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MODULE HOMOMORPHISMS ASSOCIATED WITH BANACH ALGEBRAS

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Abstract. Let A be a Banach algebra. In this paper, among the other things, we present a few results in the theory of homomorphisms on A^*. We want to find out when the equality $T(af) = aT(f)$ for every $a \in A$ and $f \in A^*$ implies the equality $T(Ff) = FT(f)$ for every $f \in A^*$ and $F \in A^{**}$. One of the main results of this paper is to introduce and study the notion of a weakly almost periodic operator.

1. Introduction

Let A be a Banach algebra. Terminologies and notations not explained in this section will be explained or referenced in the next section. The investigation of conditions which force a bounded linear map on dual a Banach algebra to be λ_a^* $(a \in A)$ has been of interest in recent literature. In [2], Baker, Lau and Pym proved that $Hom_A(A^*, A^*) = (A^*A)^*$. For some Banach algebras A, they also proved that if $T : A^* \to A^*A$ is a bounded linear map and satisfyies $T(xf) = xT(f)$ for all $x \in A$, $f \in A^*$, then there is $a \in A$ such that $T = \lambda_a^*$ (see Theorem 1.1 in [2]). For a locally compact abelian group G, the set of all bounded linear maps from $L^p(G)$ into $L^q(G)$ which commute with translation has been studied by Larsen [12]. He proved that the multipliers for $A_p(G)$ (the Herz-Figa-Talamanca algebra) can be identified with the bounded measures on G provided G is noncompact. In [7], we have studied the homomorphisms on hypergroup algebras. For a locally compact group G, the bounded linear operators on $L^\infty(G)$ into $L^\infty(G)$ which commute with convolutions and translations have been studied by Lau in [14] and by Lau and Pym in [18]. They also went further, and for several subspaces H of $L^\infty(G)$, they obtained a number of interesting and nice results. This paper is organized as

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follows: After preparation of some notations in Section 2, we study in Section 3 the operators on the dual some of Banach algebras. For a Banach algebra A, we want to find out when the equality $T(af) = aT(f)$ $(a \in A,\ f \in A^*)$ implies that $T(Ff) = FT(f)$ $(f \in A^*,\ F \in A^{**})$. For some Banach algebras A, the set of all bounded linear maps $T : A^* \to A^*$ which $T(Ff) = FT(f)$ $(F \in A^{**}$ and $f \in A^*)$ can be identified with A. In Section 4, we introduce weakly almost periodic operators on dual of a Banach algebra A. We study the relationship of weakly almost periodic operators with weakly almost periodic functionals in A^*.

2. Notation and Preliminary Results

We introduce our notations briefly; for other ideas used here we refer the reader to [2, 8] and [11]. Let A be a Banach algebra. Then A^* and A^{**} will denote the first and second conjugate spaces of A. For any $a \in A$, let λ_a be the left multiplication operators determined by a. A bounded linear map $T : A \to A$ is called a left multiplier if $T(ab) = T(a)b$ for all $a, b \in A$. Let $\mathcal{M}(A)$ denote the algebra of all left multipliers on A. The theory of multipliers studied by Larsen [12] and has received a good deal of attention from harmonic analysts.

Let A be a Banach algebra. The first Arens product on A^{**} is defined in stages as follows. Let $a, b \in A$, $f \in A^*$ and $F, G \in A^{**}$.

Define $fa \in A^*$ by $\langle fa, b\rangle = \langle f, ab\rangle$.

Define $Ff \in A^*$ by $\langle Ff, a\rangle = \langle F, fa\rangle$.

Define $GF \in A^{**}$ by $\langle GF, f\rangle = \langle G, Ff\rangle$.

Then A^{**} is a Banach algebra (for more details see [2]). For G fixed in A^{**}, the mapping $F \mapsto FG$ is weak*-weak* continuous on A^{**}. For F fixed in A^{**}, the mapping $G \mapsto FG$ is in general not weak*-weak* continuous on A^{**}.

The second Arens multiplication is defined as follows: For a, b in A, f in A^* and F, G in A^{**}, the elements $a.f$, $f.F$ of A^* and $F.G$ of A^{**} are defined by the equalities

$$\langle b, a.f\rangle = \langle ba, f\rangle, \quad \langle a, f.F\rangle = \langle a.f, F\rangle, \quad \langle f, F.G\rangle = \langle f.F, G\rangle.$$

The symbols $b(A)$, $b(A^*)$ will be used for the unit ball in A, A^*, respectively.

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure. Let $C_\circ(G)$ be the closed subspace of $C_b(G)$ consisting of all functions in $C_b(G)$ which vanish at infinity. Let $LUC(G)$ denote the space of bounded left uniformly continuous functions on G. The Banach spaces $L^p(G)$, $1 \le p \le \infty$, are as defined in [11].

3. Characterization Some of Homomorphisms

We denote by $wap(A^*)$ the space of $f \in A^*$ for which the mapping $a \mapsto af$ from A into A^* is weakly compact (see [2], [6] and [19]). This is in fact equivalent to the condition that $a \mapsto fa$ be weakly compact (see Theorem 1 in [4]). $wap(A^*)$ is a Banach subspace. It is known that

$$wap(A^*) = \{f \in A^*;\ \langle FG, f\rangle = \langle F.G, f\rangle \text{ for all } F, G \in A^{**}\}$$

(for the equivalence of these two descriptions see, for example, Theorem 1 of [4] and the remarks which follow it). In the following theorem A^* can be identified with the set of all bounded linear maps $T : A^{**} \to A^*$ which are weak*-weak* continuous and $T(FG) = FT(G)$ for all $F, G \in A^{**}$.

Theorem 3.1. *Let A be a Banach algebra with a bounded approximate identity bounded by 1. Let $T : A^{**} \to A^*$ be a bounded linear map such that $T(FG) = FT(G)$ for every $F, G \in A^{**}$. Then the following conditions hold:*

(1) *there exists a unique $f \in A^*$ such that $T(a) = af$ for all $a \in A$ and $\|T\| = \|f\|$. In addition, T is weakly compact if and only if $f \in wap(A^*)$.*

(2) *there exists a unique $f \in A^*$ such that $T(F) = Ff$ for all $F \in A^{**}$ and $\|T\| = \|f\|$. T is a weak*-weak* continuous.*

Moreover the correspondence between T and f defines a linear isometric from the set of all $T : A^{**} \to A^*$ which are weak*-weak* continuous and satisfy $T(FG) = FT(G)$ for all $F, G \in A^{**}$, onto A^*.

Proof.

(1) Let (e_α) be a bounded approximate identity bounded by 1 for A. The net $(T(e_\alpha))$ admits a subnet $(T(e_\beta))$ converging to a functional f on A in the weak* topology (see V.4.2 in [5]). For $a \in A$, we claim that $T(a) = af$. Because if $b \in A$, then

$$\begin{aligned}\langle T(a), b\rangle &= \lim_\alpha \langle T(ae_\alpha), b\rangle = \lim_\alpha \langle aT(e_\alpha), b\rangle = \lim_\alpha \langle T(e_\alpha), ba\rangle \\ &= \langle f, ba\rangle = \langle fb, a\rangle = \langle af, b\rangle.\end{aligned}$$

Thus $T(a) = af$. Let $h \in A^*$ and let $af = ah$ for all $a \in A$. In particular then for each $\alpha \in I$, $\langle f - h, e_\alpha a\rangle = \langle af - ah, e_\alpha\rangle = 0$. Hence $\langle f, a\rangle = \langle h, a\rangle$ for each $a \in A$ which implies that $f = h$, since (e_α) is a bounded approximate identity for A. Therefore f is unique. Clearly $\|T(a)\| = \|af\| \le \|a\|\|f\|$ shows that $\|T\| \le \|f\|$. Now, let $\epsilon > 0$ be given. There exists $a \in A$ with $\|a\| \le 1$ such that

$$|\langle f, a\rangle| \ge \|f\| - \epsilon.$$

For every $\alpha \in I$, we have

$$\begin{aligned} |\langle f, e_\alpha a\rangle| &= |\langle af, e_\alpha\rangle| = |\langle T(a), e_\alpha\rangle| \\ &\leq \|T(a)\| \leq \|T\|. \end{aligned}$$

It follows that

$$\|f\| - \epsilon \leq |\langle f, a\rangle| \leq \|T\|.$$

Consequently $\|T\| = \|f\|$.

If T is weakly compact, then $\{T(F);\ F \in A^{**},\ \|F\| \leq 1\}$ is relatively weakly compact in A^*. Hence $\{af;\ a \in b(A)\}$ is relatively weakly compact in A^* and so $f \in wap(A)$.

Now let $f \in wap(A)$. Then $\{af; a \in b(A)\}$ is relatively weakly compact in A^*. We claim that

$$\{T(F);\ F \in A^{**},\ \|F\| \leq 1\} \subseteq \overline{\{af;\ a \in b(A)\}}.$$

In fact, let $F \in A^{**}$, $\|F\| \leq 1$ and (a_α) be a net in A with $\|a_\alpha\| \leq 1$ such that $a_\alpha \to F$ in the weak*-topology [21]. By compactness of $\overline{\{af;\ a \in b(A)\}}$, we can assume $a_\alpha f \to g$ in A^* (in the weak topology), passing to a subnet if necessary. Clearly $a_\alpha f \to Ff$ in the weak* topology and so $g = Ff$. Consequently

$$Ff \in \overline{\{af,\ a \in b(A)\}}.$$

It follows that T is weakly compact.

(2) Let (e_α) be a bounded approximate identity for A. Then we may suppose that (e_α) converges in the weak*-topology on A^{**}, say to E [2]. It is easy to see that $FE = F$ for every $F \in A^{**}$. Put $T(E) = f$. For every $F \in A^{**}$, $T(FE) = FT(E) = Ff$. From part 1, f is unique, and $\|T\| = \|f\|$. The final assertion of the theorem is now apparent. ∎

Remark 3.2. Let G be an amenable locally compact group. Then the Fourier algebra $A(G)$ has a bounded approximate identity. Hence Theorem 3.1 is applicable in this case. Furthermore, the subspace $wap(A(G)^*) \subseteq A(G)^*$ was studied in [16] and the Banach algebra $A(G)^{**}$ was studied in [15].

Let A be a Banach algebra with a bounded approximate identity bounded by 1. By A^*A, we denote the subspace of A^* consisting of the functionals of the form fa, for all f in A^* and a in A. This is known to be a norm closed linear subspace of A^* [19].

Baker, Lau and Pym [2] proved that $Hom_A(A^*, A^*)$ (where $T \in Hom_A(A^*, A^*)$ means $T(fa) = T(f)a$ for every $f \in A^*$ and $a \in A$) can be identified isometrically isomorphic with $(A^*A)^*$. Indeed, for every $T \in Hom_A(A^*, A^*)$ there exists a unique element $n \in (A^*A)^*$ such that $T(f) = nf$ for all $f \in A^*$. It is easy to see that, A is a right ideal in $(A^*A)^*$ if and only if every $T \in Hom_A(A^*, A^*)$ is weak*-weak* continuous.

Example 3.3.

(a) Let G be a locally compact abelian group. We know that $L_0^\infty(G)$ is the space of bounded measurable functions on G which vanish at infinity [18]. As known [18], for any $F \in L_0^\infty(G)^*$ and $f \in L_0^\infty(G)$, $Ff \in L_0^\infty(G)$. In this case, the first Arens multiplication is well defined on $L_0^\infty(G)^*$ and $L_0^\infty(G)^*$ is a Banach algebra. By Theorem 2.11 in [18], $L^1(G)$ is a two-sided ideal of $L_0^\infty(G)^*$. It is known that $L^1(G)$ is a two-sided ideal in $L^1(G)^{**}$ if and only if G is a compact group [10]. Let $T \in Hom_{L^1(G)}(L_0^\infty(G), L_0^\infty(G))$, and (e_α) be a bounded approximate identity for $L^1(G)$ [11]. For $F \in L_0^\infty(G)^*$, $f \in L_0^\infty(G)$, we have

$$\begin{aligned}\langle T(Ff), \mu\rangle &= \lim_\alpha \langle T(Ff), \mu * e_\alpha\rangle = \lim_\alpha \langle e_\alpha T(Ff), \mu\rangle \\ &= \lim_\alpha \langle T(e_\alpha Ff), \mu\rangle = \lim_\alpha \langle e_\alpha FT(f), \mu\rangle \\ &= \lim_\alpha \langle FT(f), \mu * e_\alpha\rangle = \langle FT(f), \mu\rangle,\end{aligned}$$

where $\mu \in L^1(G)$. This shows that $T(Ff) = FT(f)$.

(b) Let A be a Banach algebra and let T be a bounded linear operator from A^* into A^* which is weak*-weak* continuous and satisfy $T(af) = aT(f)$ for all $a \in A$, $f \in A^*$. Then $T(Ff) = FT(f)$ for all $F \in A^{**}$ and $f \in A^*$ (see Theorem 3.1 and its proof).

(c) Let G be a locally compact abelian group. Let T be a bounded linear operator from $L^\infty(G)$ into $L^\infty(G)$ such that $T(Ff) = FT(f)$ for all $F \in L^1(G)^{**}$ and $f \in L^\infty(G)$. Then $|T|(Ff) = F|T|(f)$ ($|T|$ is the modulus of T, see [1] and [8]) for all $F \in L^1(G)^{**}$ an $f \in L^\infty(G)$. In fact, T^* is a left multiplier on $L^1(G)^{**}$. Now, let $f \in L^1(G)$ and let $\{F_\alpha\}$ be a net in $L^1(G)^{**}$ such that $F_\alpha \to F$ in the weak* topology. Since T^* is weak*-weak* continuous, $\{T^*(f)F_\alpha\}$ converges to $T^*(f)F$ in the weak* topology. Hence $T^*(f) \in L^1(G)$ see [17]. It follows that $T^* : L^1(G) \to L^1(G)$ is a left multiplier. Consequently by Theorem 1 in [23], there exists $\mu \in M(G)$ such that $T^* = \lambda_\mu^{**}$. It is easy to see that $T = \lambda_\mu^*$. By Theorem 3.5 in [8],

$$\begin{aligned}|T|(Ff) &= |\lambda_\mu^*|(Ff) = \lambda_{|\mu|}^*(Ff) \\ &= Ff|\mu| = F\lambda_{|\mu|}^*(f) \\ &= F|\lambda_\mu^*|(f) = F|T|(f).\end{aligned}$$

Moreover, if $T \in Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$ is weak*-weak* continuous, then $|T| \in Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$.

Question: Does Example 3.3 (c) remain valid when $T \in Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$ is not weak*-weak* continuous?

It is well known (and easy to prove) that for every $G \in A^{**}$ the mapping $F \mapsto FG$ from A^{**} into A^{**} is weak*-weak* continuous. The set of all G in A^{**} for which $F \mapsto GF$ from A^{**} into A^{**} is weak*-weak* continuous is called the topological center of A^{**}. The set of all F in $(A^*A)^*$ for which $G \mapsto GF$ from $(A^*A)^*$ into $(A^*A)^*$ is weak*-weak* continuous is called the topological center of $(A^*A)^*$. The topological centers of A^{**} and $(A^*A)^*$ are denoted respectively by $Z_t(A^{**})$ and $Z_t((A^*A)^*)$ (more information on this problem can be found in [2] and [19]). A. T. Lau and V. Losert [17] have proved that if G is a locally compact topological group, then the topological center of $L^1(G)^{**}$ is $L^1(G)$.

Theorem 3.4. *Let G be a locally compact abelian group. Then the following conditions are equivalent:*

(*1*) *G is a compact group.*

(*2*) *for every bounded linear operator $T \in Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$, $T(Ff) = FT(f)$ for all $F \in L^1(G)^{**}$ and $f \in L^\infty(G)$.*

Proof. Let G be a compact group. Then $L_0^\infty(G) = L^\infty(G)$. So, by Example 3.3 (a), for every $T \in Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$ we have $T(Ff) = FT(f)$ for all $F \in L^1(G)^{**}$ and $f \in L^\infty(G)$.

Conversely, let for every $T \in Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$, we have $T(Ff) = FT(f)$ for all $F \in L^1(G)^{**}$ and $f \in L^\infty(G)$. So, by Theorem 1.1 in [2], $GFf = FGf$ for every $F, G \in L^1(G)^{**}$ and $f \in L^\infty(G)$. Hence for every $\mu \in L^1(G)$, we have

$$\langle GF, f\mu\rangle = \langle FG, f\mu\rangle.$$

But $L^\infty(G)L^1(G) = LUC(G)$ [11]. Therefore $Z_t(LUC(G)^*) = LUC(G)^*$. On the other hand, $Z_t(LUC(G)^*) = M(G)$ ([13]) and $LUC(G)^* = M(G) \oplus C_0(G)^\perp$([9]). Consequently $C_0(G)^\perp = \{0\}$, we conclude that G is compact. ∎

Let A be a Banach algebra with a bounded approximate identity bounded by 1. For each a in A define $\rho_a : A^* \to [0, \infty)$ by $\rho_a(f) = \|fa\|$. The topology defined on A^* by these seminorms is denoted by τ_c (for details see [6]). It is known that $w^* \le \tau_c \le \|.\|$. Let $T \in \mathcal{B}(A^*, A^*)$ and let $T(af) = aT(f)$ for every $a \in A$ and $f \in A^*$. If T is weak*-weak* continuous, then $T(Ff) = FT(f)$ for every $F \in A^{**}$ and $f \in A^*$ (see Example 3.3). Now, let T be τ_c-τ_c continuous. Does the equality $T(af) = aT(f)$ ($a \in A$, $f \in A^*$) imply that $T(Ff) = FT(f)$ ($F \in A^{**}$, $f \in A^*$)?

Theorem 3.5. *Let A be a Banach algebra with $\mathcal{M}(A) = \{\lambda_a;\ a \in A\}$. Let $T : A^* \to A^*$ be a bounded linear map such that $T(Ff) = FT(f)$ for all $F \in A^{**}$*

and $f \in A^*$. Then $T = \lambda_a^*$ for some $a \in A$ if any one of the following conditions holds.

(*1*) T *is weak***-weak** *continuous*

(*2*) $Z_t(A^{**}) = A$.

Moreover, if A has a bounded approximate identity bounded by 1. Then $\|T\| = \|a\|$.

Proof. Let $T^* : A^{**} \to A^{**}$ be adjoint to T. Then T^* is a left multiplier on A^{**}. In fact, for $F, G \in A^{**}$, we have

$$\begin{aligned}\langle T^*(FG), f\rangle &= \langle FG, T(f)\rangle = \langle F, GT(f)\rangle = \langle F, T(Gf)\rangle \\ &= \langle T^*(F), Gf\rangle = \langle T^*(F)G, f\rangle.\end{aligned}$$

Hence $T^*(FG) = T^*(F)G$, showing that T^* is a left multiplier on A^{**}. We next show that for each $a \in A$, $T^*(a) \in A$. Let $a \in A$ and (f_α) be a net in A^* such that $f_\alpha \to f$ $(f \in A^*)$ in the weak* topology. We have

$$\begin{aligned}\lim_\alpha \langle T^*(a), f_\alpha\rangle &= \lim_\alpha \langle a, T(f_\alpha)\rangle = \lim_\alpha \langle T(f_\alpha), a\rangle \\ &= \langle T(f), a\rangle = \langle T^*(a), f\rangle,\end{aligned}$$

since T is weak*-weak* continuous. Therefore $T^*(a) \in A^{**}$ is weak*-weak* continuous. By $([21]$, Chapter 3), $T^*(a) \in A$. So T^* restricted to A is a left multiplier from A into A. Consequently by assumption, there exists $x \in A$ such that

$$T^*(a) = xa = \lambda_x^{**}(a),$$

for each $a \in A$. In particular $T^* = \lambda_x^{**}$, by weak*-continuity of the adjoint. It is clear that $T = \lambda_x^*$.

As before, the proof will be complete if we show that $T^*(a) \in A$ for all $a \in A$. To that end, suppose that $a \in A$ and (F_α) is a net in A^{**} such that $F_\alpha \to F$ $(F \in A^{**})$ in the weak* topology. Since T^* is weak*-weak* continuous and $aF_\alpha \to aF$ in the weak* topology, we have

$$T^*(aF_\alpha) \to T^*(aF) = T^*(a)F.$$

Consequently $T^*(a)F_\alpha \to T^*(a)F$, showing that $T^*(a)$ is in the topological center of A^{**}. By assumption, $T^*(a) \in A$.

Now, let (e_α) be a bounded approximate identity bounded by 1. There exists a functional $f \in A^*$ with $\|f\| \leq 1$ such that $|\langle f, a\rangle| + \epsilon \geq \|a\|$. We have

$$\begin{aligned}\|T\| &\geq \|T(f)\| = \|\lambda_a^*(f)\| = \|fa\| \geq \lim_\alpha |\langle fa, e_\alpha\rangle| \\ &= \lim_\alpha |\langle f, ae_\alpha\rangle| = |\langle f, a\rangle| \geq \|a\| - \epsilon.\end{aligned}$$

As $\epsilon > 0$ may be arbitrary, we have $\|T\| \geq \|a\|$. Clearly, $\|T\| \leq \|a\|$. Consequently $\|T\| = \|a\|$. ∎

Theorem 3.6. *Let G be a locally compact abelian group. Let T be a compact weak*-weak* continuous linear operator of $L^\infty(G)$ to itself and satisfy $T(Ff) = FT(f)$ for all $F \in L^1(G)^{**}$, $f \in L^\infty(G)$. Then every $T \in Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$ is weak*-weak* continuous.*

Proof. Let T be a compact bounded linear operator of $L^\infty(G)$ to itself and satisfy $T(Ff) = FT(f)$ for all $F \in L^1(G)^{**}$, $f \in L^\infty(G)$. It is easy to see that $T = \lambda_\mu^*$ for some $\mu \in M(G)$, since $\mathcal{M}(L^1(G)) = M(G)$ (see [12] and [23]). Indeed, T^* is a left multiplier on $L^\infty(G)^*$ (see Theorem 3.5 and its proof). Therefore $T^*|_{L^1(G)} = \lambda_\mu$ for some $\mu \in M(G)$. On the other hand, $T^* = \lambda_\mu^{**}$ is a compact left multiplier on $L^1(G)$. Therefore if (e_α) is a bounded approximate identity bounded by 1 for $L^1(G)$ [11], without loss of generality, we may assume that $\mu * e_\alpha \to \eta$ $(\eta \in L^1(G))$ in the norm topology. Now, let $f \in C_0(G)$. Since $C_0(G) \subseteq LUC(G) \subseteq L^\infty(G)L^1(G)$ [11], we have $f = g\nu$ for some $g \in L^\infty(G)$ and $\nu \in L^1(G)$. We can write

$$\begin{aligned} \langle f, \mu\rangle &= \langle g\nu, \mu\rangle = \langle g, \nu * \mu\rangle = \lim_\alpha \langle g, \nu * e_\alpha * \mu\rangle \\ &= \langle g, \nu * \eta\rangle = \langle f, \eta\rangle. \end{aligned}$$

This shows that $\mu = \eta \in L^1(G)$. On the other hand, Sakai [22] has proved that if G is a locally compact non-compact group, then the only element $\nu \in L^1(G)$ for which the operator $\eta \to \eta * \nu$ is compact, is equal to 0.

Consequently G is compact and every $T \in Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$ is weak*-weak* continuous. In fact, $L^1(G)$ is a two-sided ideal in $L^1(G)^{**}$. Let $\mu \in L^1(G)$, and let $\{F_\alpha\}$ converges to F in the weak*-topology. Let $\{e_\beta\}$ be a bounded approximate identity for $L^1(G)$ [11]. For every $f \in L^\infty(G)$ and α,

$$\begin{aligned} \langle T^*(\mu)F_\alpha, f\rangle &= \langle T^*(\mu), F_\alpha f\rangle = \langle T(F_\alpha f), \mu\rangle = \lim_\beta \langle T(F_\alpha f), e_\beta * \mu\rangle \\ &= \lim_\beta \langle \mu T(F_\alpha f), e_\beta\rangle = \lim_\beta \langle T(\mu F_\alpha f), e_\beta\rangle = \lim_\beta \langle \mu F_\alpha T(f), e_\beta\rangle \\ &= \lim_\beta \langle F_\alpha T(f), e_\beta * \mu\rangle = \lim_\beta \langle F_\alpha, T(f)e_\beta * \mu\rangle = \langle F_\alpha, T(f)\mu\rangle. \end{aligned}$$

This shows that $\langle T^*(\mu)F_\alpha, f\rangle$ converges to $\langle F, T(f)\mu\rangle$. On the other hand,

$$\begin{aligned} \langle F, T(f)\mu\rangle &= \lim_\beta \langle FT(f), e_\beta * \mu\rangle = \lim_\beta \langle T(\mu F f), e_\beta\rangle \\ &= \lim_\beta \langle \mu T(Ff), e_\beta\rangle = \langle T^*(\mu)F, f\rangle. \end{aligned}$$

Consequently $\{T^*(\mu)F_\alpha\}$ converges to $T^*(\mu)F$ in the weak*-topology, and so $T^*(\mu) \in Z_t(L^1(G)^{**}) = L^1(G)$ [17]. Now, let $\{f_\alpha\}$ converges to f in the weak*-topology. For $\mu \in L^1(G)$,

$$\lim_\alpha \langle T(f_\alpha), \mu\rangle = \lim_\alpha \langle f_\alpha, T^*(\mu)\rangle = \langle f, T^*(\mu)\rangle = \langle T(f), \mu\rangle.$$

This proves that T is weak*-weak* continuous. ∎

4. Compactness and Weakly Almost Periodicity

In the present section we state a collection of characterizations of weakly almost periodic operators and compact operators for a Banach algebra A in terms of elements in A^*.

Let T be a weakly compact operator in $Hom_A(A^*, A^*)$. For every $f \in A^*$,

$$\{T(f)a;\ a \in b(A)\} = \{T(fa);\ a \in b(A)\} \subseteq \|f\|\{T(h);\ h \in b(A^*)\}.$$

Since T is weakly compact, the last set is relatively compact with respect to the weak topology of A^*. Hence $T(f) \in wap(A^*)$.

Definition 4.1. For an operator $T : A^* \to A^*$ we are able to speak of the translate T_a, which is that operator which to each f in A^* associates the element $T(f)a$ in A^*. An operator T is said to be weakly almost periodic [almost periodic] if the set $\{T_a;\ a \in b(A)\}$ of translates of T is relatively compact with respect to weak operator topology [strong operator topology] in $\mathcal{B}(A^*, A^*)$.

Theorem 4.2. *Let A be a Banach algebra. An operator T in $Hom_A(A^*, A^*)$ is weakly almost periodic if and only if each element in $T(A^*)$ is weakly almost periodic.*

Proof. Let $\{T_a;\ a \in b(A)\}$ be a relatively compact set in $\mathcal{B}(A^*, A^*)$ (in the weak operator topology). Let $f \in A^*$ and let $\{a_\alpha\}$ be an arbitrary net in $b(A)$. Since $\{T_a;\ a \in b(A)\}$ is relatively compact, there exists an element S in $\mathcal{B}(A^*, A^*)$ and a subnet $\{T_{a_\beta}\}$ of $\{T_{a_\alpha}\}$ such that $\{T_{a_\beta}\}$ converges to S in the weak operator topology of $\mathcal{B}(A^*, A^*)$. By VI.1.3 in [5], $T(f)a_\beta$ converges to $S(f)$ in the weak topology in A^*. Thus $\{T(f)a;\ a \in b(A)\}$ is relatively weakly compact.

To prove the converse, let $T(f) \in A^*$ be weakly almost periodic for each f in A^*. To every $S \in \mathcal{B}(A^*, A^*)$ we associate the operator $N_S : f \mapsto S(f)$ on A^*. Let also

$$N : S \mapsto (N_S(f))_{f \in A^*}$$

$$\mathcal{B}(A^*, A^*) \to \prod_{f \in A^*} A^*.$$

We claim that N is a homeomorphism from $\mathcal{B}(A^*, A^*)$ with respect to weak operator topology into the product of the weak topology on the right. It is easy to see that N is injective. Now, let $S_\alpha \to S$ in the weak operator topology of $\mathcal{B}(A^*, A^*)$. Thus for each $f \in A^*$, $S_\alpha(f)$ converges to $S(f)$ in the weak topology of A^* (see VI.1.3 in [5]). This shows that N is a continuous linear map. Since every weak operator-neighborhood of 0 contains a neighborhood of the form

$$V = \{S \in \mathcal{B}(A^*, A^*);\ |\langle F_i, S(f_j)\rangle| < \epsilon \ \text{ for } \ 1 \le i \le n,\ 1 \le j \le m\}$$

where $F_i \in A^{**}$, $f_j \in A^*$ and $\epsilon > 0$, it is easy to see that N is open (or see [21]). Consequently N is a homeomorphism.

By assumption, for each $f \in A^*$, $\{T(f)a;\ a \in b(A)\}$ is relatively weakly compact. The image of $\{T_a;\ a \in b(A)\}$ lies in the subset

$$\mathcal{P} := \prod_{f \in A^*} \{T(f)a;\ a \in b(A)\}.$$

By Tychonoff 's Theorem, $\mathcal{P}$ is compact. Hence $\{T_a;\ a \in b(A)\}$ is relatively compact in the weak operator topology. ■

Remarks 4.3. The combination of Theorem 4.2 and its previously mentioned result gives: for a locally compact group G, one considers the space $wap(G)$ of all continuous weakly almost periodic functions on G, that is, all $f \in C(G)$ for which $\{_x f;\ x \in G\}$ is relatively compact in the weak topology of $C(G)$ [2]. It is well known that $wap(L^\infty(G)) = wap(G)$. Let G be an infinite locally compact group. If $I : L^\infty(G) \to L^\infty(G)$ is weakly compact, then $L^\infty(G)$ is reflexive [21]. This is contradiction (see Corollary 5.5 in [19]). We conclude that $L^\infty(G) \neq wap(G)$. By Theorem 4.2, the identity operator on $L^\infty(G)$ is not weakly almost periodic.

Completely analogous to Theorem 4.2, we also have

Theorem 4.4. *Let A be a Banach algebra. An operator T in $Hom_A(A^*, A^*)$ is almost periodic if and only if each element in $T(A^*)$ is almost periodic.*

For $f \in A^*$ define bounded linear map $T_f : A \to A^*$ by the formula: $T_f(a) = af$. By a direct verification we obtain $\|T_f\| \le \|f\|$. T_f is said to be weakly almost periodic if $\{(T_f)_a;\ a \in b(A)\}$ is relatively weakly compact in the weak operator topology in $\mathcal{B}(A, A^*)$ where $(T_f)_a(x) = T_f(xa)$ for each $x \in A$.

Theorem 4.5. *Let A be a commutative Banach algebra. For f in A^*, the following are equivalent:*

(i) *T_f is weakly almost periodic.*

(ii) *$fx \in wap(A^*)$ for every $x \in A$.*

Proof. Suppose that $\{(T_f)_a;\ a \in b(A)\}$ is relatively weakly compact in the weak operator topology of $\mathcal{B}(A, A^*)$. Let $x \in A$ and $\{fxa_\alpha\}$ be a net in $\{fxa;\ a \in b(A)\}$. Since T_f is weakly almost periodic, the net $\{(T_f)_{a_\alpha}\}$ admits a subnet $\{(T_f)_{a_\beta}\}$ converging to an element T in $\mathcal{B}(A, A^*)$ in the weak operator topology. By VI.1.3 in [5], $fxa_\beta \to T(x)$ in the weak topology in A^*. This shows that $\{fxa;\ a \in b(A)\}$ is relatively weakly compact. By [4], $fx \in wap(A^*)$.

To prove the converse, it is known that a subset $X \subseteq \mathcal{B}(A, A^*)$ is compact in the weak operator topology if and only if it is closed in the weak operator topology and the weak closure of $\{T(x);\ T \in X\}$ is weakly compact for each $x \in A$ [5]. By assumption, $\{fxa;\ a \in b(A)\}$ is weakly compact and so T_f is weakly almost periodic. ∎

Definition 4.6. Let A be a Banach algebra, X a subset of A^*. We say that X is invariant if $fa \in X$ whenever $f \in X$ and $a \in b(A)$. X is said to be equi-almost periodic if the following condition is true: for every $\epsilon > 0$, there exists a finite subset $S \subseteq b(A)$ with the property that for every $a \in b(A)$ there exists $b \in S$ such that $\|fa - fb\| < \epsilon$ for all $f \in X$.

The above definition is motivated by Lemma 3 in Loomis [20]. It is known that if G is a locally compact group, then each bounded linear map $T : L^1(G) \to L^\infty(G)$ with $T(\mu*\nu) = T(\mu)\nu$ is a convolution-type operator H_θ induced by an element θ of $L^\infty(G)$. It is known ([3], [24]) that H_θ is compact if and only if $\theta \in wap(L^1(G))$.

Theorem 4.7. *Let A be a Banach algebra with a bounded approximate identity bounded by 1. An operator T in $Hom_A(A^*, A^*)$ is compact if and only if $T(b(A^*))$ is equi-almost periodic.*

Proof. Let $T \in Hom_A(A^*, A^*)$ be compact. Let $\epsilon > 0$ be given. By hypothesis, there exist a finite number of elements $\{f_1, ..., f_n\}$ in $b(A^*)$ such that for given f in $b(A^*)$, an f_i may be found such that $\|T(f_i) - T(f)\| < \frac{\epsilon}{3}$. Since T is compact, it is easy to see that T is almost periodic. It follows from Theorem 4.4 that $T(f_i)$ $(i \in \{1, ..., n\})$ is almost periodic. It is easy to see that $\{T(f_1), ..., T(f_n)\}$ is equi-almost periodic (more information on this problem can be found in [20]). We may determine a subset $\{a_1, ..., a_m\}$ in $b(A)$ such that for each a in $b(A)$ an a_k may be found such that $\|T(f_i a) - T(f_i a_k)\| < \frac{\epsilon}{3}$ whenever $i \in \{1, ..., n\}$. For f in $b(A^*)$, there exists $f_i \in \{f_1, ..., f_n\}$ such that

$$\|T(fa) - T(f_i a)\| = \|T(f)a - T(f_i)a\| \le \|T(f) - T(f_i)\| < \frac{\epsilon}{3}$$

and also $\|T(fa_k) - T(f_i a_k)\| < \frac{\epsilon}{3}$. Using the triangle inequality, we obtain

$$\|T(f)a - T(f)a_k\| = \|T(fa) - T(fa_k)\| < \epsilon.$$

To prove the converse, let $\epsilon > 0$ be given. There exists a finite subset $\{a_1, ..., a_n\}$ in $b(A)$ such that for every $a \in b(A)$, a point a_i $(i \in \{1, ..., n\})$ may be found such that $\|T(f)a - T(f)a_i\| < \frac{\epsilon}{3}$ whenever $f \in b(A^*)$. Now, let $\{e_\alpha\}$ be a bounded approximate identity bounded by 1 for A. We have

$$\begin{aligned} |\langle T(f), a\rangle - \langle T(f), a_i\rangle| &= \lim_\alpha |\langle T(f), ae_\alpha\rangle - \langle T(f), a_i e_\alpha\rangle| \\ &= \lim_\alpha |\langle T(fa), e_\alpha\rangle - \langle T(fa_i), e_\alpha\rangle| \\ &\leq \|T(fa) - T(fa_i)\| < \frac{\epsilon}{3}, \end{aligned}$$

whenever $f \in b(A^*)$. Since T is a bounded linear map, it follows that the clousre of $\{\langle T(f), a_i\rangle;\ 1 \leq i \leq n,\ f \in b(A^*)\}$ in $\mathbb{C}$ is compact. We can choose a finite subset $\{f_1, ..., f_m\}$ in $b(A^*)$ such that for each f in $b(A^*)$ an f_k $(1 \leq k \leq m)$ may be found such that $|\langle T(f), a_i\rangle - \langle T(f_k), a_i\rangle| < \frac{\epsilon}{3}$ for all $i \in \{1, ..., n\}$. We have

$$\begin{aligned} |\langle T(f), a\rangle - \langle T(f_k), a\rangle| \leq\ & |\langle T(f), a\rangle - \langle T(f), a_i\rangle| + |\langle T(f), a_i\rangle - \langle T(f_k), a_i\rangle| \\ & + |\langle T(f_k), a_i\rangle - \langle T(f_k), a\rangle| < \epsilon \end{aligned}$$

whenever $a \in b(A)$. This shows that $\|T(f) - T(f_k)\| \leq \epsilon$. Hence the set

$$\{T(f);\ f \in b(A)\}$$

is totally bounded, which means that T is compact. ■

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