TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 3, pp. 1059-1073, June 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

REGULARITY CRITERIA FOR THE GENERALIZED MAGNETOHYDRODYNAMIC EQUATIONS AND THE QUASI-GEOSTROPHIC EQUATIONS

Jishan Fan, Hongjun Gao and Gen Nakamura

Abstract. In this paper we consider the Cauchy problem for the 3D generalized magnetohydrodynamic (MHD) equations and the quasi-geostrophic (QG) equations. We prove some new regularity criteria for weak solutions.

1. INTRODUCTION

We shall consider the following 3D generalized MHD equations v

$$u_t + u \cdot \nabla u + \nabla p - B \cdot \nabla B + \frac{1}{2}\nabla |B|^2 + (-\Delta)^\alpha u = 0, \tag{1.1}$$

$$B_t + u \cdot \nabla B - B \cdot \nabla u + (-\Delta)^\alpha B = 0, \tag{1.2}$$

$$\operatorname{div} u = \operatorname{div} B = 0 \quad in \quad (0,\infty) \times \mathbb{R}^3, \tag{1.3}$$

$$u(x,0) = u_0(x), B(x,0) = B_0(x) \quad in \quad \mathbb{R}^3, \tag{1.4}$$

where $u := (u_1, u_2, u_3)$ is the velocity field, $B := (B_1, B_2, B_3)$ is the magnetic field, $p(x,t)$ is a scalar pressure, and $u_0(x), B_0(x)$ with $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ in the sense of distribution are the initial velocity and magnetic fields. $\alpha \geq 1$ and the operator $(-\Delta)^\alpha$ is defined by

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f},$$

where $\widehat{f}$ denotes the Fourier transform of f. We will also denote $\Lambda := (-\Delta)^{\frac{1}{2}}$.

Received February 9, 2009, accepted November 18, 2009.
Communicated by Jong-Shenq Guo.
2000 *Mathematics Subject Classification*: 35Q35, 76D05.
Key words and phrases: Generalized MHD equations, Regularity criteria, Multiplier spaces, Quasi-geostrophic equations.
This work supported Partially supported by a China NSF Grant No. 10871097, National Basic Research Program of China (973 Program) No. 2007CB814800, China Postdoctoral Research Foundation No. 20080441062 and Jiangsu Planned Projects for Postdoctoral Research Foundation No. 0802020C.

It is easy to prove that problem (1.1)-(1.4) is locally well-posed for any given initial data $u_0, B_0 \in H^s(\mathbb{R}^3), s \geq 3$. Moreover, it is proved by Wu [1] that (1.1)-(1.4) has a weak solution for any $u_0, B_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ in $\mathbb{R}^3$. But whether the unique local solution can exist globally or the weak solution is regular and unique is an outstanding open problem.

Note that if $(u(x,t), B(x,t))$ is a solution to (1.1)-(1.4), then (u_λ, B_λ) with any $\lambda > 0$ is also a solution, where $u_\lambda(x,t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t)$ and $B_\lambda(x,t) = \lambda^{2\alpha-1}B(\lambda x, \lambda^{2\alpha}t)$. We will say that the norm $\|u\|_{L^p(0,\infty;L^q(\mathbb{R}^3))}$ is scaling dimension zero for $\frac{2\alpha}{p} + \frac{3}{q} = 2\alpha - 1$ in the sense that $\|u_\lambda\|_{L^p(0,\infty;L^q)} = \|u\|_{L^p(0,\infty;L^q)}$ holds for all $\lambda > 0$ if and only if $\frac{2\alpha}{p} + \frac{3}{q} = 2\alpha - 1$. Very recently, Y. Zhou [2] proved that

Theorem 1.1. ([2]). *Let $1 \leq \alpha \leq \frac{5}{4}$ and assume that $u_0, B_0 \in H^3(\mathbb{R}^3)$. If one of the following conditions is satisfied.*

(1.5) $$(i)\quad u \in L^p(0,T;L^q(\mathbb{R}^3)), \quad with\quad \frac{2\alpha}{p} + \frac{3}{q} = 2\alpha - 1, \frac{3}{2\alpha-1} < q \leq \infty,$$

(1.6) $$(ii)\quad \Lambda^\alpha u \in L^p(0,T;L^q(\mathbb{R}^3)), \quad with\quad \frac{2\alpha}{p} + \frac{3}{q} = 3\alpha - 1, \frac{3}{3\alpha-1} < q < \frac{3}{\alpha-1}.$$

Then the solution remains smooth on $(0,T]$.

Remark 1.1. If $\alpha \geq \frac{5}{4}$ and $u_0, B_0 \in H^3(\mathbb{R}^3)$, then all the global weak solutions to (1.1)-(1.4) are actually strong and unique ([1],[2]).

Remark 1.2. When $\alpha = 1$, Theorem 1.1 reduces to the results obtained by C. He and Z. P. Xin [3] and Y. Zhou [4].

Remark 1.3. The global well-posedness and regularity conditions for the Navier-Stokes and the related equations were considered in [15] and [20].

The pointwise multipliers between different spaces of differentiable functions have been studied by Maz'ya and co-workers [5, 6, 7, 8]. They are a useful tool for stating minimal regularity requirements on the coefficients of partial differential operators for proving regularity or uniqueness of solutions.

More precisely, we define the space $\dot{X}_{r,s}(\mathbb{R}^d)$ of pointwise multipliers which map $\dot{H}^r$ into $\dot{H}^{-s}$. The norm in $\dot{X}_{r,s}$ is given by the operator norm of pointwise multiplication:

(1.7) $$\|f\|_{\dot{X}_{r,s}} := \sup\left\{\frac{\|fg\|_{\dot{H}^{-s}}}{\|g\|_{\dot{H}^r}}, g \neq 0\right\}.$$

When $s = 0$, we simply denote $\dot{X}_r \equiv \dot{X}_{r,0}$.

Now we are in a position to state the main result in this paper.

Theorem 1.2. *Let* $1 \le \alpha \le \frac{5}{4}, 0 < r < \alpha$ *and assume that* $u_0, B_0 \in H^3(\mathbb{R}^3)$. *If one of the following three conditions is satisfied*

(1.8) $$(i)\ \ u \in L^{\frac{2\alpha}{\alpha-r}}(0,T;\dot{X}_{r,\alpha-1}),$$

(1.9) $$(ii)\ \ \nabla u \in L^{\frac{2\alpha}{2\alpha-r}}(0,T;\dot{X}_{r}),$$

(1.10) $$(iii)\ \ \nabla u \in L^{\frac{2\alpha}{\alpha-r}}(0,T;\dot{X}_{r,\alpha}),$$

(1.11) $$(iv)\ \ \nabla u \in L^{1}(0,T;\dot{B}^0_{\infty,\infty}).$$

Then the solution (u, B) *remains smooth on* $(0, T]$.

Remark 1.4. Since $L^{\frac{d}{r+s}}(\mathbb{R}^d) \subset L^{\frac{d}{r+s},\infty}(\mathbb{R}^d) \subset \dot{X}_{r,s}(\mathbb{R}^d)$, our result improve Theorem 1.1.

2. Preliminaries

We first recall the definition of the homogeneous Littlewood-Paley decomposition which will be used to define function spaces. We follow [9]. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}$, its Fourier transform $\mathcal{F}(f) = \hat{f}$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx.$$

We consider $\varphi \in \mathcal{S}$ satisfying $supp\hat{\varphi} \subset \left\{\xi \in \mathbb{R}^n \middle| \frac{1}{2} \le |\xi| \le 2\right\}$, and $\hat{\varphi} > 0$ if $\frac{2}{3} < |\xi| < \frac{3}{2}$. Setting $\hat{\varphi}_j = \hat{\varphi}(2^{-j}\xi)$ (in other words, $\varphi_j(x) = 2^{jn}\varphi(2^j x)$), we can adjust the normalization constant in front of $\hat{\varphi}$ so that

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}_j(\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Given $k \in \mathbb{Z}$, we define the function $S_k \in \mathcal{S}$ by its Fourier transform

$$\hat{S}_k(\xi) = 1 - \sum_{j \ge k+1} \hat{\varphi}_j(\xi).$$

We observe

$$supp\hat{\varphi}_j \cap supp\hat{\varphi}_{j'} = empty \quad set \quad if \quad |j - j'| \ge 2.$$

Let $s \in \mathbb{R}, (p,q) \in [0,\infty) \times [0,\infty]$. Given $f \in S'$, we denote $\Delta_j f = \varphi_j * f$, and then the homogeneous Triebel-Lizorkin semi-norm $\|f\|_{\dot{F}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{F}^s_{p,q}} = \begin{cases} \left\| \left(\sum_{j\in\mathbb{Z}} 2^{jqs} |\Delta_j f(\cdot)|^q \right)^{1/q} \right\|_{L^p} & if \quad q \in [1,\infty), \\ \left\| \sup_{j\in\mathbb{Z}} \left(2^{js} |\Delta_j f(\cdot)| \right) \right\| & if \quad q = \infty. \end{cases}$$

The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ is a quasi-normed space with the quasi-norm given by $\|\cdot\|_{\dot{F}^s_{p,q}}$. For $s>0, (p,q)\in[1,\infty)\times[1,\infty]$. We define the inhomogeneous Triebel-Lizorkin space norm $\|f\|_{F^s_{p,q}}$ of $f\in\mathcal{S}'$ as

$$\|f\|_{F^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{F}^s_{p,q}}.$$

The inhomogeneous Triebel-Lizorkin space is a Banach space equipped with the norm, $\|\cdot\|_{F^s_{p,q}}$. Similarly, for $s\in\mathbb{R}, (p,q)\in[0,\infty]^2$, the homogeneous Besov norm $\|f\|_{\dot{B}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum\limits_{-\infty}^{+\infty} 2^{jqs}\|\varphi_j * f\|^q_{L^p}\right) & if \quad q\in[1,\infty), \\ \sup\limits_j (2^{js}\|\varphi_j * f\|_{L^p}) & if \quad q=\infty. \end{cases}$$

The homogeneous Besov space $\dot{B}^s_{p,q}$ is a quasi-normed space with the quasi-norm given by $\|\cdot\|_{\dot{B}^s_{p,q}}$. For $s>0$ we define the inhomogeneous Besov space norm $\|f\|_{B^s_{p,q}}$ of $f\in\mathcal{S}'$ as $\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}$.

Lemma 2.1. (1) *Let $1<p<\infty, 1<q<\infty$ and let $s>0, \alpha>0, \beta>0$. We take $1<p_1<\infty, 1<p_2\le\infty$ and $1<r_1\le\infty, 1<r_2<\infty$ so that $1/p = 1/p_1+1/p_2 = 1/r_1+1/r_2$. Then there is a constant C such that for every $f\in\dot{F}^{s+\alpha}_{p_1,q}\cap\dot{F}^{-\beta}_{r_1,\infty}$ and $g\in\dot{F}^{-\alpha}_{p_2,\infty}\cap\dot{F}^{s+\beta}_{r_2,q}$ there holds $f\cdot g\in\dot{F}^s_{p,q}$ with the estimate*

$$\|f\cdot g\|_{\dot{F}^s_{p,q}} \le C\left(\|f\|_{\dot{F}^{s+\alpha}_{p_1,q}}\|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{r_1,\infty}}\|g\|_{\dot{F}^{s+\beta}_{r_2,q}}\right). \tag{2.12}$$

(2) *Let $1<p\le\infty$ and let $s>0, \alpha>0, \beta>0$. We take $1<p_1,p_2,r_1,r_2\le\infty$ so that $1/p=1/p_1+1/p_2=1/r_1+1/r_2$. Then there is a constant C such that for every $f\in\dot{F}^{s+\alpha}_{p_1,\infty}\cap\dot{F}^{-\beta}_{r_1,\infty}$ and $g\in\dot{F}^{-\beta}_{p_2,\infty}\cap\dot{F}^{s+\alpha}_{r_2,\infty}$ there holds $f\cdot g\in\dot{F}^s_{p,\infty}$ with the estimate*

$$\|f\cdot g\|_{\dot{F}^s_{p,\infty}} \le C\left(\|f\|_{\dot{F}^{s+\alpha}_{p_1,\infty}}\|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{r_1,\infty}}\|g\|_{\dot{F}^{s+\beta}_{r_2,\infty}}\right). \tag{2.13}$$

For the proof see [10].

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, first we show

$$u, B\in L^\infty(0,T;H^1)\cap L^2(0,T;H^{\alpha+1}). \tag{3.1}$$

Multiplying (1.1) by u and (1.2) by B, after integration by parts and taking the divergence free property into account, and adding up the resulting equality give

$$\begin{aligned}\tag{3.2} &\frac{1}{2}(\|u(t)\|_{L^2}^2+\|B(t)\|_{L^2}^2)+\int_0^T\|\Lambda^\alpha u(t)\|_{L^2}^2+\|\Lambda^\alpha B(t)\|_{L^2}^2dt\\ \leq\ &\frac{1}{2}(\|u_0\|_{L^2}^2+\|B_0\|_{L^2}^2).\end{aligned}$$

Testing (1.1) by Δu and using (1.3) yield

$$\begin{aligned}\tag{3.3} &\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\|\Lambda^{\alpha+1}u\|_{L^2}^2\\ =\ &-\sum_{i,j,k}\int\partial_iu_k\cdot\partial_ku_j\cdot\partial_iu_jdx+\sum_{i,j,k}\int\partial_iB_k\cdot\partial_kB_j\cdot\partial_iu_jdx\\ &-\sum_{i,j,k}\int B_k\cdot\partial_i\partial_ku_j\cdot\partial_iB_jdx.\end{aligned}$$

Similarly, testing (1.2) by ΔB, we get

$$\begin{aligned}\tag{3.4} &\frac{1}{2}\frac{d}{dt}\|\nabla B\|_{L^2}^2+\|\Lambda^{\alpha+1}B\|_{L^2}^2\\ =\ &-\sum_{i,j,k}\int\partial_iu_k\cdot\partial_kB_j\cdot\partial_iB_jdx+\sum_{i,j,k}\int\partial_iB_k\cdot\partial_ku_j\cdot\partial_iB_jdx\\ &+\sum_{i,j,k}\int B_k\cdot\partial_k\partial_iu_j\cdot\partial_iB_jdx.\end{aligned}$$

Combining (3.3) and (3.4) gives

$$\begin{aligned}\tag{3.5} &\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2)+\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\alpha+1}B\|_{L^2}^2\\ =\ &-\sum_{i,j,k}\int\partial_iu_k\cdot\partial_ku_j\cdot\partial_iu_jdx+\sum_{i,j,k}\int\partial_iB_k\cdot\partial_kB_j\cdot\partial_iu_jdx\\ &-\sum_{i,j,k}\int\partial_iu_k\cdot\partial_kB_j\cdot\partial_iB_jdx+\sum_{i,j,k}\int\partial_iB_k\cdot\partial_ku_j\cdot\partial_iB_jdx\\ =:\ &I_1+I_2+I_3+I_4.\end{aligned}$$

Each term I_i can be bounded as follows.

Firstly we assume (1.8) holds true.

$$\begin{aligned} I_1 &= -\sum_{i,j,k}\int\partial_iu_k\cdot\partial_ku_j\cdot\partial_iu_jdx\\ &= \sum_{i,j,k}\int u_k\cdot(\partial_i\partial_ku_j\cdot\partial_iu_j+\partial_ku_j\cdot\partial_i\partial_iu_j)dx\\ &= \sum_{i,j,k}\int u_k\cdot\partial_ku_j\cdot\partial_i\partial_iu_jdx\\ &\leq \sum_{i,j,k}\|u_k\cdot\partial_ku_j\|_{\dot H^{1-\alpha}}\|\partial_i\partial_iu_j\|_{\dot H^{\alpha-1}}\end{aligned}$$

$$\begin{aligned}
&\le \sum_{i,j,k} \|u_k\|_{\dot X_{r,\alpha-1}} \|\partial_k u_j\|_{\dot H^r} \|\partial_i\partial_i u_j\|_{\dot H^{\alpha-1}}\\
&\le C\|u\|_{\dot X_{r,\alpha-1}} \|\nabla u\|_{\dot H^r} \|\Delta u\|_{\dot H^{\alpha-1}}\\
&\le C\|u\|_{\dot X_{r,\alpha-1}} \|\nabla u\|_{\dot H^r} \|\Lambda^{\alpha+1} u\|_{L^2}\\
&\le C\|u\|_{\dot X_{r,\alpha-1}} \|\nabla u\|_{L^2}^{1-\frac{r}{\alpha}} \|\Lambda^{\alpha+1} u\|_{L^2}^{1+\frac{r}{\alpha}},
\end{aligned}$$

by the interpolation inequality

$$\|w\|_{\dot H^r} \le C\|w\|_{L^2}^{1-\frac{r}{\alpha}} \|w\|_{\dot H^\alpha}^{\frac{r}{\alpha}}, \tag{3.6}$$

and hence

$$I_1 \le \epsilon\|\Lambda^{\alpha+1}u\|_{L^2}^2 + C\|u\|_{\dot X_{r,\alpha-1}}^{\frac{2\alpha}{\alpha-r}} \|\nabla u\|_{L^2}^2, \tag{3.7}$$

for any $\epsilon > 0$ by the Young's inequality.

Similarly, one can obtain

$$\begin{aligned}
I_2, I_3, I_4 &\le C\|u\|_{\dot X_{r,\alpha-1}} \|\nabla B\|_{\dot H^r} \|\Delta B\|_{\dot H^{\alpha-1}}\\
&\le C\|u\|_{\dot X_{r,\alpha-1}} \|\nabla B\|_{\dot H^r} \|\Lambda^{\alpha+1} B\|_{L^2}\\
&\le C\|u\|_{\dot X_{r,\alpha-1}} \|\nabla B\|_{L^2}^{1-\frac{r}{\alpha}} \|\Lambda^{\alpha+1} B\|_{L^2}^{1+\frac{r}{\alpha}} \quad (\text{by} \quad (3.6))\\
&\le \epsilon\|\Lambda^{\alpha+1}B\|_{L^2}^2 + C\|u\|_{\dot X_{r,\alpha-1}}^{\frac{2\alpha}{\alpha-r}} \|\nabla B\|_{L^2}^2,
\end{aligned} \tag{3.8}$$

for any $\epsilon > 0$.

Inserting (3.7) and (3.8) into (3.5) and taking ϵ small and then the Gronwall's inequality yield (3.1).

Next we assume (1.9) holds true.

$$\begin{aligned}
I_1 &\le \sum_{i,j,k} \|\partial_i u_k\|_{L^2} \cdot \|\partial_k u_j \cdot \partial_i u_j\|_{L^2}\\
&\le \sum_{i,j,k} \|\partial_i u_k\|_{L^2} \cdot \|\partial_k u_j\|_{\dot X_r} \|\partial_i u_j\|_{\dot H^r}\\
&\le C\|\nabla u\|_{L^2} \|\nabla u\|_{\dot X_r} \|\nabla u\|_{\dot H^r}\\
&\le C\|\nabla u\|_{L^2}^{2-\frac{r}{\alpha}} \|\nabla u\|_{\dot X_r} \|\nabla u\|_{\dot H^\alpha}^{\frac{r}{\alpha}} \quad (\text{by} \quad (3.6))\\
&\le \epsilon\|\Lambda^{\alpha+1}u\|_{L^2}^2 + C\|\nabla u\|_{\dot X_r}^{\frac{2\alpha}{2\alpha-r}} \|\nabla u\|_{L^2}^2,
\end{aligned} \tag{3.9}$$

for any $\epsilon > 0$ by the Young's inequality.

Similarly, one can get

$$
\begin{aligned}
I_2, I_3, I_4 &\le C\|\nabla B\|_{L^2}\|\nabla u\|_{\dot{X}_r}\|\nabla B\|_{\dot{H}^r} \\
&\le C\|\nabla B\|_{L^2}^{2-\frac{r}{\alpha}}\|\nabla u\|_{\dot{X}_r}\|\Lambda^{\alpha+1}B\|_{L^2}^{\frac{r}{\alpha}} \\
&\le \epsilon\|\Lambda^{\alpha+1}B\|_{L^2}^2 + C\|\nabla u\|_{\dot{X}_r}^{\frac{2\alpha}{2\alpha-r}}\|\nabla B\|_{L^2}^2,
\end{aligned} \tag{3.10}
$$

for any $\epsilon > 0$.

Putting (3.9) and (3.10) into (3.5) and taking ϵ small enough and then the Gronwall's inequality give (3.1).

We assume that (1.10) holds true.

$$
\begin{aligned}
I_1 &\le \sum_{i,j,k}\|\partial_i u_k\|_{\dot{H}^\alpha}\cdot\|\partial_k u_j\cdot\partial_i u_j\|_{\dot{H}^{-\alpha}} \\
&\le \sum_{i,j,k}\|\partial_i u_k\|_{\dot{H}^\alpha}\cdot\|\partial_k u_j\|_{\dot{X}_{r,\alpha}}\|\partial_i u_j\|_{\dot{H}^r} \\
&\le C\|\nabla u\|_{\dot{H}^\alpha}\|\nabla u\|_{\dot{X}_{r,\alpha}}\|\nabla u\|_{\dot{H}^r} \\
&\le C\|\nabla u\|_{\dot{X}_{r,\alpha}}\|\nabla u\|_{L^2}^{1-\frac{r}{\alpha}}\|\Lambda^{\alpha+1}u\|_{L^2}^{1+\frac{r}{\alpha}} \quad (by \quad (3.6)) \\
&\le \epsilon\|\Lambda^{\alpha+1}u\|_{L^2}^2 + C\|\nabla u\|_{\dot{X}_{r,\alpha}}^{\frac{2\alpha}{\alpha-r}}\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{3.11}
$$

for any $\epsilon > 0$ by the Young's inequality.

Similarly, we deduce

$$
\begin{aligned}
I_2, I_3, I_4 &\le C\|\nabla B\|_{\dot{H}^\alpha}\|\nabla u\|_{\dot{X}_{r,\alpha}}\cdot\|\nabla B\|_{\dot{H}^r} \\
&\le C\|\nabla u\|_{\dot{X}_{r,\alpha}}\|\nabla B\|_{L^2}^{1-\frac{r}{\alpha}}\|\Lambda^{\alpha+1}B\|_{L^2}^{1+\frac{r}{\alpha}} \quad (by \quad (3.6)) \\
&\le \epsilon\|\Lambda^{\alpha+1}B\|_{L^2}^2 + C\|\nabla u\|_{\dot{X}_{r,\alpha}}^{\frac{2\alpha}{\alpha-r}}\|\nabla B\|_{L^2}^2,
\end{aligned} \tag{3.12}
$$

for any $\epsilon > 0$.

Inserting (3.11) and (3.12) into (3.5) and taking ϵ small enough and then the Gronwall's inequality gives (3.1).

Finally we assume (1.11) holds true. Using the Littlewood-Paley decomposition, we decompose $\partial_i u_j$ as follows:

$$
\partial_i u_j = \sum_{\ell=-\infty}^{+\infty}\Delta_\ell\partial_i u_j = \sum_{\ell<-N}\Delta_\ell\partial_i u_j + \sum_{\ell=-N}^{N}\Delta_\ell\partial_i u_j + \sum_{j>N}\Delta_\ell\partial_i u_j,
$$

where N is a positive integer to be chosen later. Substituting this decomposition into I_1, we obtain

$$
\begin{aligned}
I_1 =& \sum_{i,j,k}\sum_{\ell<-N}\int \partial_i u_k\cdot\partial_k u_j\cdot\Delta_\ell\partial_i u_j dx \\
&+\sum_{i,j,k}\sum_{\ell=-N}^{N}\int \partial_i u_k\cdot\partial_k u_j\cdot\Delta_\ell\partial_i u_j dx \\
&+\sum_{i,j,k}\sum_{\ell>N}\int \partial_i u_k\cdot\partial_k u_j\cdot\Delta_\ell\partial_i u_j dx \\
=&: J_1(t)+J_2(t)+J_3(t).
\end{aligned}
\tag{3.13}
$$

Next, we estimate each $J_i(i=1,2,3)$. First, recalling

$$
\|\Delta_j f\|_{L^q}\le C2^{3j\left(\frac{1}{p}-\frac{1}{q}\right)}\|\Delta_j f\|_{L^p}, 1\le p\le q\le\infty, \tag{3.14}
$$

with C being a positive constant independent of f and j, we apply Hölder's inequality and (3.14) to infer that

$$
\begin{aligned}
J_1(t) \le& C\|\nabla u\|_{L^2}^2\sum_{i,j}\sum_{\ell<-N}\|\Delta_\ell\partial_i u_j\|_{L^\infty} \\
\le& C\|\nabla u\|_{L^2}^2\sum_{i,j}\sum_{\ell<-N}2^{\frac{3}{2}\ell}\|\Delta_\ell\partial_i u_j\|_{L^2} \\
\le& C2^{-\frac{3}{2}N}\|\nabla u\|_{L^2}^3.
\end{aligned}
$$

For $J_2(t)$, we use Hölder's inequality and (3.14) to conclude that

$$
J_2(t)\le C\|\nabla u\|_{L^2}^2\sum_{i,j}\sum_{\ell=-N}^{N}\|\Delta_\ell\partial_i u_j\|_{L^\infty}\le CN\|\nabla u\|_{\dot B^0_{\infty,\infty}}\|\nabla u\|_{L^2}^2.
$$

For $J_3(t)$ we make use of Hölder's inequality and (3.14) to deduce that

$$
\begin{aligned}
J_3(t) \le& C\|\nabla u\|_{L^{\frac{6}{3-\alpha}}}^2\sum_{i,j}\sum_{\ell>N}\|\Delta_\ell\partial_i u_j\|_{L^{\frac{3}{\alpha}}} \\
\le& C\|\nabla u\|_{L^{\frac{6}{3-\alpha}}}^2\sum_{i,j}\sum_{\ell>N}2^{3\ell\left(\frac{1}{2}-\frac{\alpha}{3}\right)}\|\Delta_\ell\partial_i u_j\|_{L^2} \\
\le& C\|\nabla u\|_{L^{\frac{6}{3-\alpha}}}^2\sum_{i,j}\left(\sum_{\ell>N}2^{-(4\alpha-3)\ell}\right)^{1/2}\left(\sum_{\ell>N}2^{2\alpha\ell}\|\Delta_\ell\partial_i u_j\|_{L^2}^2\right)^{1/2} \\
\le& C\|\nabla u\|_{L^{\frac{6}{3-\alpha}}}^2 2^{-(4\alpha-3)N}\|\Lambda^{1+\alpha}u\|_{L^2} \\
\le& C2^{-(4\alpha-3)N}\|\nabla u\|_{L^2}\|\Lambda^{1+\alpha}u\|_{L^2}^2,
\end{aligned}
$$

due to the Gagliardo-Nirenberg inequality

$$\|\nabla u\|^2_{L^{\frac{6}{3-\alpha}}} \leq C\|\nabla u\|_{L^2}\|\Lambda^{1+\alpha}u\|_{L^2}. \tag{3.15}$$

Now we choose N so that $C2^{-(4\alpha-3)N}(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2}) \leq \frac{1}{8}$, i.e.,

$$N \geq \frac{1}{4\alpha-3}\frac{\log^+(8C(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2}))}{\log 2},$$

to conclude

$$\begin{aligned} I_1 &\leq C\|\nabla u\|^2_{L^2} + C\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}\|\nabla u\|^2_{L^2}\log^+(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2}) \\ &\quad +\frac{1}{8}\|\Lambda^{1+\alpha}u\|^2_{L^2}. \end{aligned} \tag{3.16}$$

Similarly, we can get

$$\begin{aligned} & I_2, I_3, I_4 \\ \leq\ & C(\|\nabla u\|^2_{L^2}+\|\nabla B\|^2_{L^2}) + \frac{1}{8}(\|\Lambda^{1+\alpha}u\|^2_{L^2}+\|\Lambda^{1+\alpha}B\|^2_{L^2}) \\ & +C\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}(\|\nabla u\|^2_{L^2}+\|\nabla B\|^2_{L^2})\log^+(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2}). \end{aligned} \tag{3.17}$$

Putting (3.16) and (3.17) into (3.5) and we apply the Gronwall's inequality to get (3.1).

After we have (3.1), the estimates for higher order derivatives can be obtained by an inductive procedure.

This completes the proof of Theorem 1.2. ■

4. The QG Equations

In this section we use the similar method to the previous section to study the regularity of the dissipative quasi-geostrophic equations:

$$\partial_t\theta + v\cdot\nabla\theta = -\kappa\Lambda^\alpha\theta, \tag{4.1}$$

$$v(x,t) = -\nabla^\perp(-\Delta)^{-\frac{1}{2}}\theta = -\int_{\mathbb{R}^2}\frac{\nabla^\perp\theta(x+y,t)}{|y|}dy, \tag{4.2}$$

$$\theta(x,0) = \theta_0(x),\ x\in\mathbb{R}^2, \tag{4.3}$$

where $\theta(x,t)$ is a scalar function representing temperature, $v(x,t)$ is the velocity field of the fluid, $\kappa \geq 0$ is the diffusion coefficient, $\Lambda^\alpha := (-\Delta)^{\frac{\alpha}{2}}$, and $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$. (4.1)-(4.3) is an important model in geophysical fluid dynamics, they are special cases of the general quasi-geostrophic approximations for atmospheric and oceanic fluid flow with small Rossby and Ekman numbers. See, e.g., [11, 12, 13, 14] for the instructive discussions and the physical and mathematical motivations of the study of (4.1)-(4.3), in particular of the inviscid case $(k = 0)$. The case $\alpha > 1$ is called the subcritical case, the case $\alpha = 1$ is critical, the case $0 \le \alpha < 1$ is supercritical. For $\alpha > 1$, the global regularity of the solution of (4.1)-(4.3) is well-known [16]. For $\alpha = 1$, the global regularity of the solution of (4.1)-(4.3) has been proved very recently in [17, 18]. On the other hand, for $0 < \alpha < 1$, the question of global regularity/finite time singularity is still a challenging open question. In particular, the critical dissipation case $(\alpha = 1)$ has similar features to the 3-D Navier-Stokes equations and could be considered as its model problem. In order to see the similarities to the 3-D Navier-Stokes equations (with fractional powers of Laplacian) more apparently we apply the operator $\nabla^{\perp}$ to (4.1) to get

$$\partial_t \nabla^{\perp}\theta + (\upsilon \cdot \nabla)\nabla^{\perp}\theta = (\nabla^{\perp}\theta \cdot \nabla)\upsilon - k\Lambda^{\alpha}\nabla^{\perp}\theta. \tag{4.4}$$

Then we observe that $\nabla^{\perp}\theta$ has the role of vorticist, and (4.2) corresponds to the Biot-Savart law for the 3-D Navier-Stokes equations. In this mote we are concerned with the sufficient conditions to guarantee regularity of solutions to the quasi-geostrophic equations. Constantin, Majda, and Tabak [13] proved that:

$$\lim_{t\to T}\sup \|\theta(t)\|_{H^m} < \infty \quad \text{if and only if} \quad \int_0^T \|\nabla^{\perp}\theta(t)\|_{L^\infty} dt < \infty, \tag{4.5}$$

where $m > 2$, which holds for solutions of both viscous and inviscid $(k = 0)$ equations. Very recently, Chae [19] generalizes (4.5) to obtain the following theorem:

Theorem 4.1. *Let $\theta(x,t)$ be a solution of the quasi-geostrophic equation (4.1)-(4.3) with $\alpha \in (0,1], k > 0$, and its derivative $\nabla^{\perp}\theta$ satisfies*

$$\nabla^{\perp}\theta \in L^r(0,T;L^p(\mathbb{R}^2)) \quad \text{for some} \quad (r,p) \quad \text{with} \quad \frac{2}{p} + \frac{\alpha}{r} \le \alpha, \frac{2}{\alpha} < p < \infty, \tag{4.6}$$

then there is no singularity up to T.

Our theorem generalizes this as follows.

Theorem 4.2. *Let $\theta(x,t)$ be a solution of (4.1)-(4.3) with $\alpha \in (0,1], k > 0$, and its derivative $\nabla^{\perp}\theta$ satisfies one of the following conditions:*

$$(i)\ \nabla^{\perp}\theta \in L^{\frac{\alpha}{\alpha-r}}(0,T;\dot{X}_r), \tag{4.7}$$

$$(ii)\ \nabla^{\perp}\theta \in L^{\frac{2\alpha}{\alpha-2r}}(0,T;\dot{X}_{r,\alpha/2}), \tag{4.8}$$

$$(iii)\ 0 < s < \alpha/2, \nabla^{\perp}\theta \in L^{\frac{\alpha}{\alpha-s}}(0,T;\dot{F}^{-s}_{\infty,\infty}) \tag{4.9}$$

then there is no singularity up to T.

Remark 4.1. Very recently, J.Yuan [21] refines (4.6) as

$$\nabla^{\perp}\theta \in L^r(0,T;\dot{B}^0_{p,\infty}) \quad with \quad \frac{2}{p}+\frac{\alpha}{r}\leq \alpha, \frac{2}{\alpha}<p\leq\infty. \tag{4.10}$$

Remark 4.2. Since $\dot{B}^0_{p,\infty}\subset \dot{F}^{-s}_{\infty,\infty}$ with $\frac{2}{p}=s$ our condition (4.9) generalizes (4.10).

Proof of Theorem 4.2. We plan to show that our integrability condition for $\nabla^{\perp}\theta$ in Theorem 4.2 implies

$$\int_0^T \|\nabla^{\perp}\theta(t)\|_{L^\infty}dt<\infty, \tag{4.11}$$

thus guaranteeing the desived regularity until T by (4.5). Multiplying (4.4) by $\Delta\nabla^{\perp}\theta$ and integrating by parts, we see that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Lambda^2\theta(t)\|^2_{L^2}+k\|\Lambda^{2+\frac{\alpha}{2}}\theta\|^2_{L^2}\\ =&\int(\upsilon\cdot\nabla)\nabla^{\perp}\theta\cdot\Delta\nabla^{\perp}\theta dx-\int(\nabla^{\perp}\theta\cdot\nabla)\upsilon\cdot\Delta\nabla^{\perp}\theta dx=:I+J.\end{aligned} \tag{4.12}$$

Integrating by parts, we have

$$\begin{aligned}I&=-\int\nabla[(\upsilon\cdot\nabla)\nabla^{\perp}\theta]\cdot\nabla\nabla^{\perp}\theta dx\\ &=-\int(\nabla\upsilon)\cdot(\nabla\nabla^{\perp}\theta)\cdot\nabla\nabla^{\perp}\theta dx-\int(\upsilon\cdot\nabla)\nabla\nabla^{\perp}\theta\cdot\nabla\nabla^{\perp}\theta dx\\ &=:I_1+I_2.\end{aligned} \tag{4.13}$$

Integrating by parts again, and using the fact that $\operatorname{div}\upsilon=0$, we get

$$I_2=-\frac{1}{2}\int(\upsilon\cdot\nabla)|\nabla\nabla^{\perp}\theta|^2dx=\frac{1}{2}\int|\nabla\nabla^{\perp}\theta|^2\operatorname{div}\upsilon dx=0.$$

Now we assume that (4.7) holds true. Then

$$\begin{aligned}I_1&\leq\int|\nabla\upsilon||\nabla\nabla^{\perp}\theta|^2dx\leq\|\nabla\nabla^{\perp}\theta\|_{L^2}\||\nabla\upsilon|\cdot\nabla\nabla^{\perp}\theta\|_{L^2}\\ &\leq C\|\nabla\nabla^{\perp}\theta\|_{L^2}\|\nabla\upsilon\|_{\dot{X}_r}\|\nabla\nabla^{\perp}\theta\|_{\dot{H}^r}\\ &\leq C\|\nabla\upsilon\|_{\dot{X}_r}\|\nabla\nabla^{\perp}\theta\|^{2-\frac{2r}{\alpha}}_{L^2}\|\nabla\nabla^{\perp}\theta\|^{\frac{2r}{\alpha}}_{\dot{H}^\alpha}\quad(\text{by}\quad(3.6))\\ &\leq\frac{k}{4}\|\Lambda^{2+\frac{\alpha}{2}}\theta\|^2_{L^2}+C\|\nabla^{\perp}\theta\|^{\frac{\alpha}{\alpha-r}}_{\dot{X}_r}\|\Lambda^2\theta\|^2_{L^2}\end{aligned} \tag{4.14}$$

In order to estimate J we first integrate by parts:

$$\begin{aligned} J &= \int \nabla[(\nabla^{\perp}\theta\cdot\nabla)v]\cdot\nabla\nabla^{\perp}\theta dx \\ &= \int \nabla\nabla^{\perp}\theta\cdot\nabla v\cdot\nabla\nabla^{\perp}\theta dx + \int(\nabla^{\perp}\theta\cdot\nabla)\nabla v\cdot\nabla\nabla^{\perp}\theta dx, \end{aligned}$$

since $\|\nabla\nabla v\|_{L^q} \le C\|\nabla\nabla^{\perp}\theta\|_{L^q}, 1 < q < \infty$, due to the Calderon-Zygmund inequality, we observe that the estimate of J is the same as the estimate of I_1, and we have

$$J \le \frac{k}{4}\|\Lambda^{2+\frac{\alpha}{2}}\theta\|_{L^2}^2 + C\|\nabla^{\perp}\theta\|_{\dot{X}_r}^{\frac{\alpha}{\alpha-r}}\|\Lambda^2\theta\|_{L^2}^2. \tag{4.15}$$

Combining the estimates (4.14)-(4.15) and absorbing the diffusion term into the left hand side, we obtain

$$\frac{d}{dt}\|\Lambda^2\theta\|_{L^2}^2 + k\|\Lambda^{2+\frac{\alpha}{2}}\theta\|_{L^2}^2 \le C\|\nabla^{\perp}\theta\|_{\dot{X}_r}^{\frac{\alpha}{\alpha-r}}\|\Lambda^2\theta\|_{L^2}^2. \tag{4.16}$$

By Gronwall's lemma,

$$\|\Lambda^2\theta(t)\|_{L^2} \le \|\Lambda^2\theta_0\|_{L^2}\exp\left[C\int_0^t\|\nabla^{\perp}\theta\|_{\dot{X}_r}^{\frac{\alpha}{\alpha-r}}dt\right], \quad \forall t\in[0,T].$$

Hence, $\|\Lambda^2\theta\|_{L^2}\in L^{\infty}(0,T)$. Integrating (4.16) over $[0,T]$, we have

$$\|\Lambda^2\theta(t)\|+k\int_0^T\|\Lambda^{2+\frac{\alpha}{2}}\theta(t)\|_{L^2}^2dt \le C\int_0^T\|\nabla^{\perp}\theta\|_{\dot{X}_r}^{\frac{\alpha}{\alpha-r}}dt\sup_{0\le t\le T}\|\Lambda^2\theta(t)\|_{L^2}^2+\|\Lambda^2\theta_0\|_{L^2}^2,$$

which implies $\int_0^T\|\Lambda^{2+\frac{\alpha}{2}}\theta(t)\|_{L^2}^2dt < \infty$. Applying the Gagliardo-Nirenberg inequality,

$$\|\nabla\theta\|_{L^\infty} \le C\|\theta\|_{L^2}^{\frac{\alpha}{4+\alpha}}\|\Lambda^{2+\frac{\alpha}{2}}\theta\|_{L^2}^{\frac{4}{4+\alpha}},$$

in $\mathbb{R}^2$, we have (4.11).

We assume that (4.8) holds true. Then

$$\begin{aligned} I_1 &\le \|\nabla\nabla^{\perp}\theta\|_{\dot{H}^{\alpha/2}}\||\nabla v|\cdot\nabla\nabla^{\perp}\theta\|_{\dot{H}^{-\alpha/2}} \\ &\le \|\nabla v\|_{\dot{X}_{r,\alpha/2}}\|\nabla\nabla^{\perp}\theta\|_{\dot{H}^{\alpha/2}}\|\nabla\nabla^{\perp}\theta\|_{\dot{H}^r} \\ &\le C\|\nabla^{\perp}\theta\|_{\dot{X}_{r,\alpha/2}}\|\nabla\nabla^{\perp}\theta\|_{L^2}^{1-\frac{2r}{\alpha}}\|\nabla\nabla^{\perp}\theta\|_{\dot{H}^{\alpha/2}}^{1+\frac{2r}{\alpha}} \quad (\text{by } (3.6)) \\ &\le \frac{k}{4}\|\Lambda^{2+\frac{\alpha}{2}}\theta\|_{L^2}^2 + C\|\nabla^{\perp}\theta\|_{\dot{X}_{r,\alpha/2}}^{\frac{2\alpha}{\alpha-2r}}\|\Lambda^2\theta\|_{L^2}^2. \end{aligned} \tag{4.17}$$

Since $\|\nabla\nabla v\|_{\dot{H}^r} \le C\|\nabla\nabla^{\perp}\theta\|_{\dot{H}^r}$, due to the Calderon-Zygmund inequality, we observe that the estimate of J is the same as that of I_1, and we have

$$J \leq \frac{k}{4}\|\Lambda^{2+\frac{\alpha}{2}}\theta\|_{L^2}^2 + C\|\nabla^\perp\theta\|_{\dot{X}_{r,\alpha/2}}^{\frac{2\alpha}{\alpha-2r}}\|\Lambda^2\theta\|_{L^2}^2, \tag{4.18}$$

and hence we similarly get (4.11).

Finally, we assume that (4.9) holds true. Using Lemma 2.1 and the interpolation inequality (3.6), we bound J as follows.

$$\begin{aligned} J &= \int (\nabla^\perp\theta\cdot\nabla)v\cdot\Lambda^2\nabla^\perp\theta dx = \int \Lambda^{1-s}(\nabla^\perp\theta\cdot\nabla v)\cdot\Lambda^{1+s}\nabla^\perp\theta dx \\ &\leq \|\Lambda^{1-s}(\nabla^\perp\theta\cdot\nabla v)\|_{L^2}\cdot\|\Lambda^{1+s}\nabla^\perp\theta\|_{L^2} \\ &\leq C\|\nabla^\perp\theta\cdot\nabla v\|_{\dot{F}^{1-s}_{2,2}}\|\Lambda^{1+s}\nabla^\perp\theta\|_{L^2} \\ &\leq C\left(\|\nabla^\perp\theta\|_{\dot{F}^{-s}_{\infty,\infty}}\|\nabla v\|_{\dot{F}^1_{2,2}} + \|\nabla^\perp\theta\|_{\dot{F}^1_{2,2}}\|\nabla v\|_{\dot{F}^{-s}_{\infty,\infty}}\right)\|\Lambda^{1+s}\nabla^\perp\theta\|_{L^2} \qquad (4.19) \\ &\leq C\|\nabla^\perp\theta\|_{\dot{F}^{-s}_{\infty,\infty}}\|\Lambda^2\theta\|_{L^2}\|\Lambda^{2+s}\theta\|_{L^2} \\ &\leq C\|\nabla^\perp\theta\|_{\dot{F}^{-s}_{\infty,\infty}}\|\Lambda^2\theta\|_{L^2}^{2-\frac{2s}{\alpha}}\|\Lambda^{2+\frac{\alpha}{2}}\theta\|_{L^2}^{\frac{2s}{\alpha}} \\ &\leq \frac{1}{4}\|\Lambda^{2+\frac{\alpha}{2}}\theta\|_{L^2}^2 + C\|\nabla^\perp\theta\|_{\dot{F}^{-s}_{\infty,\infty}}^{\frac{\alpha}{\alpha-s}}\|\Lambda^2\theta\|_{L^2}^2, \end{aligned}$$

where we have used the following inequalities [22]:

$$\|\nabla v\|_{\dot{F}^1_{2,2}} \leq C\|\nabla^\perp\theta\|_{\dot{F}^1_{2,2}},$$

and

$$\|\nabla v\|_{\dot{F}^{-s}_{\infty,\infty}} \leq C\|\nabla^\perp\theta\|_{\dot{F}^{-s}_{\infty,\infty}}.$$

On the other hand,

$$\begin{aligned} I &= \sum_i \int v_i\partial_i\nabla^\perp\theta\cdot\Delta\nabla^\perp\theta dx = -\sum_{i,k}\int v_i\nabla^\perp\theta\cdot\partial_k^2\partial_i\nabla^\perp\theta dx \\ &= \sum_{i,k}\int \partial_k v_i\cdot\nabla^\perp\theta\cdot\partial_k\partial_i\nabla^\perp\theta dx \end{aligned}$$

and now I can be bounded by the same method as that of J, we obtain

$$I \leq \frac{1}{4}\|\Lambda^{2+\frac{\alpha}{2}}\theta\|_{L^2}^2 + C\|\nabla^\perp\theta\|_{\dot{F}^{-s}_{\infty,\infty}}^{\frac{\alpha}{\alpha-s}}\|\Lambda^2\theta\|_{L^2}^2. \tag{4.20}$$

Inserting (4.19) and (4.20) into (4.12) and using the Gronwall's inequality gives

$$\int_0^T \|\Lambda^{2+\frac{\alpha}{2}}\theta\|_{L^2}^2 dt \leq C,$$

which implies (4.11).

This completes the proof. ∎

REFERENCES

1. J. Wu, Generalized MHD equations, *J. Differential Equations*, **195** (2003), 284-312.
2. Y. Zhou, Regularity criteria for the generalized viscous MHD equations, *Ann. I. H. Poincaré-AN*, **24** (2007), 491-505.
3. C. He and Z. Xin, On the regularity of solutions to the magnetohydrodynamic equations, *J. Differential Equations*, **213(2)** (2005), 235-254.
4. Y. Zhou, Remarks on regularities for the 3D MHD equations, *Discrete Contin. Dynam. Syst.*, **12(5)** (2005), 881-886.
5. V. G. Maz'ya, On the theory of the n-dimensional Schrödinger operator, *Izv. Akad. Nauk SSSR* (*Ser. Mat.*), **28** (1964), 1145-1172.
6. V. G. Maz'ya and T. O. Shaposhnikova, *Theory of Multipliers in Spaces of Differentiable Functions*, Pitman, 1985.
7. V. G. Maz'ya and I. E. Verbitsky, Capacitary estimates for fractional integrals, with applications to partial differential equations and Sobolev multipliers, *Ark. Mat.*, **33** (1995), 81-115.
8. V. G. Maz'ya and I. E. Verbitsky, The Schrödinger operator on the energy space: boundedness and compactness criteria, *Acta Math.*, **188** (2002), 263-302.
9. H. Triebel, *Theory of Function Spaces*, Birkäuser Verlag, Boston, 1983.
10. H. Kozono and Y. Shimada, Bilinear estimates in homogeneous Triebel-Lizorkin spaces and the Navier-Stokes equations, *Math. Nachr.*, **276** (2004), 63-74.
11. P. Constantin, Geometric statistics in turbulence, *SIAM Rev.*, **36** (1994), 73-98.
12. A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge, University, Press, Cambridge, UK, 2002.
13. P. Constantin, A. Majda and E. Tabak, Formation of strong fronts in the 3-D quasi-geostrophic thermal active scalar, *Nonlinearity*, **7** (1994), 1459-1533.
14. D. Chae, On the continuation principles for the Euler equations and the quasi-geostrophic equation, *J. Differential Equations*, **227** (2006), 640-651.
15. D. Chae and J. Lee, *On the global well-posedness and stability of the Navier-Stokes and the related equations*, Advances in Math. Fluid Mech., Contributions to Current Challenges in Mathematical Fluid Mechanics, Birkhauser, 2004, pp. 31-51.
16. P. Constantin and J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, *SIAM J. Math. Anal.*, **30** (1999), 937-948.
17. L. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Arxiv preprint math/0608447.
18. A. Kiselev, F. Nazarov and A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, *Inventiones Math.*, **167** (2007), 445-453.

19. D. Chae, On the regularity conditions for the dissipative quasi-geostrophic equations, *SIAM J. Math. Anal.*, **37** (2006), 1649-1656.

20. D. Chae, On the Regularity Conditions for the Navier-Stokes and the Related Equations, *Revista Mat. Iberoamericana*, **23** (2007), 371-384.

21. J. Yuan, On regularity criterionfor the dissipative quasi-geostrophic equations, *J. Math. Anal. Appl.*, **340** (2008), 334-339.

22. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.

Jishan Fan[a,b], Hongjun Gao[a,c] and Gen Nakamura[d]
[a] Institute of Mathematics
School of Mathematical Sciences
Nanjing Normal University
Nanjing 210046
P. R. China
E-mail: fanjishan@njfu.edu.cn
[b] Department of Applied Mathematics
Nanjing Forestry University
Nanjing 210037
P. R. China
[c] Center of Nonlinear Science
Nanjing University
Nanjing 210093
P. R. China
E-mail: gaohj@njnu.edu.cn
[d] Department of Mathematics
Hokkaido University
Sapporo 060-0810
Japan
E-mail: gnaka@math.sci.hokudai.ac.jp