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HIGHER-ORDER GENERALIZED ADJACENT DERIVATIVE AND APPLICATIONS TO DUALITY FOR SET-VALUED OPTIMIZATION

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Abstract. A new notion of the higher-order generalized adjacent derivative for a set-valued map is defined. By virtue of the derivative, a higher-order Mond-Weir type dual problem is introduced for a constrained set-valued optimization problem. The weak duality, strong duality and converse duality theorems are established.

1. INTRODUCTION

The duality theory has been shown to be useful in mathematical economics, numerical analysis, engineering and other fields involve vector-valued maps (or set-valued maps) as constraints and objectives (see [8]). So the theory of duality has received many attentions (see 3-5, 7, 9-13, 17-26). In [25], Weir and Mond obtained weak, strong and converse duality for weak minimal solutions of multiple objective optimization problems under different pseudo-convexity and quasi-convexity assumptions. In [18], Preda and Koller introduced a Mond-Weir duality scheme for optimization problems involving set functions and discussed the Mond-Weir type duality under generalized pseudo-convexity and generalized quasi-convexity assumptions. In [19], by virtue of the tangent derivative of set-valued maps introduced in [1], Sach and Craven obtained Mond-Weir type duality theorems of set-valued optimization under the condition that set-valued maps satisfy an invex property. In [20], by using the codifferential of set-valued maps introduced in [2],

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Sach et al. discussed Mond-Weir type weak duality and strong duality of set-valued optimization problems where set-valued maps satisfy generalized invex properties.

Since higher-order tangent sets introduced in [1], in general, are not cones and convex sets, there are some difficulties in studying higher-order optimality conditions and duality for general set-valued optimization problems. Until now, there are only a few papers to deal with higher-order optimality conditions and duality of set-valued optimization problems by virtue of the higher-order derivatives or epiderivatives introduced by the higher-order tangent sets. In [15], Li et al. studied some properties of higher-order tangent sets and higher-order derivatives introduced in [1], and then obtained higher-order Fritz John type necessary and sufficient optimality conditions of (weak) maximal solutions for set-valued optimization problems in terms of the higher-order derivatives under cone-concavity assumptions. By using these concepts, they also discussed higher-order Mond-Weir duality for a set-valued optimization problem in [16]. In [14], Li and Chen introduced higher-order generalized epiderivatives of set-valued maps, and established higher-order Fritz John type necessary and sufficient conditions for Henig efficient solutions to a constrained set-valued optimization problem.

Motivated by the work reported in [14, 15, 16], we introduce a notion of higher-order generalized adjacent derivative for a set-valued map. Then, by virtue of the derivative, we discuss higher-order Mond-Weir type duality for a constrained set-valued optimization problem.

The rest of the paper is organized as follows. In Section 2, we collect some concepts and some of their properties required for the paper. In Section 3, we introduce the generalized higher-order adjacent set of a set and the higher-order generalized adjacent derivative of a set-valued map, and study some of their properties. In Section 4, by virtue of the derivative, we propose a higher-order Mond-Weir type dual problem for a constrained set-valued optimization problem, and then establish the weak duality, strong duality and inverse duality theorems, respectively.

2. Preliminaries and Notations

Throughout this paper, let X, Y and Z be three real normed spaces, where the spaces Y and Z are partially ordered by nontrivial pointed closed convex cones $C \subset Y$ and $D \subset Z$ with $intC \neq \emptyset$ and $intD \neq \emptyset$, respectively. We assume that $0_X, 0_Y, 0_Z$ denote the origins of X, Y, Z, respectively, Y^* denotes the topological dual space of Y and C^* denotes the dual cone of C. Let M be a nonempty set in Y. The cone hull of M is defined by $cone(M) = \{ty \mid t \geq 0, y \in M\}$. Let E be a nonempty subset of X, $F : E \to 2^Y$ and $G : E \to 2^Z$ be two set-valued maps. The domain, the graph and the epigraph of F are defined respectively by $dom(F) = \{x \in E \mid F(x) \neq \emptyset\}$, $gph(F) = \{(x, y) \in X \times Y \mid x \in E, y \in F(x)\}$ and $epi(F) = \{(x, y) \in X \times Y \mid x \in E, y \in F(x) + C\}$. The profile map

$F_+ : E \to 2^Y$ is defined by $F_+(x) := F(x) + C$, for every $x \in dom(F)$. Let $y_0 \in Y$. Denote $F(E) = \bigcup_{x \in E} F(x)$ and $(F - y_0)(x) = F(x) - y_0$.

Definition 2.1. Let $F : E \to 2^Y$ be a set-valued map, $(x_0, y_0) \in gph(F)$.

(i) F is said to be C-convex on a convex set E, if $epi(F)$ is convex.

(ii) F is said to be generalized C-convex at (x_0, y_0) on a nonempty subset E, if $cone(epi(F) - (x_0, y_0))$ is convex.

Remark 2.1. If F is C-convex on a convex set E, then F is generalized C-convex at (x_0, y_0) on E. But the converse does not hold. For example, let $X = Y = R, E = [-1, 2], C = R_+, F(x) = \{y \in Y | y \geq x^{\frac{2}{5}}\}$, $(x_0, y_0) = (0, 0) \in gph(F)$. Then F is generalized C-convex at (x_0, y_0) on E, but F is not C-convex on E.

Suppose that m is a positive integer, X is a normed space supplied with a distance d and K is a subset of X. We denote by $d(x, K) = \inf_{y \in K} d(x, y)$ the distance from x to K, where we set $d(x, \emptyset) = +\infty$. Now we recall the definitions in [1], [16] and [14].

Definition 2.2. ([1]) Let x belong to a subset K of a normed space X and $v_1, \cdots, v_{m-1}$ be elements of X. We say that the set

$$\begin{aligned} & T_K^{\flat(m)}(x, v_1, \cdots, v_{m-1}) \\ = \; & Liminf_{h \to 0^+} \frac{K - x - hv_1 - \cdots - h^{m-1} v_{m-1}}{h^m} \\ = \; & \{y \in X \mid \lim_{h \to 0^+} d(y, \frac{K - x - hv_1 - \cdots - h^{m-1} v_{m-1}}{h^m}) = 0\} \end{aligned}$$

is the mth-order adjacent set of K at $(x, v_1, \cdots, v_{m-1})$.

From Proposition 3.2 in [15], we have the following result.

Proposition 2.1. *If K is convex, $x \in K$, and $v_i \in X$, $i = 1, \cdots, m-1$, then $T_K^{\flat(m)}(x, v_1, \cdots, v_{m-1})$ is convex.*

Definition 2.3. ([1]) The mth-order adjacent derivative $D^{\flat(m)} F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in gph(F)$ with respect to (in short, w.r.t.) $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} & gph(D^{\flat(m)} F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})) \\ = \; & T_{gph(F)}^{\flat(m)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1}). \end{aligned}$$

Definition 2.4. ([16]). The C-directed mth-order adjacent derivative $D_C^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in gph(F)$ w.r.t. $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ is the mth-order adjacent derivative of set-valued map F_+ at $(x_0, y_0) \in gph(F)$ w.r.t. $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$.

Definition 2.5. Let $M \subset Y$. An element $y \in M$ is said to be a minimal point of M if $M \bigcap (y - C) = \{y\}$. The set of all minimal points of M is denoted by $Min_C M$.

Definition 2.6. ([14]). The mth-order generalized adjacent epiderivative $D_g^{\flat(m)} F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in gph(F)$ w.r.t. $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} &D_g^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x) \\ &= Min_C\{y \in Y \mid (x, y) \in T_{epi(F)}^{\flat(m)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})\}, \end{aligned}$$

for any $x \in dom[D^{\flat(m)}F_+(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})]$.

However there are set-valued maps for which the mth-order adjacent derivatives, the C-directed mth-order adjacent derivatives and the mth-order generalized adjacent epiderivatives do not exist. The following example highlights one such a set-valued map.

Example 2.1. Let $X = Y = R$, $E = X$, $C = R_+$, $n \in (1, 2)$, $F(x) = \{y \in R \mid y \geq |x|^n\}, \forall x \in E$, $(x_0, y_0) = (0, 0)$ and $(u, v) = (1, 0)$. Then, for any $x \in E$,

$$T_{gph(F)}^{\flat(2)}(x_0, y_0, u, v) = T_{epi(F)}^{\flat(2)}(x_0, y_0, u, v) = \emptyset,$$

Therefore, for any $x \in E$, $D^{\flat(2)}F(x_0, y_0, u, v)(x-x_0)$, $D_C^{\flat(2)}F(x_0, y_0, u, v)(x-x_0)$ and $D_g^{\flat(2)}F(x_0, y_0, u, v)(x - x_0)$ do not exist.

To deal with the situation where these derivatives do not exist, we introduce higher-order generalized adjacent derivatives of set-valued maps in the following section.

3. Higher-order Generalized Adjacent Derivatives

In this section, we first introduce higher-order generalized adjacent derivatives of set-valued maps, and then investigate their some properties.

Definition 3.1. Let x belong to a subset K of X and $v_1, \cdots, v_{m-1}$ be elements of X. The subset

$$G\text{-}T_K^{\flat(m)}(x, v_1, \cdots, v_{m-1})$$
$$= Liminf_{h\to 0^+} \frac{cone(K-x) - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}$$
$$= \{y \in X \mid \lim_{h\to 0^+} d(y, \frac{cone(K-x) - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}) = 0\}$$

is said to be the generalized mth-order adjacent set of K at $(x, v_1, \cdots, v_{m-1})$.

Definition 3.2. The mth-order generalized adjacent derivative $D_G^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in gph(F)$ w.r.t. $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$D_G^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)$$
$$= Min_C\{y \in Y \mid (x, y) \in G\text{-}T_{epi(F)}^{\flat(m)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})\}.$$

From properties of higher-order adjacent sets [1], we have the following result.

Proposition 3.1. *If* $D_G^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0) \neq \emptyset$ *and the set* $\{y \in Y \mid (x - x_0, y) \in G\text{-}T_{epi(F)}^{\flat(m)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})\}$ *fulfills the domination property for all* $x \in E$*, then for all* $x \in E$,

(*i*) $D^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0) \subseteq$

$$D_G^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0) + C;$$

(*ii*) $D_C^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0) \subseteq$

$$D_G^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0) + C;$$

(*iii*) $D_g^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0) \subseteq$

$$D_G^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0) + C.$$

Remark 3.1. The reverse inclusions in Proposition 3.1 may not hold. The following examples explain the case, where we only take $m = 2$.

Example 3.2. Let X, Y, E, C, $F(\cdot)$, R, (x_0, y_0), (u, v) be as in Example 2.1. From Example 2.1, we know that, for any $x \in E$, $D^{\flat(2)}F(x_0, y_0, u, v)(x - x_0)$, $D_C^{\flat(2)}F(x_0, y_0, u, v)(x - x_0)$, and $D_g^{\flat(2)}F(x_0, y_0, u, v)(x - x_0)$ do not exist. However, $G\text{-}T_{epi(F)}^{\flat(2)}(x_0, y_0, u, v) = \{y \mid y \geq 0\}$. Therefore, for any $x \in E$,

$$D_G^{\flat(2)}F(x_0, y_0, u, v)(x - x_0) = \{0\}.$$

Example 3.3. Let $X = E = R$, $Y = R^2$, $E = X$, $C = R^2_+$, $F(x) = \{(y_1, y_2) \in R^2 \mid y_1 \geq x^{\frac{4}{3}}, y_2 \geq x^2\}, \forall x \in E$, $(x_0, y_0) = (0, (0, 0))$ and $(u, v) = (1, (0, 0))$. By direct calculation, we know, for any $x \in E$, $D^{\flat(2)} F(x_0, y_0, u, v)(x - x_0)$, $D_C^{\flat(2)} F(x_0, y_0, u, v)(x - x_0)$ and $D_g^{\flat(2)} F(x_0, y_0, u, v)(x - x_0)$ do not exist.

However, $G\text{-}T^{\flat(2)}_{epi(F)}(x_0, y_0, u, v) = \{(x, (y_1, y_2)) \in R \times R^2 \mid x \in R, y_1 \geq 0, y_2 \geq 0\}$. Therefore, for any $x \in E$,

$$D_G^{\flat(2)} F(x_0, y_0, u, v)(x - x_0) = \{(0, 0)\}.$$

Definition 3.3. ([14]).

(i) The cone C is called Daniell, if any decreasing sequence in Y having a lower bound converges to its infimum.

(ii) A subset H of Y is said to be minorized, if there is a $y \in Y$ so that $H \subset \{y\} + C$.

(iii) The domination property is said to hold for a subset H of Y if $H \subset Min_C H + C$.

Now we give an existence theorem of $D_G^{\flat(m)} F$.

Theorem 3.1. *Let C be a closed convex pointed cone and let C be Daniell. Suppose that the set $P(x) := \{y \in Y \mid (x, y) \in G\text{-}T^{\flat(m)}_{epi(F)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})\}$ is minorized. Then $D_G^{\flat(m)} F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)$ exists.*

Proof. It follows from the definition that $G\text{-}T^{\flat(m)}_{epi(F)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ is closed. Then we can prove it as the proof of Theorem 2 in [6]. ■

Now we discuss some crucial propositions of the mth-order generalized adjacent derivative.

Proposition 3.2. *Let $x, x_0 \in E$, $y_0 \in F(x_0)$, $(u_i, v_i) \in \{0_X\} \times C$. If the set $P(x - x_0) := \{y \in Y \mid (x - x_0, y) \in G\text{-}T^{\flat(m)}_{epi(F)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})\}$ fulfills the domination property for all $x \in E$, then, for all $x \in E$,*

$$F(x) - y_0 \subset D_G^{\flat(m)} F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0) + C.$$

Proof. Take any $x \in E, y \in F(x)$ and an arbitrary sequence $\{h_n\}$ with $h_n \to 0^+$. Since $y_0 \in F(x_0)$, $(x - x_0, y - y_0) \in epi(F) - (x_0, y_0)$, and

$$h_n^m (x - x_0, y - y_0) \in cone(epi(F) - (x_0, y_0)).$$

It follows from $(u_i, v_i) \in \{0_X\} \times C, i = 1, \cdots, m - 1$, and C being a convex cone that

$$h_n(u_1, v_1) + \cdots + h_n^{m-1}(u_{m-1}, v_{m-1}) \in \{0_X\} \times C,$$

and

$$(x_n, y_n) := h_n(u_1, v_1) + \cdots + h_n^{m-1}(u_{m-1}, v_{m-1}) + h_n^m(x - x_0, y - y_0) \\ \in cone(epi(F) - (x_0, y_0)).$$

So

$$(x - x_0, y - y_0) = \frac{(x_n, y_n) - h_n(u_1, v_1) - \cdots - h_n^{m-1}(u_{m-1}, v_{m-1})}{h_n^m},$$

which implies that

$$(x - x_0, y - y_0) \in G\text{-}T^{\flat(m)}_{epi(F)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1}),$$

i.e., $y - y_0 \in P(x - x_0)$. By the definition of mth-order generalized adjacent derivative and the domination property, we have

$$P(x - x_0) \subset D_G^{\flat(m)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0) + C.$$

Thus $F(x) - y_0 \subset D_G^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0) + C.$ ■

Remark 3.2. Since the cone-convexity and cone-concavity assumptions are omitted, Proposition 3.2 improves Theorem 4.1 in [15] and Proposition 3.1 in [14].

Proposition 3.3. *Let E be a nonempty subset of X, $x_0 \in E$, $y_0 \in F(x_0)$. Let F be generalized C-convex at (x_0, y_0) on E, $u_i \in E, v_i \in F(u_i) + C, i = 1, \cdots, m-1$. If the set $q(x - x_0) := \{y \in Y \mid (x - x_0, y) \in G\text{-}T^{\flat(m)}_{epi(F)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)\}$ fulfills the domination property for all $x \in E$, then for any $x \in E$,*

$$F(x) - y_0 \subset D_G^{\flat(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C.$$

Proof. Take any $x \in E$ and $y \in F(x)$. Let $\{h_n\}$ be an arbitrary sequence with $h_n \to 0^+$. Since E is convex and $F - y_0$ is convexlike on E, we get that $epi(F) - (x_0, y_0)$ is a convex subset and $cone(epi(F) - (x_0, y_0))$ is a convex cone. Then

$$\begin{aligned} &h_n(u_1 - x_0, v_1 - y_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0) \\ &= (h_n + \cdots + h_n^{m-1})(\frac{h_n u_1 + \cdots + h_n^{m-1} u_{m-1}}{h_n + \cdots + h_n^{m-1}} \\ &-x_0, \frac{h_n v_1 + \cdots + h_n^{m-1} v_{m-1}}{h_n + \cdots + h_n^{m-1}} - y_0) \in cone(epiF - (x_0, y_0)). \end{aligned}$$

It follows from $h_n > 0$, E is convex and $cone(epiF - (x_0, y_0))$ is a convex cone that

$$(x_n, y_n) := h_n(u_1 - x_0, v_1 - y_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0) \\ + h_n^m(x - x_0, y - y_0) \in cone(epiF - (x_0, y_0)).$$

So

$$(x-x_0, y-y_0) = \frac{(x_n, y_n) - h_n(u_1 - x_0, v_1 - y_0) - \cdots - h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0)}{h_n^m},$$

and then

$$(x - x_0, y - y_0) \in G\text{-}T^{\flat(m)}_{epi(F)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0),$$

i.e., $y - y_0 \in q(x - x_0)$. By the definition of mth-order generalized adjacent derivative and the domination property, we have

$$q(x-x_0) \subset D^{\flat(m)}_G(x_0, y_0, u_1-x_0, v_1-y_0, \cdots, u_{m-1}-x_0, v_{m-1}-y_0)(x-x_0)+C.$$

Thus $F(x)-y_0 \subset D^{\flat(m)}_G F(x_0, y_0, u_1-x_0, v_1-y_0, \cdots, u_{m-1}-x_0, v_{m-1}-y_0)(x-x_0) + C$, and the proof is complete. ■

Remark 3.3. Since the cone-convexity assumptions are replaced by the generalized cone-convexity assumptions, it follows from Remark 2.1 that Proposition 3.3 improves [14, Proposition 3.1].

4. Higher-order Duality for Set-valued Optimization

In this section, we introduce a class of higher-order Mond-Weir type dual problems for a constrained set-valued optimization problem by virtue of higher-order generalized adjacent derivatives and discuss its weak duality, strong duality and converse duality properties. Throughout this section, the notation $(F, G)(x)$ is used to denote $F(x) \times G(x)$. Let $(x_0, y_0) \in gph(F)$. Firstly, we recall the definition of interior tangent cone of a set and state a result regarding it in [9].

The interior tangent cone of K at x_0 is defined as

$$IT_K(x_0) = \{u \in X \mid \exists\lambda > 0, \forall t \in (0, \lambda), \forall u' \in B_X(u, \lambda), x_0 + tu' \in K\},$$

where $B_X(u, \lambda)$ stands for the closed ball centered at $u \in X$ and of radius λ.

Lemma 4.1. ([9]). *If $K \subset X$ is convex, $x_0 \in K$ and $intK \neq \emptyset$, then*

$$IT_{intK}(x_0) = intcone(K - x_0).$$

Consider the following set-valued optimization problem:

$$(P)\begin{cases} \min & F(x) \\ \text{s.t.} & G(x) \bigcap (-D) \neq \emptyset,\ x \in E. \end{cases}$$

Set $\bar{K} := \{x \in E \mid G(x) \bigcap (-D) \neq \emptyset\}$. A point $(x_0, y_0) \in X \times Y$ is called a feasible solution of (P) if $x_0 \in \bar{K}$ and $y_0 \in F(x_0)$.

Definition 4.1. A point (x_0, y_0) is said to be a weakly minimal solution of (P) if $(x_0, y_0) \in \bar{K} \times F(\bar{K})$ satisfying $y_0 \in F(x_0)$ and $(F(\bar{K}) - y_0) \bigcap (-intC) = \emptyset$.

Suppose that $(u_i, v_i, w_i) \in X \times Y \times Z, i = 1, \cdots, m-1$, $(x_0, y_0) \in gph(F)$, $z_0 \in G(x_0) \bigcap (-D)$, and $\Omega = dom[D_G^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \cdots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)] \bigcap E$. We introduce a higher-order dual problem (DP) of (P) as follows:

$$\begin{aligned} \max\ & y_0 \\ \text{s.t.}\ & \phi(y)+\psi(z) \geq 0, (y, z) \in D_G^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1+z_0, \cdots, u_{m-1}, v_{m-1}, \end{aligned}$$

(1) $\quad w_{m-1} + z_0)(x), x \in \Omega,$

(2) $\quad \psi(z_0) \geq 0,$

(3) $\quad \phi \in C^* \setminus \{0_{Y^*}\},$

(4) $\quad \psi \in D^*.$

Let $H = \{y_0 \in F(x_0) \mid (x_0, y_0, z_0, \phi, \psi)$ satisfies conditions $(1) - (4)\}$. A point $(x_0, y_0, z_0, \phi, \psi)$ satisfying (1)-(4) is called a feasible solution of (DP). A feasible solution $(x_0, y_0, z_0, \phi, \psi)$ is called a weakly maximal solution of (DP) if $(H - y_0) \bigcap intC = \emptyset$.

Theorem 4.1. (Weak duality). *Suppose that* $(u_i, v_i, w_i + z_0) \in \{0_X\} \times C \times D, i = 1, \cdots, m-1$ *and the set* $\{(y, z) \in Y \times Z \mid (x, y, z) \in G\text{-}T_{epi(F,G)}^{\flat(m)}(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \cdots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)\}$ *fulfills the domination property for all* $x \in \Omega$*. If* $(\bar{x}, \bar{y})$ *is a feasible solution of* (P) *and* $(x_0, y_0, z_0, \phi, \psi)$ *is a feasible solution of* (DP)*, then*

$$\phi(\bar{y}) \geq \phi(y_0).$$

Proof. It follows from Proposition 3.2 and the assumptions that

$$(F, G)(\bar{x}) - (y_0, z_0) \subset D_G^{(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \cdots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)(\bar{x} - x_0) + C \times D. \tag{5}$$

Since $(\bar{x}, \bar{y})$ is a feasible solution of (P), $G(\bar{x}) \bigcap (-D) \neq \emptyset$. Take $\bar{z} \in G(\bar{x}) \bigcap (-D)$. It follows from (2) and (4) that

$$\psi(\bar{z} - z_0) \leq 0. \tag{6}$$

By (1), (3), (4), (5) and (6), we have

$$\phi(\bar{y}) \geq \phi(y_0).$$

So the proof is complete. ■

Remark 4.1. In Theorem 4.1, the cone-convexity assumptions used in Theorem 4.1 of [16] are not required.

By the similar proof method for Theorem 4.1, it follows from Proposition 3.3 that the following theorem holds.

Theorem 4.2. *(Weak duality) Let* $(u_i, v_i, w_i+z_0) \in epi(F,G)-(x_0, y_0, z_0), i = 1, \cdots, m-1$. *Suppose that* (F, G) *is generalized* $C \times D$*-convex at* (x_0, y_0, z_0) *on a nonempty set* E *and the set* $\{(y, z) \in Y \times Z \mid (x, y, z) \in G\text{-}T^{\flat(m)}_{epi(F,G)}(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \cdots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)\}$ *fulfills the domination property for all* $x \in \Omega$. *If* $(\bar{x}, \bar{y})$ *is a feasible solution of* (P) *and* $(x_0, y_0, z_0, \phi, \psi)$ *is a feasible solution of* (DP)*, then*

$$\phi(\bar{y}) \geq \phi(y_0).$$

Lemma 4.2. *Let* $(u_i, v_i, w_i) \in X \times (-C) \times (-D), i = 1, \cdots, m-1$. *If* $(x_0, y_0) \in gph(F)$ *is a weakly minimal solution of* (P)*, then for any* $z_0 \in G(x_0) \bigcap (-D)$*,*

$$\begin{aligned}(7)\qquad &[D^{\flat(m)}_G(F,G)(x_0, y_0, z_0, u_1, v_1, w_1+z_0, \cdots, u_{m-1}, v_{m-1}, w_{m-1}+z_0)(x)\\ &+C \times D + (0_Y, z_0)] \bigcap (-int(C \times D)) = \emptyset,\end{aligned}$$

for all $x \in \Omega$.

Proof. Since (x_0, y_0) is a weakly minimal solution of (P), $(F(\bar{K})-y_0) \bigcap -intC = \emptyset$. Then

$$(8)\qquad cone(F(\bar{K}) + C - y_0) \bigcap (-intC) = \emptyset.$$

Assume that the relation (7) does not hold. Then there exist $\bar{c} \in C$, $\bar{d} \in D$ and $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Z$ with $\bar{x} \in \Omega$ such that

$$(9)\quad (\bar{y}, \bar{z}) \in D^{\flat(m)}_G(F,G)(x_0, y_0, z_0, u_1, v_1, w_1+z_0, \cdots, u_{m-1}, v_{m-1}, w_{m-1}+z_0)(\bar{x}),$$

and

$$(10)\qquad (\bar{y}, \bar{z}) + (\bar{c}, \bar{d}) + (0_Y, z_0) \in -int(C \times D).$$

It follows from (9) and the definition of mth-order generalized adjacent derivative that, for arbitrary sequence $\{h_n\}$ with $h_n \to 0^+$, there exists a sequence $\{(x_n, y_n, z_n)\} \subset cone(epi(F,G) - (x_0, y_0, z_0))$ such that

$$(11)\qquad \frac{(x_n, y_n, z_n) - h_n(u_1, v_1, w_1+z_0) - \cdots - h_n^{m-1}(u_{m-1}, v_{m-1}, w_{m-1}+z_0)}{h_n^m} \to (\bar{x}, \bar{y}, \bar{z}).$$

From (10) and (11), there exists a sufficient large number N_1 such that

$$y_n - h_n v_1 - \cdots - h_n^{m-1} v_{m-1} + h_n^m \bar{c} \in -intC, \text{ for } n > N_1 \tag{12}$$

and

$$\begin{aligned} \tilde{z}_n :&= \frac{z_n - h_n(w_1 + z_0) - \cdots - h_n^{m-1}(w_{m-1} + z_0)}{h_n^m} \\ &= \frac{h_n + \cdots + h_n^{m-1}}{h_n^m}\left(\frac{z_n - h_n w_1 - \cdots - h_n^{m-1} w_{m-1}}{h_n + \cdots + h_n^{m-1}} - z_0\right) \\ &\to \bar{z} \in -(intD + z_0 + \bar{d}) \subset -intcone(D + z_0). \end{aligned} \tag{13}$$

Since $v_1, \cdots, v_{m-1}, -\bar{c} \in -C$, $h_n > 0$ and C is a convex cone,

$$h_n v_1 + \cdots + h_n^{m-1} v_{m-1} - h_n^m \bar{c} \in -C. \tag{14}$$

It follows from (12) and (14) that

$$y_n \in -intC, \text{ for all } n > N_1. \tag{15}$$

By (13) and Lemma 4.1, we have $-\bar{z} \in IT_{intD}(-z_0)$. Then, it follows from the definition of $IT_{intD}(-z_0)$ that $\exists \lambda > 0, \forall t \in (0, \lambda), \forall u' \in B_X(-\bar{z}, \lambda), -z_0 + tu' \in intD$. Since $h_n \to 0^+$, there exists a sufficient large number N_2 such that

$$\frac{h_n^m}{h_n + \cdots + h_n^{m-1}} \in (0, \lambda), \text{ for all } n > N_2.$$

Then, from (13), we have

$$-z_0 + \frac{h_n^m}{h_n + \cdots + h_n^{m-1}}(-\tilde{z}_n) \in intD, \text{ for all } n > N_2,$$

i.e.,

$$\frac{z_n - h_n w_1 - \cdots - h_n^{m-1} w_{m-1}}{h_n + \cdots + h_n^{m-1}} \in -intD, \text{ for all } n > N_2.$$

It follows from $h_n > 0$, $w_1, \cdots, w_{m-1} \in -D$ and D is a convex cone that

$$z_n \in -intD, \text{ for all } n > N_2. \tag{16}$$

Since $z_n \in cone(G(x_n) + D - z_0)$, there exist $\lambda_n \geq 0$, $\bar{z}_n \in G(x_n)$ and $d_n \in D$ such that $z_n = \lambda_n(\bar{z}_n + d_n - z_0)$. It follows from (16) that $\bar{z}_n \in G(x_n) \bigcap (-D)$, for $n > N_2$. Then, for any $n > N_2$, $x_n \in \bar{K}$. It follows from (15) that

$$y_n \in cone(F(\bar{K}) + C - y_0) \bigcap -intC, \text{ for } n > \max\{N_1, N_2\},$$

which contradicts (8). So (7) holds and the proof is complete. ■

Theorem 4.3. (Strong duality). *Suppose that the following conditions are satisfied:*

(i) $(u_i, v_i, w_i + z_0) \in epi(F,G) - (x_0, y_0, z_0)$ *and* $(u_i, v_i, w_i) \in X \times (-C) \times (-D), i = 1, \cdots, m-1$*;*

(ii) (F,G) *is generalized* $C \times D$*-convex at* (x_0, y_0, z_0) *on* E *;*

(iii) *The pair* (x_0, y_0) *is a weakly minimal solution of* (P)*;*

(iv) $P(x) := \{(y,z) \in Y \times Z \mid (x,y,z) \in G\text{-}T^{(m)}_{epi(F,G)}(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \cdots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)\}$ *fulfills the domination property for all* $x \in E$*, and* $(0_Y, 0_Z) \in P(0_X)$*;*

(v) *There exists* $x' \in E$ *such that* $G(x') \bigcap (-intD) \neq \emptyset$*.*

Then there exists $(\phi, \psi) \in (C^* \setminus \{0_{Y^*}\}) \times D^*$ *such that* $(x_0, y_0, z_0, \phi, \psi)$ *is a weakly maximal solution of* (DP)*.*

Proof. Define

$$M = \bigcup_{x \in \Omega} D^{\flat(m)}_G(F,G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \cdots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)(x)$$
$$+ C \times D + (0_Y, z_0).$$

By the similar proof method for the convexity of M in Theorem 5.1 of [14], just replacing mth-order generalized adjacent epiderivative by mth-order generalized adjacent derivative, we have that M is a convex set. It follows from Lemma 4.2 that

$$M \bigcap (-int(C \times D)) = \emptyset.$$

By the separation theorem of convex sets, there exist $\phi \in Y^*$ and $\psi \in Z^*$, not both zero functionals, such that

$$\phi(\bar{y}) + \psi(\bar{z}) \geq \phi(y) + \psi(z), \text{ for all } (\bar{y}, \bar{z}) \in M, (y,z) \in -int(C \times D). \tag{17}$$

It follows from (17) that

$$\phi(y) \leq \psi(z), \text{ for all } (y,z) \in (-intC) \times intD. \tag{18}$$

and

$$\phi(\bar{y}) + \psi(\bar{z}) \geq 0, \text{ for all } (\bar{y}, \bar{z}) \in M. \tag{19}$$

From (18), we obtain that $\psi(z) \geq 0$, for all $z \in intD$. Thus, $\psi \in D^*$. Similarly, we get $\phi \in C^*$.

Now we show that $\phi \neq 0_{Y^*}$. Suppose that $\phi = 0_{Y^*}$. Then $\psi \in D^* \setminus \{0_{Z^*}\}$. By Proposition 3.3 and condition (v), there exists $(y', z') \in (F, G)(x')$ such that $z' \in -intD$ and

$$(y', z') - (y_0, z_0) \in D_G^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \cdots,$$
$$u_{m-1}, v_{m-1}, w_{m-1} + z_0)(x' - x_0) + C \times D.$$

Thus it follows from (19) that $\psi(z') \geq 0$. Since $z' \in -intD$ and $\psi \in D^* \setminus \{0_{Z^*}\}$, we have $\psi(z') < 0$, which leads to a contradiction. Therefore $\phi \neq 0_{Y^*}$.

From (19) and condition (iv), we have $\psi(z_0) \geq 0$. Since $z_0 \in -D$ and $\psi \in D^*$, $\psi(z_0) \leq 0$. Therefore,

$$\psi(z_0) = 0. \tag{20}$$

It follows from (19), (20), $\phi \in C^* \setminus \{0_{Y^*}\}$ and $\psi \in D^*$ that, for all $(y, z) \in D_G^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \cdots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)(x)$, $\phi(y) + \psi(z) \geq 0$. So $(x_0, y_0, z_0, \phi, \psi)$ is a feasible solution of (DP).

Finally, we prove that $(x_0, y_0, z_0, \phi, \psi)$ is a weakly maximal solution of (DP).

Suppose that $(x_0, y_0, z_0, \phi, \psi)$ is not a weakly maximal solution of (DP). Then there exists a feasible solution $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\phi}, \tilde{\psi})$ of (DP) such that $\tilde{y} - y_0 \in intC$. It follows from $\phi \in C^* \setminus \{0_{Y^*}\}$ that

$$\phi(\tilde{y}) > \phi(y_0). \tag{21}$$

Since (x_0, y_0) is a weakly minimal solution of (P), by Theorem 4.2, we have $\phi(\tilde{y}) \leq \phi(y_0)$, which contradicts (21). Thus the conclusion holds and the proof is complete. ■

Remark 4.2. Since the cone-convexity assumptions are replaced by the generalized cone-convexity assumptions, Theorem 4.3 improves Theorem 4.2 in [16].

Theorem 4.4. (Converse duality). *Suppose that the following conditions are satisfied:*

(*i*) $(u_i, v_i, w_i + z_0) \in \{0_X\} \times C \times D$, $i = 1, \cdots, m-1$;

(*ii*) *There exists* $(\phi, \psi) \in (C^* \setminus \{0_{Y^*}\}) \times D^*$ *such that* $(x_0, y_0, z_0, \phi, \psi)$ *is a weakly maximal solution of* (DP);

(*iii*) $P(x) := \{(y, z) \in Y \times Z \mid (x, y, z) \in G\text{-}T_{epi(F,G)}^{\flat(m)}(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \cdots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)\}$ *fulfills the domination property for all* $x \in \bar{K}$, *and* $(0_Y, 0_Z) \in P(0_X)$.

Then the pair (x_0, y_0) *is a weakly minimal solution of* (P).

Proof. It follows from assumptions (i), (iii) and Proposition 3.2 that

$$(y-y_0, z-z_0) \in D_G^{\flat(m)}(F,G)(x_0,y_0,z_0,u_1,v_1,w_1+z_0,\cdots,$$
$$u_{m-1},v_{m-1},w_{m-1}+z_0)(x-x_0)+C\times D,$$

for all $x \in \bar{K}$, $y \in F(x)$ and $z \in G(x)$. Then

(22) $\phi(y-y_0)+\psi(z-z_0) \geq 0,$ for all $x \in \bar{K},\ y \in F(x), z \in G(x).$

It follows from $x \in \bar{K}$ that there exists $\bar{z} \in G(x)$ such that $\bar{z} \in -D$. So $\psi(\bar{z}) \leq 0$. Then, from (2) and (22), we have

(23) $\phi(y) \geq \phi(y_0),$ for all $x \in \bar{K},\ y \in F(x).$

We now show that (x_0, y_0) is a weakly minimal solution of (P). Assume that (x_0, y_0) is not a weakly minimal solution of (P). Then there exists $y_1 \in F(\bar{K})$ such that $y_1 - y_0 \in -intC$. It follows from $\phi \in C^* \setminus \{0_{Y^*}\}$ that $\phi(y_1) < \phi(y_0)$, which contradicts (23). So (x_0, y_0) is a weakly minimal solution of (P) and the proof is complete. ■

Remark 4.3. In Theorem 4.4, the cone-convexity assumptions used in Theorem 4.3 of [16] are omitted.

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