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STRUCTURE OF THE FIXED-POINT SET OF ASYMPTOTICALLY REGULAR MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. It is shown that the set of fixed-points of some asymptotically regular mappings in uniformly convex Banach spaces is not only connected but even a retract of a domain. The results presented in this paper improve and extend some results in [11, 13].

1. Introduction

The concept of asymptotically regular mapping is due to Browder and Petryshyn [3].

Definition 1. Let (M, d) be a metric space. A mapping $T : M \to M$ is called *asymptotically regular* if $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ for all $x \in M$.

Example 2. Let $T : [0, 1] \to [0, 1]$ be an arbitrary nonexpansive mapping. It is easy to check that $S = \frac{1}{2}(I + T)$ is also nonexpansive. Thus, for each $x \in [0, 1]$,

$$|S^{n+1}x - S^n x| \leqslant \ldots \leqslant |S^2x - Sx| \leqslant |Sx - x|.$$

Furthermore S is nondecreasing function. Indeed, if $x \leqslant y$ and $Sx > Sy$, then we have $\frac{1}{2}(x + Tx) > \frac{1}{2}(y + Ty)$ which implies

$$|Tx - Ty| \geqslant Tx - Ty > y - x = |x - y|.$$

Thus, for each $x \in [0, 1]$ we have $x \leqslant Sx$ or $Sx \leqslant x$. So, we have

$$1 \geqslant |S^{n+1}x - x| = \sum_{k=1}^{n} |S^{k+1}x - S^k x| \geqslant n \cdot |S^{n+1}x - S^n x|$$

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which implies $|S^{n+1}x - S^n x| \leqslant \frac{1}{n}$. Then S is asymptotically regular. ■

In 1976, Ishikawa obtained a surprising result, a special case of which may be stated as follows: Let C be an arbitrary nonempty bounded closed convex subset of a Banach space E, $T : C \to C$ nonexpansive, and $\lambda \in (0,1)$. Set $T_\lambda = (1-\lambda)I + \lambda T$. Then for each $x \in C$, $\|T_\lambda^{n+1}x - T_\lambda^n x\| \to 0$ as $n \to \infty$, and $Fix\ T = Fix\ T_\lambda$. This result holds for any above set regardless of its geometrical regularity. In 1978, Edelstein and O'Brien proved that $\{T_\lambda^{n+1}x - T_\lambda^n x\}$ converges to 0 uniformly for $x \in C$, and, in 1983 Goebel and Kirk proved that this convergence is even uniform for $T \in \mathcal{T}$, where $\mathcal{T}$ denotes the collection of all nonexpansive self mappings of C, see [1, 10].

In [15] and [21] one can find two very interesting examples of asymptotically regular mappings without fixed points.

If T is a mapping from a set C into itself, then we use the symbol $\|T\|$ to denote the Lipschitz constant of T, that is

$$\|T\| = \sup\Big\{\frac{\|Tx - Ty\|}{\|x-y\|} : x, y \in C, x \neq y\Big\}.$$

The present author [12] proved the following extension of the well-known Lifshitz Theorem to asymptotically regular mappings, where $k(M)$ denotes the Lifshitz constant of a metric space M, see [1].

Theorem 3. *Let M be a complete metric space with $k(M) > 1$ and T be a mapping from M to M. If T is asymptotically regular,*

$$\liminf_{n\to\infty} \|T^n\| < k(M),$$

and for some $x \in M$ the sequence $\{T^n x\}$ is bounded then T has a fixed point in C.

Domînguez Benavides, Japón Pineda and Xu (see [1], [6] and the references therein) proved that the Lifshitz characteristic can be replaced by another geometric coefficient which can be computed in certain classes of Banach spaces.

We recall some observations about the structure of the fixed-point sets. Suppose $C \subset E$ is a bounded closed convex set and $T : C \to C$ a nonexpansive mapping (that is, $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$). Assume $Fix\ T = \{x \in C : Tx = x\} \neq \varnothing$. Obviously $Fix\ T$ is a closed set, and if E is strictly convex, then $Fix\ T$ is a convex set. In general the fixed-point set of a nonexpansive mapping need not be convex and can be extremely irregular. Bruck [4] asserts that if a nonexpansive mapping $T : C \to C$ has a fixed point in every nonempty closed convex subset of C which is invariant under T and if C is convex and weakly compact, then

$Fix\ T$, the set of fixed-points, is nonexpansive retract of C (that is, there exists a nonexpansive mapping $R : C \to Fix\ T$ such that $R_{|Fix\ T} = I$), [5], [10], and the references given there. The Bruck results was extended by Domínguez Benavides and Lorenzo Ramírez [7] to the case of asymptotically nonexpansive mappings if the space E was sufficiently regular.

Recently the present author, using the method of Hilbert space, proved that the set of fixed-points of some asymptotically regular mappings in a Hilbert space is a retract of a domain [13]:

Theorem 4. *Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. If $T : C \to C$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} \|T^n\| < \sqrt{2},$$

then T has a fixed point in C and $Fix\ T$ is a retract of C.

In this paper, by means of techniques of asymptotic center, we establish some results on the structure of the fixed-point set of asymptotically regular mappings in uniformly convex Banach spaces. The results obtained in this paper improve and extend some results in [11, 13].

2. Uniformly Convex Banach Spaces

Let us recall that a space E is *uniformly convex* if its modulus of convexity

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| \leqslant 1, \|y\| \leqslant 1, \|x-y\| \geqslant \varepsilon\right\}$$

is strictly positive for all $\varepsilon > 0$. A Hilbert space H is uniformly convex. This fact is a direct consequence of parallelogram identity. Also spaces l^p and L^p, $1 < p < \infty$, are uniformly convex. It is well known that δ_E is continuous on $[0, 2)$ and strictly increasing in uniformly convex Banach spaces [10].

Recall the concept and the notion of *asymptotic center* due to Edelstein (1972), see [10]. Let C be a nonempty closed convex subset of a Banach space E and $\{x_n\} \subset E$ be a bounded sequence. Then the asymptotic radius and asymptotic center of $\{x_n\}$ with respect to C are the number

$$r(C, \{x_n\}) = \inf_{y\in C}\left(\limsup_{n\to\infty}\|y - x_n\|\right)$$

and the (possibly empty) set

$$A(C, \{x_n\}) = \left\{y \in C : \limsup_{n\to\infty}\|y - x_n\| = r(C, \{x_n\})\right\},$$

respectively. It is well known that if E is reflexive, then $A(C,\{x_n\})$ is bounded closed convex and nonempty, and if E is uniformly convex, then $A(C,\{x_n\})$ consist only a single point, $\{z\}=A(C,\{x_n\})$, i.e., other words $z\in C$ is the unique point which minimizes the functional

$$\limsup_{n\to\infty}\|y-x_n\|$$

over y in C.

We start from some result for all uniformly convex Banach spaces.

Theorem 5. *Let E be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Suppose $T:C\to C$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty}\|T^n\|<\gamma,$$

where $\gamma>1$ is the unique solution of the equation

$$\gamma\Big(1-\delta_E\Big(\frac{1}{\gamma}\Big)\Big)=1. \tag{1}$$

Then T has a fixed point in C and $Fix\ T$ is a retract of C. (Note that in a Hilbert space, $\gamma=\frac{1}{2}\sqrt{5}$ and in L^p-spaces $(2\leqslant p<\infty)$, $\gamma_p=(1+2^{-p})^{\frac{1}{p}}$.)

Proof. Let $\{n_i\}$ be a sequence of natural numbers such that

$$\liminf_{n\to\infty}\|T^n\|=\lim_{i\to\infty}\|T^{n_i}\|=k<\gamma.$$

We may assume that $k\geqslant 1$ since if $k<1$, the well known Banach Contraction Principle guarantees a fixed point of T.

For an $x=x_0\in C$ we can inductively define a sequence $\{x_m\}\subset C$ in the following manner: x_{m+1} is the asymptotic center of the sequence $\{T^{n_i}x_m\}_{i\geqslant 1}$, that is, x_{m+1} is the unique point in C that minimizes the functional

$$\limsup_{i\to\infty}\|y-T^{n_i}x_m\|$$

over y in C. For each $m\geqslant 0$ we set

$$r_m=\limsup_{i\to\infty}\|x_{m+1}-T^{n_i}x_m\|,$$
$$d(x_m)=\limsup_{i\to\infty}\|x_m-T^{n_i}x_m\|.$$

Obviously,

$$r_m\leqslant d(x_m) \text{ for } m\geqslant 0. \tag{2}$$

Existence of fixed points. If $r_m = 0$ for some $m \geqslant 0$, then $Tx_{m+1} = x_{m+1}$. Indeed, we choose $\varepsilon > 0$. Since $r_m = 0$, there exists $i_0 \in \mathbb{N}$ such that

$$\|T^{n_i}x_m - x_{m+1}\| < \frac{\varepsilon}{2}$$

if $i \geqslant i_0$. From the assumption "$\liminf_{n\to\infty} \|T^n\| < \gamma$" there exists a positive integer p such that $\|T^p\| < \infty$. By asymptotic regularity of T there exists $k_0 \in \mathbb{N}$, $k_0 \geqslant i_0$, such that

$$\begin{aligned}\|T^{p+n_i}x_m - x_{m+1}\| &\leqslant \|T^{p+n_i}x_m - T^{n_i}x_m\| + \|T^{n_i}x_m - x_{m+1}\| \\ &\leqslant \sum_{j=0}^{p-1} \|T^{n_i+j+1}x_m - T^{n_i+j}x_m\| + \|T^{n_i}x_m - x_{m+1}\| < \varepsilon\end{aligned}$$

if $i \geqslant k_0$. Hence for $i \geqslant k_0$, we obtain

$$\begin{aligned}\|T^p x_{m+1} - x_{m+1}\| &\leqslant \|T^p x_{m+1} - T^{p+n_i}x_m\| + \|T^{p+n_i}x_m - x_{m+1}\| \\ &\leqslant \|T^p\| \cdot \|x_{m+1} - T^{n_i}x_m\| + \|T^{p+n_i}x_m - x_{m+1}\| \\ &\leqslant \varepsilon(\frac{1}{2}\|T^p\| + 1) \to 0\end{aligned}$$

as $\varepsilon \downarrow 0$, so $T^p x_{m+1} = x_{m+1}$. It is easily verified by induction that $T^{ps}x_{m+1} = x_{m+1}$ for $s = 1, 2, \ldots$. Therefore

$$\|Tx_{m+1} - x_{m+1}\| = \|T^{ps+1}x_{m+1} - T^{ps}x_{m+1}\| \to 0$$

as $s \to \infty$, so $Tx_{m+1} = x_{m+1}$. Analogically, if $d(x_m) = 0$, then $T^{n_i}x_m \to x_m$ as $i \to \infty$, yielding $Tx_m = x_m$.

Assume $r_m > 0$ for all $m \geqslant 0$. Let $m \geqslant 0$ be fixed and $\varepsilon > 0$ be small enough. First choose $p \in \mathbb{N}$ such that

$$\|T^{n_p}x_{m+1} - x_{m+1}\| > d(x_{m+1}) - \varepsilon$$

and

$$\|T^{n_p}\| \leqslant k + \frac{\varepsilon}{2},$$

and then choose $s_0 \in \mathbb{N}$ so large that

$$\|T^{n_s}x_m - x_{m+1}\| < r_m + \varepsilon < (k+\varepsilon)(r_m+\varepsilon)$$

and

$$\begin{aligned}\|T^{n_s}x_m - T^{n_p}x_{m+1}\| &\leqslant \|T^{n_s}x_m - T^{n_s+n_p}x_m\| + \|T^{n_s+n_p}x_m - T^{n_p}x_{m+1}\| \\ &\leqslant \sum_{j=0}^{n_p-1} \|T^{n_s+j+1}x_m - T^{n_s+j}x_m\| + \|T^{n_p}\| \cdot \|T^{n_s}x_m - x_{m+1}\| \\ &\leqslant \frac{\varepsilon}{2} \cdot (r_m+\varepsilon) + \|T^{n_p}\| \cdot (r_m+\varepsilon) \leqslant (k+\varepsilon)(r_m+\varepsilon)\end{aligned}$$

for $s \geqslant s_0$ (by the asymptotic regularity of T). It follows from the properties of δ_E, that

$$\begin{aligned}&\Big\|T^{n_s}x_m - \frac{1}{2}(x_{m+1} + T^{n_p}x_{m+1})\Big\| \\ &\leqslant \Big(1 - \delta_E\Big(\frac{d(x_{m+1}) - \varepsilon}{(k+\varepsilon)(r_m+\varepsilon)}\Big)\Big)(k+\varepsilon)(r_m+\varepsilon)\end{aligned}$$

for $s \geqslant s_0$. Hence

$$r_m \leqslant \Big(1 - \delta_E\Big(\frac{d(x_{m+1}) - \varepsilon}{(k+\varepsilon)(r_m+\varepsilon)}\Big)\Big)(k+\varepsilon)(r_m+\varepsilon).$$

Taking the limit as $\varepsilon \downarrow 0$ we obtain by continuity of δ_E,

$$r_m \leqslant \Big(1 - \delta_E\Big(\frac{d(x_{m+1})}{k \cdot r_m}\Big)\Big) \cdot k \cdot r_m$$

and

$$\Big(1 - \delta_E\Big(\frac{d(x_{m+1})}{k \cdot r_m}\Big)\Big) \cdot k \geqslant 1.$$

This implies

$$\begin{aligned}d(x_{m+1}) &\leqslant k \cdot \delta_E^{-1}\Big(1 - \frac{1}{k}\Big) \cdot r_m \\ &\leqslant k \cdot \delta_E^{-1}\Big(1 - \frac{1}{k}\Big) \cdot d(x_m) = B \cdot d(x_m),\end{aligned}$$

where $B = k \cdot \delta_E^{-1}\Big(1 - \frac{1}{k}\Big) < 1$. From (1) and $k < \gamma$, we have $0 \leqslant B < 1$. Hence

$$d(x_m) \leqslant B \cdot d(x_{m-1}) \leqslant \ldots \leqslant B^m \cdot d(x_0). \tag{3}$$

Noticing

$$\|x_{m+1} - x_m\| \leqslant \|x_{m+1} - T^{n_i}x_m\| + \|T^{n_i}x_m - x_m\|$$

and taking the limit superior as $i \to \infty$ on both sides, we get

$$\begin{aligned}\|x_{m+1} - x_m\| &\leqslant r_m + d(x_m) \leqslant 2d(x_m) \\ &\leqslant 2 \cdot B^m \cdot d(x_0)\end{aligned} \tag{4}$$

and hence, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+m} - x_m\| &\leqslant \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \ldots + \|x_{m+1} - x_m\| \\ &\leqslant 2B^{n+m-1}d(x_0) + 2B^{n+m-2}d(x_0) + \ldots + 2B^m d(x_0) + \ldots \\ &= \frac{2B^m d(x_0)}{1 - B}.\end{aligned}$$

So, we see that $\{x_m\}$ is norm Cauchy and hence strong convergent. Let $z = \lim_{m\to\infty} x_m$. Then one can easily see that

$$\|z - T^{n_i}z\| \leqslant \|z - x_m\| + \|x_m - T^{n_i}x_m\| + \|T^{n_i}x_m - T^{n_i}z\|$$
$$\leqslant (1 + \|T^{n_i}\|)\|z - x_m\| + \|x_m - T^{n_i}x_m\|.$$

Taking the limit superior as $i \to \infty$ on each side, we get

$$d(z) = \limsup_{i\to\infty} \|z - T^{n_i}z\| \leqslant (1 + k)\|z - x_m\| + d(x_m)$$
$$\leqslant (1 + k)\|z - x_m\| + B^m \cdot d(x_0) \to 0 \text{ as } m \to \infty.$$

Therefore $d(z) = 0$ and this implies $Tz = z$.

Retraction. Let $A : C \to C$ denote a mapping which associates with a given $x \in C$ a unique $\xi \in A(C, \{T^{n_i}x\}_{i\geqslant 1})$, that is, $\xi = Ax$. Sędłak and Wiśnicki [18] proved the following Lemma which is very useful in our proof.

Lemma 6. *Let E be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Then the mapping $A : C \to C$ is continuous.*

Note that $x_m = A^m x$ for $m = 1, 2, \ldots$ and by (4),

$$\|A^{m+1}x - A^m x\| \leqslant 2 \cdot B^m \cdot d(x) \leqslant 2 \cdot B^m \cdot diamC$$

for $m = 0, 1, 2, \ldots$ Thus

$$\sup_{x\in C} \|A^m x - A^i x\| \leqslant \frac{2 \cdot B^m}{1 - B} diamC \to 0 \text{ if } m, i \to \infty$$

which implies that the sequence $\{A^m x\}$ converges uniformly to a function

$$Rx = \lim_{m\to\infty} A^m x,\ x \in C.$$

It follows from Lemma 6, that $R : C \to C$ is continuous. Moreover,

$$\|Rx - T^{n_i}Rx\| \leqslant \|Rx - A^m x\| + \|A^m x - T^{n_i}A^m x\| + \|T^{n_i}A^m x - T^{n_i}Rx\|$$
$$\leqslant (1 + \|T^{n_i}\|)\|Rx - A^m x\| + \|A^m x - T^{n_i}A^m x\|.$$

Taking the limit superior as $i \to \infty$ on each side, we get

$$d(Rx) = \limsup_{i\to\infty} \|Rx - T^{n_i}Rx\| \leqslant (1 + k)\|Rx - A^m x\| + d(A^m x)$$
$$\leqslant (1 + k)\|Rx - A^m x\| + B^m \cdot d(x) \to 0 \text{ as } m \to \infty.$$

Thus $d(Rx) = 0$ and as earlier, $Rx = TRx$ for every $x \in C$, and R is a retraction of C onto $Fix\ T$. ■

3. $p-$UNIFORMLY CONVEX BANACH SPACES

In $p-$uniformly convex Banach spaces we can establish a more general result. Let $p > 1$ be a real number. A Banach space E is said to be $p-$*uniformly convex* (or E is said to have the modulus of convexity of power type p) if there exists a constant $d > 0$ such that the modulus of convexity $\delta_E(\varepsilon) \geqslant d \cdot \varepsilon^p$ for $0 \leqslant \varepsilon \leqslant 2$. We note that a Hilbert space is $2-$uniformly convex (indeed, $\delta_H(\varepsilon) = 1 - \sqrt{1 - (\frac{\varepsilon}{2})^2} \geqslant \frac{1}{8}\varepsilon^2$) and an L^p-space ($1 < p < \infty$) is $\max\{p, 2\}-$uniformly convex.

In [14, 22] the following inequalities was proved.

Lemma 7.

(a) *Let $p > 1$ be a real number and let E be a $p-$uniformly convex Banach space. Then there exists a constant $c_p > 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leqslant \lambda\|x\|^p + (1-\lambda)\|y\|^p - c_p \cdot W_p(\lambda) \cdot \|x-y\|^p$$

for all $x, y \in E$, $0 \leqslant \lambda \leqslant 1$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

(b) *If H is a Hilbert space, then*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

When E is particularly an L^p-space, we have the following

Lemma 8. *Suppose E is an L^p-space.*

(a) *If $1 < p \leqslant 2$, then*

$$\|\lambda x + (1-\lambda)y\|^2 \leqslant \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - (p-1) \cdot \lambda \cdot (1-\lambda) \cdot \|x-y\|^2$$

for all $x, y \in E$ and $0 \leqslant \lambda \leqslant 1$ ($c_p = p-1$);

(b) *If $2 < p < \infty$, then*

$$\|\lambda x + (1-\lambda)y\|^p \leqslant \lambda\|x\|^p + (1-\lambda)\|y\|^p - c_p \cdot W_p(\lambda) \cdot \|x-y\|^p$$

for all $x, y \in E$, $0 \leqslant \lambda \leqslant 1$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$ and

$$c_p = \frac{1 + t_p^{p-1}}{(1+t_p)^{p-1}} = (p-1)(1+t_p)^{2-p}$$

with t_p being the unique solution of the equation

$$(p-2)t^{p-1} + (p-1)t^{p-2} - 1 = 0, \ 0 < t < 1.$$

All constants appeared in the above inequalities are best possible.

The following theorem is an extension of Theorem 3 from [11].

Theorem 9. *Let $p > 1$ be a real number and let E be a $p-$uniformly convex Banach space, C a nonempty bounded closed convex subset of E. If $T : C \to C$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} \|T^n\| < (1+c_p)^{\frac{1}{p}},$$

then T has a fixed point in C and $Fix\ T$ is a retract of C.

Proof. Let $\{n_i\}$ be a sequence of natural numbers such that

$$\liminf_{n\to\infty} \|T^n\| = \lim_{i\to\infty} \|T^{n_i}\| = k < (1+c_p)^{\frac{1}{p}}.$$

We may assume that $k \geqslant 1$ since if $k < 1$, the well known Banach Contraction Principle guarantees a fixed point of T.

For an $x = x_0 \in C$ we can inductively define a sequence $\{x_m\} \subset C$ in the following manner: x_{m+1} is the asymptotic center of the sequence $\{T^{n_i} x_m\}_{i\geqslant 1}$, that is, x_{m+1} is the unique point in C that minimizes the functional

$$\limsup_{i\to\infty} \|y - T^{n_i} x_m\|$$

over y in C. For each $m \geqslant 0$ we set

$$r_{m+1} = \limsup_{i\to\infty} \|x_{m+1} - T^{n_i} x_m\|,$$
$$d(x_m) = \limsup_{i\to\infty} \|x_m - T^{n_i} x_m\|.$$

Obviously,

$$r_{m+1} \leqslant d(x_m) \text{ for all } m \geqslant 0.$$

For $0 < \lambda < 1$, $n_i, n_j \in \mathbb{N}$, from Lemma 7 (a), we have

$$\begin{aligned}
&\|(1-\lambda)x_{m+1} + \lambda \cdot T^{n_j} x_{m+1} \\
&-T^{n_i} x_m\|^p + c_p \cdot W_p(\lambda) \cdot \|x_{m+1} - T^{n_j} x_{m+1}\|^p \\
\leqslant\ & (1-\lambda)\|x_{m+1} - T^{n_i} x_m\|^p + \lambda \|T^{n_j} x_{m+1} - T^{n_i} x_m\|^p \\
\leqslant\ & (1-\lambda)\|x_{m+1} - T^{n_i} x_m\|^p \\
&+\lambda\Big(\|T^{n_j} x_{m+1} - T^{n_j+n_i} x_m\| + \|T^{n_j+n_i} x_m - T^{n_i} x_m\|\Big)^p \\
\leqslant\ & (1-\lambda)\|x_{m+1} - T^{n_i} x_m\|^p \\
&+\lambda\Big(\|T^{n_j}\| \cdot \|x_{m+1} - T^{n_i} x_m\| + \sum_{v=0}^{n_j-1} \|T^{n_i+v+1} x_m - T^{n_i+v} x_m\|\Big)^p.
\end{aligned}$$

Taking the limit superior as $i \to \infty$ on both sides, by the asymptotic regularity of T, we get that for each n_j,

$$\begin{aligned}
&\limsup_{i\to\infty} \|x_{m+1} - T^{n_i} x_m\|^p + c_p \cdot W_p(\lambda) \cdot \|x_{m+1} - T^{n_j} x_{m+1}\|^p \\
&\leqslant \limsup_{i\to\infty} \|(1-\lambda)x_{m+1} + \lambda \cdot T^{n_j} x_{m+1} - T^{n_i} x_m\|^p \\
&\quad + c_p \cdot W_p(\lambda) \cdot \|x_{m+1} - T^{n_j} x_{m+1}\|^p \\
&\leqslant (1-\lambda) \cdot \limsup_{i\to\infty} \|x_{m+1} - T^{n_i} x_m\|^p + \lambda \cdot \|T^{n_j}\|^p \cdot \limsup_{i\to\infty} \|x_{m+1} - T^{n_i} x_m\|^p
\end{aligned}$$

and

$$\begin{aligned}
&r_{m+1}^p + c_p \cdot W_p(\lambda) \cdot \|x_{m+1} - T^{n_j} x_{m+1}\|^p \\
&\leqslant (1-\lambda) \cdot r_{m+1}^p + \lambda \cdot \|T^{n_j}\|^p \cdot r_{m+1}^p.
\end{aligned}$$

So, we have

$$\|x_{m+1} - T^{n_j} x_{m+1}\|^p \leqslant \frac{\lambda(\|T^{n_j}\|^p - 1)}{c_p \cdot W_p(\lambda)} \cdot r_{m+1}^p,$$

and hence

$$\|x_{m+1} - T^{n_j} x_{m+1}\|^p \leqslant \frac{\|T^{n_j}\|^p - 1}{c_p \cdot [\lambda^{p-1} \cdot (1-\lambda) + (1-\lambda)^p]} \cdot r_{m+1}^p.$$

Letting $\lambda \downarrow 0$ and next taking the limit superior as $j \to \infty$ on both sides, we get

$$[d(x_{m+1})]^p \leqslant \frac{k^p - 1}{c_p} \cdot r_{m+1}^p \leqslant \frac{k^p - 1}{c_p} \cdot [d(x_m)]^p,$$

where $B = \frac{k^p-1}{c_p} < 1$ by assumption of Theorem. Hence

$$d(x_m) \leqslant D \cdot d(x_{m-1}) \leqslant \ldots \leqslant D^m \cdot d(x_0),$$

where $D = B^{\frac{1}{p}} < 1$. From this inequalities, noticing,

$$\|x_{m+1} - x_m\| \leqslant \|x_{m+1} - T^{n_i} x_m\| + \|T^{n_i} x_m - x_m\|$$

and taking the limit superior as $i \to \infty$ on both sides,

$$\|x_{m+1} - x_m\| \leqslant r_{m+1} + d(x_m) \leqslant 2 \cdot d(x_m) \leqslant \ldots \leqslant 2 \cdot D^m \cdot d(x_0).$$

So, we see that $\{x_m\}$ is a Cauchy sequence and its limit is a fixed point of T. A similar argument as in the corresponding part the proof of Theorem 5 concludes that $Fix\ T$ is a retract of C. ■

Remark 10. From Theorem 9, by Lemma 7 (b) we immediately obtain Theorem 4. Now, let $E = L^p$, $2 < p < \infty$, and $\gamma_p = (1 + 2^{-p})^{\frac{1}{p}}$ be the solution of the equation (1). Because an easy computation shows the bounds

$$2^{2-p} < c_p < (p-1)2^{2-p} \text{ for all } 2 < p < \infty,$$

[19], so in L^p-spaces $\gamma_p < (1 + c_p)^{\frac{1}{p}}$ for $2 < p < \infty$. For example, $(1 + c_3)^{\frac{1}{3}} = (3 - \sqrt{2})^{\frac{1}{3}}$, $(1 + c_4)^{\frac{1}{4}} = \left(\frac{4}{3}\right)^{\frac{1}{4}}$. Thus in L^p-spaces $(2 < p < +\infty)$ Theorem 9 is an extension of Theorem 5. Moreover from Theorem 9, by Lemma 8 (a), we have the following result:

Corollary 11. *Let C be a nonempty bounded closed convex subset of L^p, $1 < p \leqslant 2$. If $T : C \to C$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} \|T^n\| < \sqrt{p},$$

then T has a fixed point in C and $Fix\ T$ is a retract of C.

Remark 12. Corollary 11 improve Corollary 1 presented in the paper [11]. By the way we note that the proof of Theorem 2 in [11] is unfortunately not correct because the inequality

$$diam_a(\{x_n\}) \leqslant D(\{x_n\}) = \limsup_{n\to\infty}(\limsup_{m\to\infty} \|x_n - x_m\|)$$

is false (H.K. Xu, personal communication, 1995). Indeed, if $E = l^2$ and $y_n = e_n - e_{n+1}$, $n = 1, 2, \ldots$, where $\{e_n\}$ denotes the standard basis in l^2, then $diam_a(\{y_n\}) = \sqrt{6} > 2 = D(\{y_n\})$. Actually, it is easy to see that for each bounded sequence $\{x_n\}$ in a Banach space E, $r_a(\{x_n\}) \leqslant D(\{x_n\}) \leqslant diam_a(\{x_n\})$.

Using the result of Prus and Smarzewski [17], [20] and Xu [22] we can obtain from Theorem 9 some additional corollaries, for example, for Hardy and Sobolev spaces.

Let H^p, $1 < p < \infty$, denote the Hardy space [9] of all analytic functions x in the unit disc $|z| < 1$ of the complex plane such that

$$\|x\| = \lim_{r\to 1_-} \left(\frac{1}{2\pi}\int_0^{2\pi} |x(re^{i\Theta}|^p d\Theta\right)^{\frac{1}{p}} < +\infty.$$

Now, let Ω be an open subset of $\mathbb{R}^n$. Denote by $W^{r,p}(\Omega)$, $r \geqslant 0$, $1 < p < \infty$, the Sobolev space [2] of distributions x such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \leqslant k$ equipped with the norm

$$\|x\| = \Big(\sum_{|\alpha|\leqslant k}\int_\Omega |D^\alpha x(\omega)|^p d\omega\Big)^{\frac{1}{p}}.$$

Let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, be a sequence of positive measure spaces, where the index Λ is finite or countable. Given a sequence of linear subspaces X_α in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < \infty$ and $q = \max\{2, p\}$, [16], the linear space of all sequences $x = \{x_\alpha \in X_\alpha : \alpha \in \Lambda\}$ equipped with the norm

$$\|x\| = \Big(\sum_{\alpha \in \Lambda} \left(\|x_\alpha\|_{p,\alpha} \right)^q \Big)^{\frac{1}{q}},$$

where $\|\cdot\|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$.

Finally, let $L^p = L^p(S_1, \Sigma_1, \mu_1)$ and $L^q = L^q(S_2, \Sigma_2, \mu_2)$, where $1 < p < \infty$, $q = \max\{2, p\}$, and (S_i, Σ_i, μ_i) are positive measure spaces. Denote by $L_q(L_p)$ the Banach space [8] of all measurable L^p-value functions x on S_2 such that

$$\|x\| = \Big(\int_{S_2} \left(\|x(s)\|_p \right)^q \mu_2(ds) \Big)^{\frac{1}{q}}.$$

All these spaces are $q-$uniform convex with $q = \max\{2, p\}$, [17], [20], and the norm in these spaces satisfies

$$\|\lambda x + (1-\lambda) y\|^q \leqslant \lambda \|x\|^q + (1-\lambda)\|y\|^q - d \cdot W_q(\lambda) \cdot \|x - y\|^q$$

with a constant

$$d = d_p = \begin{cases} \dfrac{p-1}{8} & \text{if } 1 < p \leqslant 2, \\ \dfrac{1}{p \cdot 2^p} & \text{if } 2 < p < \infty. \end{cases}$$

Hence the following result follows from Theorem 9:

Corollary 13. *Let C be a nonempty bounded closed convex subset of the space E, where $E = H^p$ or $E = W^{r,p}(\Omega)$ or $E = L_{q,p}$ or $E = L_q(L_p)$ and $1 < p < \infty$, $q = \max\{2, p\}$, $r \geqslant 0$. If $T : C \to C$ is an asymptotically regular mapping such that*

$$\liminf_{n \to \infty} \|T^n\| < (1 + d)^{\frac{1}{q}},$$

then T has a fixed point in C and $Fix\ T$ is a retract of C.

Remark 14. Note that Theorem 3 was significantly generalized by Dom´nguez Benavides, Japón Pineda and Xu, (see [1, 6]), but it is not very clear whether our statements are also valid in these cases.

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