x \in T_k$. Thus,

$$\bigcup_{i=1}^{d-1} S_i \subseteq \bigcup_{j=2, j\equiv 0 \pmod 2}^{n-1} T_j. \tag{2.6}$$

Conversely, arbitrarily choose j $(2 \leq j \leq n-1, j \equiv 0 \pmod 2)$ and

$$x = (d-1)^{n_{d-1}} \dots t^{n_t} \alpha x_{k+2} x_{k+3} \dots x_n \in T_j,$$

where $\alpha \in I_d \setminus \{1, 2, \dots, t\}$, $(n_t, \dots, n_{d-1}) \in \mathsf{P}_{k,d-t}$, $n_t \neq 0, 1 \leq t \leq d-1$. Let $i = \max\{\ell | \ n_\ell \neq 0, t \leq \ell \leq d-1\}$. Then $x \in S_i$, and so

$$\bigcup_{j=2, j\equiv 0 \pmod 2}^{n-1} T_j \subseteq \bigcup_{i=1}^{d-1} S_i. \tag{2.7}$$

Combining (2.6) with (2.7) yields the equality in (2.5). $S_d \subseteq T_n$ clearly from the definitions of S_d and T_n, and so the conclusion in (2.5) follows.

It follows from (2.3) and (2.5) that

$$|S| = \sum_{j=0, j\equiv 0 \pmod 2}^{n-1} |T_j| + |S_d|,$$

and so

$$|\overline{S}| = \sum_{j=1, j\equiv 1 \pmod 2}^{n-1} |T_j| + |T_n - S_d|.$$

The equation (2.4) follows and the proof of the theorem is complete. ■

Lemma 2.4. *For each k with $1 \leqslant k < n$, we have*

$$|T_k| = \sum_{t=1}^{d-1} \binom{k+d-t-2}{k-1} (d-t) d^{n-k-1} \tag{2.8}$$

and

$$|T_n| = \binom{n+d-2}{d-2}. \tag{2.9}$$

Proof. From the definition of T_k, for a vertex

$$x = (d-1)^{n_{d-1}} \dots t^{n_t} \alpha x_{k+2} x_{k+3} \dots x_n \in T_k, \tag{2.10}$$

the first k positions depend on the choice of $(n_t, n_{t+1}, \dots, n_{d-1}) \in \mathsf{P}_{k,d-t}$ with $n_t \neq 0$. Let

$$\mathsf{P}^{\,t}_{k,d-t} = \{(n_t, n_{t+1}, \dots, n_{d-1}) \in \mathsf{P}_{k,d-t} |\ n_t \neq 0\}.$$

Then $(n_t, n_{t+1}, \dots, n_{d-1}) \in \mathsf{P}^{\,t}_{k,d-t} \Leftrightarrow (n_t, n_{t+1}, \dots, n_{d-1}) \in \mathsf{P}_{k-1,d-t}$.

It follows from Lemma 2.1 that for each k with $1 \leq k \leq n$,

$$|\mathsf{P}^{\,t}_{k,d-t}| = |\mathsf{P}_{k-1,d-t}| = \binom{k+d-t-2}{k-1}. \tag{2.11}$$

Since $\alpha \in I_d \setminus \{1, 2, \dots, t\}$, there are $(d-t)$ choices of α in (2.10). Also since $x_j \in I_d$, there are d^{n-k-1} of the subsequence $x_{k+2}x_{k+3}\dots x_n$ in (2.10). Therefore, for a fixed t with $1 \leq t \leq d-1$, there are $\binom{k+d-t-2}{k-1}(d-t)d^{n-k-1}$ choices of the vertex x in (2.10). Thus, for a fixed k with $1 \leqslant k < n$, we have

$$|T_k| = \sum_{t=1}^{d-1} \binom{k+d-t-2}{k-1}(d-t)d^{n-k-1},$$

and so (2.8) follows. For $k = n$, using (2.11), we have that

$$|T_n| = \sum_{t=1}^{d-1} \binom{n+d-t-2}{n-1} = \binom{n+d-2}{n} = \binom{n+d-2}{d-2},$$

where the second equality is obtained by using the combinatorial equality (Pascal's formula)

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}. \tag{2.12}$$

The equality (2.9) follows. ■

Lemma 2.5. *If n is even, then for $d = 2$, $|T_n - S_2| = 0$, and for any $d \geq 3$,*

$$|T_n - S_d| = \sum_{k \equiv 1 \,(\mathrm{mod}\, 2)}^{n-1} \binom{k+d-2}{d-3}. \tag{2.13}$$

Proof. For each t with $1 \le t \le d-1$, let

$$B_t = \left\{ \begin{array}{c} \{(d-1)^{n_{d-1}}(d-2)^{n_{d-2}} \ldots t^{n_t} |\text{ and } n_t \equiv 1 \pmod 2, \\ (n_t, n_{t+1}, \ldots, n_{d-1}) \in \mathsf{P}_{n,d-t}, n \equiv 0 \pmod 2\} \end{array} \right\}.$$

From the definitions of S_d and T_n, we have

$$\bigcup_{t=1}^{d-1} B_t = T_n - S_d.$$

If $t = d-1$ then $n_{d-1} = n$. Since $n \equiv 0 \pmod 2$, we have $|B_{d-1}| = 0$, and so $|T_n - S_2| = 0$ for $d = 2$.

For $d \ge 3$ and $1 \le t \le d-2$, if $(n_t, n_{t+1}, \ldots, n_{d-1}) \in \mathsf{P}_{n,d-t}$ with $n \equiv 0 \pmod 2$ and $n_t \equiv 1 \pmod 2$ then $(n_{t+1}, \ldots, n_{d-1})$ is a solution of the indefinite equation $z_{t+1} + z_{t+2} + \ldots + z_{d-1} = n - n_t$. Let

$$\mathsf{P}^{o}_{n,d-t} = \{(n_t, n_{t+1}, \ldots, n_{d-1}) \in \mathsf{P}_{n,d-t} |\ n_t \equiv 1 (\mathrm{mod}\, 2), n \equiv 0 (\mathrm{mod}\, 2)\}.$$

Then $(n_t, n_{t+1}, \ldots, n_{d-1}) \in \mathsf{P}^{o}_{n,d-t} \Leftrightarrow (n_{t+1}, \ldots, n_{d-1}) \in \mathsf{P}_{n-n_t,d-t-1}$.

Since n_t can be taken over all odd numbers in $\{1, 2, \ldots, n-1\}$, for $1 \le t \le d-2$, we have that

$$\begin{aligned} |B_t| &= |\mathsf{P}^{o}_{n,d-t}| = \sum_{k \equiv 1 \,(\mathrm{mod}\, 2)}^{n-1} |\mathsf{P}_{k,d-t-1}| \\ &= \sum_{k \equiv 1 \,(\mathrm{mod}\, 2)}^{n-1} \binom{k+d-t-2}{k}, \end{aligned}$$

and so

$$\begin{aligned} |T_n - S_d| &= \sum_{t=1}^{d-1} |B_t| = \sum_{t=1}^{d-2} \sum_{k \equiv 1 \,(\mathrm{mod}\, 2)}^{n-1} \binom{k+d-t-2}{k} \\ &= \sum_{k \equiv 1 \,(\mathrm{mod}\, 2)}^{n-1} \sum_{t=1}^{d-2} \binom{k+d-t-2}{k} \\ &= \sum_{k \equiv 1 \,(\mathrm{mod}\, 2)}^{n-1} \binom{k+d-2}{k+1} \\ &= \sum_{k \equiv 1 \,(\mathrm{mod}\, 2)}^{n-1} \binom{k+d-2}{d-3}. \end{aligned}$$

The last equality is obtained by using the combinatorial equality (2.12). The lemma follows. ∎

3. Bounds on Feedback Numbers

In this section, we give the bounds on the feedback number $f(d, n)$.

Lemma 3.1. (Beineke and Vandell [5]). *For feedback vertex set F in a graph G with υ vertices, ε edges and maximum degree $\triangle$, it holds that*

$$|F| \geq \left\lceil \frac{\varepsilon - \upsilon + 1}{\triangle - 1} \right\rceil.$$

Theorem 3.1. *For any $d \geq 3$ and $n \geq 1$:*

$$\left\lceil \frac{d^{n+1} - d - \frac{d(d-1)}{2} - d^n + 1}{2d-1} \right\rceil \leq f(d,n) \leq d^n \left(1 - \left(\frac{d}{1+d}\right)^{d-1}\right) + \binom{n+d-2}{d-2}.$$

Proof. Substituting $\upsilon = d^n$, $\varepsilon = d^{n+1} - d - \frac{1}{2}d(d-1)$ and $\Delta = 2d$ into Lemma 3.1, we immediately have

$$f(d,n) \geq \left\lceil \frac{d^{n+1} - d - \frac{d(d-1)}{2} - d^n + 1}{2d - 1} \right\rceil.$$

We now show that

$$f(d,n) \leq d^n \left(1 - \left(\frac{d}{1+d}\right)^{d-1}\right) + \binom{n+d-2}{d-2}.$$

By Theorem 2.1, we have $f(d,n) \leq |\overline{S}|$. To estimate the value of $|\overline{S}|$, we consider two cases according to the parity of n£¬ that is,

$$|\overline{S}| = |T_1| + |T_3| + |T_5| + \ldots + \begin{cases} |T_{n-1}| + |T_n - S_d| & \text{if } n \text{ is even;} \\ |T_{n-2}| + |T_n| & \text{if } n \text{ is odd.} \end{cases}$$

Noting from Lemma 2.4 and Lemma 2.5 that

$$\begin{aligned} |T_n - S_d| &= \sum_{k \equiv 1 \pmod 2}^{n-1} \binom{k+d-2}{d-3} \\ &< \sum_{k=1}^{n-1} \binom{k+d-2}{d-3} \\ &< \binom{n+d-2}{d-2} = |T_n|, \end{aligned}$$

we only need to estimate the value of $|\overline{S}|$ when n is odd. From (2.8) we have that

$$\begin{aligned}
\sum_{k=1,k\equiv 1\,(\mathrm{mod}\,2)}^{n-2} |T_k| &= \sum_{k=1,k\equiv 1\,(\mathrm{mod}\,2)}^{n-2} \sum_{t=1}^{d-1} \binom{k+d-t-2}{k-1}(d-t)\, d^{n-k-1} \\
&= \sum_{k=1,k\equiv 1\,(\mathrm{mod}\,2)}^{n-2} \frac{(d+k-2)!(kd-k+1)}{(k+1)!(d-2)!}\, d^{n-k-1} \\
&\leq \sum_{k=1,k\equiv 1\,(\mathrm{mod}\,2)}^{+\infty} \frac{(d+k-2)!(kd-k+1)}{(k+1)!(d-2)!}\, d^{n-k-1} \\
&= \sum_{k=1,k\equiv 1\,(\mathrm{mod}\,2)}^{+\infty} \frac{(d+k-2)!}{k!(d-2)!}\, d^{n-k} \\
&\quad - \sum_{k=1,k\equiv 1\,(\mathrm{mod}\,2)}^{+\infty} \frac{(d+k-1)!}{(k+1)!(d-2)!}\, d^{n-k-1} \\
&= \sum_{k=1,k\equiv 1\,(\mathrm{mod}\,2)}^{+\infty} \frac{(d+k-2)!}{k!(d-2)!}\, d^{n-k} \\
&\quad - \sum_{t=2,t\equiv 0\,(\mathrm{mod}\,2)}^{+\infty} \frac{(d+t-2)!}{t!(d-2)!}\, d^{n-t} \\
&= -\sum_{k=1}^{+\infty} \frac{(d+k-2)!}{k!(d-2)!}\, (-d)^{n-k} \\
&= d^n \left(1 - \sum_{k=0}^{+\infty} \frac{(d+k-2)!}{k!(d-2)!}\, (-d)^{-k}\right).
\end{aligned}$$

Substituting $z = -\frac{1}{d}$ and $t = d-1$ into the generating function

$$\frac{1}{(1-z)^t} = \sum_{k=0}^{+\infty} \binom{k+t-1}{k} z^k$$

immediately yields that

$$\sum_{k=1,k\equiv 1\,(\mathrm{mod}\,2)}^{n-2} |T_k| \leq d^n \left(1 - \frac{d^{d-1}}{(1+d)^{d-1}}\right).$$

Thus,

$$|\overline{S}| \leq d^n \left(1 - \frac{d^{d-1}}{(1+d)^{d-1}}\right) + \binom{n+d-2}{d-2}.$$

The proof of the theorem is complete. ■

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