TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 3, pp. 927-959, June 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

A GENERAL SYSTEM OF GENERALIZED NONLINEAR MIXED COMPOSITE-TYPE EQUILIBRIA IN HILBERT SPACES

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Abstract. Very recently, Ceng and Yao [L. C. Ceng, J. C. Yao, A relaxed extragradient-like method for a generalized mixed equilibrium problem, a general system of generalized equilibria and a fixed point problem, *Nonlinear Anal.*, **72** (2009), 1922-1937, suggested and analyzed a relaxed extragradient-like method for finding a common solution of a generalized mixed equilibrium problem, a general system of generalized equilibria and a fixed point problem of a strict pseudocontractive mapping in a Hilbert space. In this paper, based on the authors' iterative method, we introduce a modification of the relaxed extragradient-like method for finding a common solution of a generalized mixed equilibrium problem with perturbed mapping, a general system of generalized nonlinear mixed composite-type equilibria and a fixed point problem of a strict pseudocontractive mapping in a Hilbert space, and then obtain a strong convergence theorem. Utilizing this theorem, we establish some new strong convergence results in fixed point problems, variational inequalities, mixed equilibrium problems and systems of generalized nonlinear mixed composite-type equilibria in Hilbert spaces.

1. INTRODUCTION

It is well known that the equilibrium problem includes, as special cases, variational inequalities, optimization problems, minimax problems, Nash equilibrium

Received April 20, 2009, accepted November 20, 2009.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: 49J30, 49J40, 47J25, 47H09.
Key words and phrases: Strictly pseudocontractive mapping, Fixed point, Inverse-strongly-monotone mapping, Variational inequality, Generalized mixed equilibrium problem with perturbed mapping, System of generalized nonlinear mixed composite-type equilibria.
[1]This research was partially supported by the National Science Foundation of China (10771141).
[2]This research was partially supported by the Leading Academic Discipline Project of Shanghai Normal University (DZL707), Innovation Program of Shanghai Municipal Education Commission Grant (09ZZ133), National Science Foundation of China (10771141), Ph.D. Program Foundation of Ministry of Education of China (20070270004), Science and Technology Commission of Shanghai Municipality Grant (075105118), and Shanghai Leading Academic Discipline Project (S30405).
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problems in noncooperative games, saddle point problems, fixed point problems and complementarity problems. Up to now it has been widely studied by many authors; see, for example, [3, 4, 16-21] and the references therein.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $S : C \to H$ be a mapping on C. We denote by $F(S)$ the set of fixed points of S and by P_C the metric projection of H onto C. Moreover, we also denote by **R** the set of all real numbers. Consider the following generalized mixed equilibrium problem with perturbed mapping, which consists of finding $\bar{x} \in C$ such that

$$(1.1a) \qquad \Theta(\bar{x}, y) + \varphi(y) - \varphi(\bar{x}) + \langle (F + T)\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C,$$

where $F : C \to H$ is a nonlinear mapping, $T : C \to H$ is a perturbed mapping, $\varphi : C \to \mathbf{R}$ is a function and $\Theta : C \times C \to \mathbf{R}$ is a bifunction. We denote by $GMEP$ the set of solutions of problem (1.1a). Here some special cases of problem (1.1a) are stated as follows:

If $T = 0$, then problem (1.1a) reduces to the following generalized mixed equilibrium problem of finding $\bar{x} \in C$ such that

$$(1.1b) \qquad \Theta(\bar{x}, y) + \varphi(y) - \varphi(\bar{x}) + \langle F\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C,$$

which was recently introduced and studied by Peng and Yao [1]. The set of solutions of problem (1.1b) is denoted by $GMEP(\Theta, \varphi, F)$. Subsequently, Yao, Liou and Yao [2] also considered this problem.

If $F = 0$, then problem (1.1b) reduces to the following mixed equilibrium problem of finding $\bar{x} \in C$ such that

$$\Theta(\bar{x}, y) + \varphi(y) - \varphi(\bar{x}) \geq 0, \quad \forall y \in C,$$

which was considered by Ceng and Yao [3]. The set of solutions of this problem is denoted by *MEP*.

If $\varphi = 0$, then problem (1.1b) reduces to the following generalized equilibrium problem of finding $\bar{x} \in C$ such that

$$(1.2) \qquad \Theta(\bar{x}, y) + \langle F\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C,$$

which was studied by Takahashi and Takahashi [4].

If $\Theta = 0$, $\varphi = 0$ and $F = A$, then problem (1.1b) reduces to the following classical variational inequality problem of finding $\bar{x} \in C$ such that

$$(1.3) \qquad \langle A\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C.$$

The set of solutions of problem (1.3) is denoted by *VI(A,C)*. The variational inequality problem has been extensively studied in the literature; see [5-15] and the references therein. Recently, in order to solve problem (1.1b), Peng and Yao [1] developed a CQ method. They established some strong convergence results for finding a common element of the set of solutions of problem (1.1b), the set of solutions of problem (1.3), and the set of fixed points of a nonexpansive mapping.

On the other hand, let C be a nonempty closed convex subset of a real Hilbert space H. Let $G_1, G_2 : C \times C \to \mathbf{R}$ be two bifunctions, $B_1, B_2, T_1, T_2 : C \to H$ be four nonlinear mappings and $\psi_1, \psi_2 : C \to \mathbf{R}$ be two functions. Consider the following problem of finding $(\bar{x}, \bar{y}) \in C \times C$ such that

$$(1.4)\quad \begin{cases} \mu_1 G_1(\bar{x}, x) + \langle \mu_1(B_1+T_1)\bar{y} + \bar{x} - \bar{y}, x - \bar{x}\rangle \geq \mu_1\psi_1(\bar{x}) - \mu_1\psi_1(x), \ \forall x \in C, \\ \mu_2 G_2(\bar{y}, y) + \langle \mu_2(B_2+T_2)\bar{x} + \bar{y} - \bar{x}, y - \bar{y}\rangle \geq \mu_2\psi_2(\bar{y}) - \mu_2\psi_2(y), \ \forall y \in C, \end{cases}$$

which is called a general system of generalized nonlinear mixed composite-type equilibria where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants. We denote by $\mho$ the set of solutions of problem (1.4).

Next we present some special cases of problem (1.4) as follows:

If $G_1 = G_2 = \Theta$, $B_1 = B_2 = A$, $T_1 = T_2 = T$ and $\psi_1 = \psi_2 = \varphi$, then problem (1.4) reduces to the following problem of finding $(\bar{x}, \bar{y}) \in C \times C$ such that

$$(1.5)\quad \begin{cases} \mu_1 \Theta(\bar{x}, x) + \langle \mu_1(A+T)\bar{y} + \bar{x} - \bar{y}, x - \bar{x}\rangle \geq \mu_1\varphi(\bar{x}) - \mu_1\varphi(x), \ \forall x \in C, \\ \mu_2 \Theta(\bar{y}, y) + \langle \mu_2(A+T)\bar{x} + \bar{y} - \bar{x}, y - \bar{y}\rangle \geq \mu_2\varphi(\bar{y}) - \mu_2\varphi(y), \ \forall y \in C, \end{cases}$$

which is called a new system of generalized nonlinear mixed composite-type equilibria where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants.

If $C = H$, $G_1 = G_2 = 0$ and $\psi_1 = \psi_2 = \varphi$, then problem (1.4) reduces to the following new system of generalized nonlinear mixed variational inequalities: Find $(\bar{x}, \bar{y}) \in H \times H$ such that

$$(1.6)\quad \begin{cases} \langle \mu_1(B_1 + T_1)\bar{y} + \bar{x} - \bar{y}, x - \bar{x}\rangle \geq \mu_1\varphi(\bar{x}) - \mu_1\varphi(x), \quad \forall x \in H, \\ \langle \mu_2(B_2 + T_2)\bar{x} + \bar{y} - \bar{x}, y - \bar{y}\rangle \geq \mu_2\varphi(\bar{y}) - \mu_2\varphi(y), \quad \forall y \in H, \end{cases}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants, which is introduced and considered by Kim and Kim [29].

If $T_1 = T_2 = 0$ and $\psi_1 = \psi_2 = 0$, then problem (1.4) reduces to the following general system of generalized equilibria: Find $(\bar{x}, \bar{y}) \in C \times C$ such that

$$(1.7)\quad \begin{cases} G_1(\bar{x}, x) + \langle B_1\bar{y}, x - \bar{x}\rangle + \frac{1}{\mu_1}\langle \bar{x} - \bar{y}, x - \bar{x}\rangle \geq 0, \quad \forall x \in C, \\ G_2(\bar{y}, y) + \langle B_2\bar{x}, y - \bar{y}\rangle + \frac{1}{\mu_2}\langle \bar{y} - \bar{x}, y - \bar{y}\rangle \geq 0, \quad \forall y \in C, \end{cases}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants, which is introduced and studied by Ceng and Yao [30]. We denote by Ω the set of solutions of problem (1.7).

If $T = 0$ and $\varphi = 0$ in problem (1.5), then problem (1.5) reduces to the following new system of generalized equilibria: Find $(\bar{x}, \bar{y}) \in C \times C$ such that

$$(1.8) \qquad \begin{cases} \Theta(\bar{x}, x) + \langle A\bar{y}, x - \bar{x}\rangle + \frac{1}{\mu_1}\bar{x} - \bar{y}, x - \bar{x}\rangle \geq 0, \quad \forall x \in C, \\ \Theta(\bar{y}, y) + \langle A\bar{x}, y - \bar{y}\rangle + \frac{1}{\mu_2}\langle \bar{y} - \bar{x}, y - \bar{y}\rangle \geq 0, \quad \forall y \in C, \end{cases}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants, which is introduced and considered by Ceng and Yao [30].

If $G_1 = G_2 = \Theta$, $B_1 = B_2 = F$, $T_1 = T_2 = T$, $\psi_1 = \psi_2 = \varphi$ and $\bar{x} = \bar{y}$, then problem (1.4) reduces to problem (1.1a).

If $G_1 = G_2 = 0$, then problem (1.7) reduces to the following general system of variational inequalities: Find $(\bar{x}, \bar{y}) \in C \times C$ such that

$$(1.9) \qquad \begin{cases} \langle \mu_1 B_1\bar{y} + \bar{x} - \bar{y}, x - \bar{x}\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 B_2\bar{x} + \bar{y} - \bar{x}, y - \bar{y}\rangle \geq 0, \quad \forall y \in C, \end{cases}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants, which is introduced and studied by Ceng, Wang and Yao [22].

If $B_1 = B_2 = A$, then problem (1.9) reduces to the following new system of variational inequalities: Find $(\bar{x}, \bar{y}) \in C \times C$ such that

$$(1.10) \qquad \begin{cases} \langle \mu_1 A\bar{y} + \bar{x} - \bar{y}, x - \bar{x}\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 A\bar{x} + \bar{y} - \bar{x}, y - \bar{y}\rangle \geq 0, \quad \forall y \in C, \end{cases}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants, which is defined and studied by Verma [23] (see also [24]).

If $\bar{x} = \bar{y}$, then problem (1.10) reduces to the classical variational inequality (1.3).

We remark that Zeng and Yao introduced a system of variational inequalities in [25] similar to but different from (1.9). Recently, Ceng, Wang and Yao [22] introduced and studied a relaxed extragradient method for finding solutions of problem (1.9). It is clear that the authors' results unifies and extends many results in the literature. Later on, Yao, Liou and Yao [2] proposed a new iterative method based on the relaxed hybrid method and the extragradient method for finding a common element of the set of solutions of problem (1.1b), the set of fixed points of a strictly pseudocontractive mapping and the set of solutions of problem (1.9).

Very recently, Ceng and Yao [30] introduced and considered a relaxed extragradient-like method for finding a common element of the set of solutions of problem (1.1b), the set of fixed points of a strictly pseudocontractive mapping and the set of solutions of problem (1.7). The authors' results [30] include, as special cases, the corresponding ones of Takahashi and Takahashi [4], Ceng, Wang and Yao [22], Peng and Yao [1], and Yao, Liou and Yao [2].

Theorem CY. (see [30, Theorem 3.1]). Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ, $G_1, G_2 : C \times C \to \mathbf{R}$ be three bifunctions satisfying conditions (H1)-(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with assumptions (A1) or (A2), where

$(H1)$ $\Theta(x,x) = 0,\ \forall x \in C$;

$(H2)$ Θ is monotone, i.e., $\Theta(x,y) + \Theta(y,x) \le 0,\ \forall x, y \in C$;

$(H3)$ for each $y \in C,\ x \mapsto \Theta(x,y)$ is weakly upper semicontinuous;

$(H4)$ for each $x \in C,\ y \mapsto \Theta(x,y)$ is convex and lower semicontinuous;

$(A1)$ for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

$(A2)$ C is a bounded set.

Let the mappings $F, B_1, B_2 : C \to H$ be α-inverse-strongly monotone, $\tilde{\beta}_1$-inverse-strongly monotone and $\tilde{\beta}_2$-inverse-strongly monotone, respectively. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\Xi := F(S) \cap \Omega \cap GMEP(\Theta, \varphi, F) \neq \emptyset$. For fixed $u \in C$ and $x_0 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by

$$(1.11)\quad \begin{cases} \Theta(z_n, z) + \varphi(z) - \varphi(z_n) \\ \quad +\langle Fx_n, z - z_n\rangle + \frac{1}{\lambda_n}\langle z - z_n, z_n - x_n\rangle \ge 0,\ \forall z \in C, \\ G_2(u_n, u) + \langle B_2 z_n, u - u_n\rangle + \frac{1}{\mu_2}\langle u - u_n, u_n - z_n\rangle \ge 0,\ \forall u \in C, \\ G_1(y_n, y) + \langle B_1 u_n, y - y_n\rangle + \frac{1}{\mu_1}\langle y - y_n, y_n - u_n\rangle \ge 0,\ \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \ge 0, \end{cases}$$

where $\mu_1 \in (0, 2\tilde{\beta}_1),\ \mu_2 \in (0, 2\tilde{\beta}_2)$, and $\{\lambda_n\} \subset [0, 2\alpha],\ \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and $(\gamma_n + \delta_n)k \le \gamma_n$ for all $n \ge 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ and $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$;

(v) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\alpha$ and $\liminf_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Then, $\{x_n\}$ converges strongly to $\bar{x} = P_{\Xi}u$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.7), where

$$G_2(\bar{y}, y) + \langle B_2\bar{x}, y - \bar{y}\rangle + \frac{1}{\mu_2}\langle y - \bar{y}, \bar{y} - \bar{x}\rangle \geq 0, \quad \forall y \in C.$$

Throughout this paper, suppose that S is a k-strictly pseudocontractive self-mapping on a nonempty closed convex subset C of a real Hilbert space H. Inspired by Takahashi and Takahashi [4], Ceng, Wang and Yao [22], Peng and Yao [1], Yao, Liou and Yao [2], Kim and Kim [30], Ceng and Yao [29], we introduce a new relaxed extragradient-like algorithm for finding a common solution of problem (1.1a), problem (1.4) and the fixed point problem of S,

$$\begin{cases} z_n &= T_{\lambda_n}^{(\Theta,\varphi)}(x_n - \lambda_n(F+T)x_n), \\ y_n &= T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n) \\ & \qquad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n)], \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $\Theta, G_1, G_2 : C \times C \to \mathbf{R}$ satisfy conditions (H1)-(H4), $\varphi, \psi_1, \psi_2 : C \to \mathbf{R}$ are three lower semicontinuous and convex functions with assumption (A1) or (A2), $F, B_1, B_2 : C \to H$ are α-inverse-strongly monotone, $\tilde{\beta}_1$-inverse-strongly monotone and $\tilde{\beta}_2$-inverse-strongly monotone, respectively, and $T, T_1, T_2 : C \to H$ are η-Lipschitz continuous, η_1-Lipschitz continuous and η_2-Lipschitz continuous, respectively, and then derive a strong convergence result. Utilizing this theorem, we establish some new strong convergence theorems in several aspects:

(1) problem (1.1a), problem (1.4) and the fixed point problem of nonexpansive mapping S;

(2) the mixed equilibrium problem, problem (1.4) and the fixed point problem of k-strictly pseudocontractive mapping S;

(3) problem (1.3), problem (1.4) and the fixed point problem of k-strictly pseudocontractive mapping S;

(4) problem (1.1a), problem (1.4) and the fixed point problem of k-strictly pseudocontractive mapping S, where $F = T = (I - A)/2$ and A is $\tilde{\kappa}$-strictly pseudocontractive mapping on C;

(5) different conditions imposed on the iterative parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\lambda_n\}$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. We denote by $\omega_w(\{x_n\})$ the weak ω-limit set of $\{x_n\}$. For every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C, i.e.,

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

It is also known that, $P_C x$ is characterized by the following property:

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \quad \forall x \in H \text{ and } y \in C.$$

In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

A mapping $S : C \to C$ is called a strictly pseudocontractive if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.2}$$

In this case, we say that S is a k-strict pseudocontraction. A mapping $A : C \to H$ is called α-inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that any inverse-strongly monotone mapping is Lipschitz continuous. Meantime, observe that (2.2) is equivalent to

$$\langle Sx - Sy, x - y \rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.3}$$

From [26], we know that if S is a k-strict pseudocontractive mapping, then S is Lipschitz continuous with constant $\frac{1+k}{1-k}$, i.e., $\|Sx - Sy\| \leq \frac{1+k}{1-k}\|x - y\|$ for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. It is clear that the class of strict pseudocontractions strictly includes the one of nonexpansive mappings which are mappings $S : C \to C$ such that $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$.

In order to prove our main results in the next section, we need the following lemmas and propositions.

Lemma 2.1. (see [3]). *Let C be a nonempty closed convex subset of H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (H1)-(H4) and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$T_r^{(\Theta,\varphi)}(x) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in C\}$$

for all $x \in H$. Assume that either (A1) or (A2) holds. Then the following statements hold:

(*i*) $T_r^{(\Theta,\varphi)}(x) \neq \emptyset$ *for each* $x \in H$ *and* $T_r^{(\Theta,\varphi)}$ *is single-valued;*

(*ii*) $T_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\|^2 \leq \langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x-y\rangle;$$

(*iii*) $F(T_r^{(\Theta,\varphi)}) = MEP(\Theta,\varphi)$*;*

(*iv*) $MEP(\Theta,\varphi)$ *is closed and convex.*

Remark 2.1. If $\varphi = 0$, then $T_r^{(\Theta,\varphi)}$ is rewritten as T_r^{Θ}.

Lemma 2.2. *Let* C *be a nonempty closed convex subset of* H. *Let* $G_1, G_2 : C\times C \to \mathbf{R}$ *be two bifunctions satisfying conditions (H1)-(H4) and let the mappings* $B_1, B_2 : C \to H$ *be* $\tilde{\beta}_1$*-inverse-strongly monotone and* $\tilde{\beta}_2$*-inverse-strongly monotone, respectively, and* $T_1, T_2 : C \to H$ *be* η_1*-Lipschitz continuous and* η_2*-Lipschitz continuous, respectively. Let* $\mu_1 \in (0, 2\tilde{\beta}_1)$ *and* $\mu_2 \in (0, 2\tilde{\beta}_2)$*, respectively. Let* $\psi_1, \psi_2 : C \to \mathbf{R}$ *be two lower semicontinuous and convex functions with assumption* (*A1*) *or* (*A2*). *Then, for given* $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ *is a solution of problem* (*1.4*) *if and only if* $\bar{x}$ *is a fixed point of the mapping* $\Gamma : C \to C$ *defined by*

$$\begin{aligned}\Gamma(x) &= T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(x-\mu_2(B_2+T_2)x)\\ &\quad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(x-\mu_2(B_2+T_2)x)], \quad \forall x \in C,\end{aligned}$$

where $\bar{y} = T_{\mu_2}^{(G_2,\psi_2)}(\bar{x}-\mu_2(B_2+T_2)\bar{x})$.

Proof. Observe that

$$\begin{cases} \mu_1 G_1(\bar{x},x) + \langle \mu_1(B_1+T_1)\bar{y} + \bar{x} - \bar{y}, x-\bar{x}\rangle \geq \mu_1\psi_1(\bar{x}) - \mu_1\psi_1(x), & \forall x\in C,\\ \mu_2 G_2(\bar{y},y) + \langle \mu_2(B_2+T_2)\bar{x} + \bar{y} - \bar{x}, y-\bar{y}\rangle \geq \mu_2\psi_2(\bar{y}) - \mu_2\psi_2(y), & \forall y\in C,\end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \bar{x} = T_{\mu_1}^{(G_1,\psi_1)}(\bar{y}-\mu_1(B_1+T_1)\bar{y}),\\ \bar{y} = T_{\mu_2}^{(G_2,\psi_2)}(\bar{x}-\mu_2(B_2+T_2)\bar{x})\end{cases}$$

$$\Updownarrow$$

$$\bar{x}=T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(\bar{x}-\mu_2(B_2+T_2)\bar{x})-\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(\bar{x}-\mu_2(B_2+T_2)\bar{x})]. \blacksquare$$

Corollary 2.1. (see [30, Lemma 2.2]). *Let* C *be a nonempty closed convex subset of* H. *Let* $G_1, G_2 : C\times C \to \mathbf{R}$ *be two bifunctions satisfying conditions* (*H1*)*-*(*H4*) *and let the mappings* $B_1, B_2 : C \to H$ *be* $\tilde{\beta}_1$*-inverse-strongly monotone and* $\tilde{\beta}_2$*-inverse-strongly monotone, respectively. Let* $\mu_1 \in (0, 2\tilde{\beta}_1)$ *and* $\mu_2 \in (0, 2\tilde{\beta}_2)$*,* *respectively. Then, for given* $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ *is a solution of problem* (1.7) *if and only if* $\bar{x}$ *is a fixed point of the mapping* $\Gamma : C \to C$ *defined by*

$$\Gamma(x) = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(x - \mu_2 B_2 x) - \mu_1 B_1 T_{\mu_2}^{G_2}(x - \mu_2 B_2 x)], \quad \forall x \in C,$$

where $\bar{y} = T_{\mu_2}^{G_2}(\bar{x} - \mu_2 B_2 \bar{x})$.

Corollary 2.2. (see [22, Lemma 2.1]). *For given* $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ *is a solution of problem* (1.9) *if and only if* $\bar{x}$ *is a fixed point of the mapping* $G : C \to C$ *defined by*

$$G(x) = P_C[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)], \quad \forall x \in C,$$

where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

Remark 2.2. From the proof of Theorem 3.1 in Section 3, we know that if $G_1, G_2 : C \times C \to \mathbf{R}$ are two bifunctions satisfying (H1)-(H4), the mappings $B_1, B_2 : C \to H$ are $\tilde{\beta}_1$-inverse-strongly monotone and $\tilde{\beta}_2$-inverse-strongly monotone, respectively, $T_1, T_2 : C \to H$ are η_1-Lipschitz continuous and η_2-Lipschitz continuous, respectively, and $\psi_1, \psi_2 : C \to \mathbf{R}$ are two lower semicontinuous and convex functions with assumption (A1) or (A2), then $\Gamma : C \to C$ is a nonexpansive mapping provided $\mu_1 \in (0, 2\tilde{\beta}_1)$ and $\mu_2 \in (0, 2\tilde{\beta}_2)$.

Throughout this paper, the set of fixed points of the mapping Γ is denoted by $\mho$.

Lemma 2.3. (see [27]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Proposition 2.1. (see [30, Proposition 2.1]). *Let* C, H, Θ, φ *and* $T_r^{(\Theta,\varphi)}$ *be as in Lemma 2.1. Then the following holds:*

$$\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\|^2 \leq \frac{s-t}{s}\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\rangle$$

for all $s, t > 0$ *and* $x \in H$.

Corollary 2.3. (see [4, Lemma 2.3]). *Let* C, H, Θ *and* T_r^{Θ} *be as in Remark 2.1. Then the following holds:*

$$\|T_s^{\Theta}x - T_t^{\Theta}x\|^2 \leq \frac{s-t}{s}\langle T_s^{\Theta}x - T_t^{\Theta}x, T_s^{\Theta}x - x\rangle$$

for all $s, t > 0$ *and* $x \in H$.

Lemma 2.4. (see [26]). *Demiclosedness Principle. Assume that T is a k-strictly pseudocontractive self-mapping on a nonempty closed convex subset C of a real Hilbert space H. Then, the mapping $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x^* \in C$ (for short, $x_n \rightharpoonup x^* \in C$), and the sequence $\{(I-T)x_n\}$ converges strongly to some y (for short, $(I-T)x_n \to y$), it follows that $(I-T)x^* = y$. Here I is the identity mapping of H.*

Lemma 2.5. (see [26]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(*i*) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(*ii*) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

The following Lemma is an immediate consequence of an inner product.

Lemma 2.6. *In a real Hilbert space H, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

3. Main Results

We are now in a position to prove the main result of this paper.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta, G_1, G_2 : C \times C \to \mathbf{R}$ be three bifunctions which satisfy assumptions (H1)-(H4) and $\varphi, \psi_1, \psi_2 : C \to \mathbf{R}$ be three lower semicontinuous and convex functions with assumption (A1) or (A2). Let the mappings $F, B_1, B_2 : C \to H$ be α-inverse-strongly monotone, $\tilde{\beta}_1$-inverse-strongly monotone and $\tilde{\beta}_2$-inverse-strongly monotone, respectively, and $T, T_1, T_2 : C \to H$ be η-inverse-strongly monotone, η_1-inverse-strongly monotone and η_2-inverse-strongly monotone, respectively. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $F(S) \cap GMEP \cap \mho \neq \emptyset$. For fixed $u \in C$ and $x_0 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by*

$$(3.1) \qquad \begin{cases} z_n = T_{\lambda_n}^{(\Theta,\varphi)}(x_n - \lambda_n(F+T)x_n), \\ y_n = T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n) \\ \qquad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n)], \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $0 < \mu_1 < \min\{\tilde{\beta}_1, \eta_1\},\ 0 < \mu_2 < \min\{\tilde{\beta}_2, \eta_2\}$*, and* $0 \le \lambda_n \le \min\{\alpha, \eta\}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\} \subset [0,1]$ *satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \le \gamma_n$ *for all* $n \ge 0$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$*;*

(iv) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$*;*

(v) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \min\{\alpha, \eta\}$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Then, $\{x_n\}$ *converges strongly to* $\bar{x} = P_{F(S)\cap GMEP\cap\mho} u$ *and* $(\bar{x}, \bar{y})$ *is a solution of problem* (*1.4*)*, where* $\bar{y} = T_{\mu_2}^{(G_2,\psi_2)}(\bar{x} - \mu_2(B_2 + T_2)\bar{x})$.

Proof. We divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, take $z \in F(S) \cap GMEP \cap \mho$ arbitrarily. Since $z = T_{\lambda_n}^{(\Theta,\varphi)}(z - \lambda_n(F+T)z) = Sz$, F is α-inverse-strongly monotone and T is η-inverse-strongly monotone, we know from $0 \le \lambda_n \le \min\{\alpha, \eta\}$ that for any $n \ge 0$

$$
\begin{aligned}
&\|(x_n - z) - \lambda_n((F+T)x_n - (F+T)z)\|^2 \\
&= \|\tfrac{1}{2}[(x_n - z) - 2\lambda_n(Fx_n - Fz)] + \tfrac{1}{2}[(x_n - z) - 2\lambda_n(Tx_n - Tz)]\|^2 \\
&\le \tfrac{1}{2}\|(x_n - z) - 2\lambda_n(Fx_n - Fz)\|^2 + \tfrac{1}{2}\|(x_n - z) - 2\lambda_n(Tx_n - Tz)\|^2 \\
&= \tfrac{1}{2}[\|x_n - z\|^2 - 4\lambda_n\langle x_n - z, Fx_n - Fz\rangle + 4\lambda_n^2\|Fx_n - Fz\|^2] \\
&\quad + \tfrac{1}{2}[\|x_n - z\|^2 - 4\lambda_n\langle x_n - z, Tx_n - Tz\rangle + 4\lambda_n^2\|Tx_n - Tz\|^2] \\
&\le \tfrac{1}{2}[\|x_n - z\|^2 - 4\lambda_n\alpha\|Fx_n - Fz\|^2 + 4\lambda_n^2\|Fx_n - Fz\|^2] \\
&\quad + \tfrac{1}{2}[\|x_n - z\|^2 - 4\lambda_n\eta\|Tx_n - Tz\|^2 + 4\lambda_n^2\|Tx_n - Tz\|^2] \\
&= \tfrac{1}{2}[\|x_n - z\|^2 - 4\lambda_n(\alpha - \lambda_n)\|Fx_n - Fz\|^2] \\
&\quad + \tfrac{1}{2}[\|x_n - z\|^2 - 4\lambda_n(\eta - \lambda_n)\|Tx_n - Tz\|^2] \\
&= \|x_n - z\|^2 - 2\lambda_n(\alpha - \lambda_n)\|Fx_n - Fz\|^2 - 2\lambda_n(\eta - \lambda_n)\|Tx_n - Tz\|^2 \\
&\le \|x_n - z\|^2,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\|z_n - z\|^2 \\
&= \|T_{\lambda_n}^{(\Theta,\varphi)}(x_n - \lambda_n(F+T)x_n) - T_{\lambda_n}^{(\Theta,\varphi)}(z - \lambda_n(F+T)z)\|^2 \\
&\le \|(x_n - \lambda_n(F+T)x_n) - (z - \lambda_n(F+T)z)\|^2 \\
&= \|(x_n - z) - \lambda_n((F+T)x_n - (F+T)z)\|^2 \\
&\le \|x_n - z\|^2 - 2\lambda_n(\alpha - \lambda_n)\|Fx_n - Fz\|^2 - 2\lambda_n(\eta - \lambda_n)\|Tx_n - Tz\|^2 \\
&\le \|x_n - z\|^2,
\end{aligned} \tag{3.2}
$$

Also, since $z = T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(z - \mu_2(B_2+T_2)z) - \mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z - \mu_2(B_2+T_2)z)]$, and $B_1, B_2, T_1, T_2 : C \to H$ are $\tilde{\beta}_1$-inverse-strongly monotone and $\tilde{\beta}_2$-inverse-strongly monotone, η_1-inverse-strongly monotone and η_2-inverse-strongly monotone, respectively, we deduce from $0 < \mu_1 < \min\{\tilde{\beta}_1, \eta_1\}$ and $0 < \mu_2 < \min\{\tilde{\beta}_2, \eta_2\}$ that for any $n \ge 0$

$$
\begin{aligned}
&\|y_n - z\|^2 \\
&= \|T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n) \\
&\quad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n)] \\
&\quad -T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(z - \mu_2(B_2+T_2)z) \\
&\quad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z - \mu_2(B_2+T_2)z)]\|^2 \\
&\le \|[T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n) \\
&\quad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n)] \\
&\quad -[T_{\mu_2}^{(G_2,\psi_2)}(z - \mu_2(B_2+T_2)z) \\
&\quad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z - \mu_2(B_2+T_2)z)]\|^2 \\
&= \|[T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n) - T_{\mu_2}^{(G_2,\psi_2)}(z - \mu_2(B_2+T_2)z)] \\
&\quad -\mu_1[(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n) \\
&\quad -(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z - \mu_2(B_2+T_2)z)]\|^2 \\
&\le \tfrac{1}{2}\|[T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n) - T_{\mu_2}^{(G_2,\psi_2)}(z - \mu_2(B_2+T_2)z)] \\
&\quad -2\mu_1[B_1T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n) \\
&\quad -B_1T_{\mu_2}^{(G_2,\psi_2)}(z - \mu_2(B_2+T_2)z)]\|^2 \\
&\quad +\tfrac{1}{2}\|[T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n) - T_{\mu_2}^{(G_2,\psi_2)}(z - \mu_2(B_2+T_2)z)] \\
&\quad -2\mu_1[T_1T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n)
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
&-T_1T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)]\|^2\\
&\le \tfrac{1}{2}[\|T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)-T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2\\
&\quad -4\mu_1(\tilde{\beta}_1-\mu_1)\|B_1T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)\\
&\quad -B_1T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2]\\
&\quad +\tfrac{1}{2}[\|T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)-T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2\\
&\quad -4\mu_1(\eta_1-\mu_1)\|T_1T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)\\
&\quad -T_1T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2]\\
&=\|T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)-T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2\\
&\quad -2\mu_1(\tilde{\beta}_1-\mu_1)\|B_1T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)\\
&\quad -B_1T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2\\
&\quad -2\mu_1(\eta_1-\mu_1)\|T_1T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)\\
&\quad -T_1T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2\\
&\le\|T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)-T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2\\
&\le\|(z_n-\mu_2(B_2+T_2)z_n)-(z-\mu_2(B_2+T_2)z)\|^2\\
&=\|(z_n-z)-\mu_2((B_2+T_2)z_n-(B_2+T_2)z)\|^2\\
&\le\tfrac{1}{2}\|(z_n-z)-2\mu_2(B_2z_n-B_2z)\|^2+\tfrac{1}{2}\|(z_n-z)-2\mu_2(T_2z_n-T_2z)\|^2\\
&\le\tfrac{1}{2}[\|z_n-z\|^2-4\mu_2(\tilde{\beta}_2-\mu_2)\|B_2z_n-B_2z\|^2]\\
&\quad +\tfrac{1}{2}[\|z_n-z\|^2-4\mu_2(\eta_2-\mu_2)\|T_2z_n-T_2z\|^2]\\
&=\|z_n-z\|^2-2\mu_2(\tilde{\beta}_2-\mu_2)\|B_2z_n-B_2z\|^2-2\mu_2(\eta_2-\mu_2)\|T_2z_n-T_2z\|^2\\
&\le\|z_n-z\|^2.
\end{aligned}
$$

Furthermore, from (3.1) we have

$$
\begin{aligned}
\|x_{n+1}-z\| &= \|\alpha_n(u-z)+\beta_n(x_n-z)+\gamma_n(y_n-z)+\delta_n(Sy_n-z)\|\\
&\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\|\gamma_n(y_n-z)+\delta_n(Sy_n-z)\|.
\end{aligned}
\tag{3.4}
$$

Combining (2.2) with (2.3), we have

$$
\begin{aligned}
&\|\gamma_n(y_n-z)+\delta_n(Sy_n-z)\|^2\\
&=\gamma_n^2\|y_n-z\|^2+\delta_n^2\|Sy_n-z\|^2+2\gamma_n\delta_n\langle Sy_n-z,y_n-z\rangle\\
&\le\gamma_n^2\|y_n-z\|^2+\delta_n^2[\|y_n-z\|^2+k\|y_n-Sy_n\|^2]\\
&\quad +2\gamma_n\delta_n[\|y_n-z\|^2-\tfrac{1-k}{2}\|y_n-Sy_n\|^2]
\end{aligned}
$$

$$
\begin{aligned}
&= (\gamma_n+\delta_n)^2\|y_n-z\|^2+[\delta_n^2k-(1-k)\gamma_n\delta_n]\|y_n-Sy_n\|^2\\
&= (\gamma_n+\delta_n)^2\|y_n-z\|^2+\delta_n[(\gamma_n+\delta_n)k-\gamma_n]\|y_n-Sy_n\|^2\\
&\le (\gamma_n+\delta_n)^2\|y_n-z\|^2,
\end{aligned}
$$

which implies that

$$
\|\gamma_n(y_n-z)+\delta_n(Sy_n-z)\|\le(\gamma_n+\delta_n)\|y_n-z\|. \tag{3.5}
$$

From (3.2)-(3.5) it follows that

$$
\begin{aligned}
\|x_{n+1}-z\| &\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\|\gamma_n(y_n-z)+\delta_n(Sy_n-z)\|\\
&\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+(\gamma_n+\delta_n)\|y_n-z\|\\
&\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+(\gamma_n+\delta_n)\|z_n-z\|\\
&\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+(\gamma_n+\delta_n)\|x_n-z\|\\
&= \alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\|.
\end{aligned}
$$

By induction, we obtain that for all $n\ge 0$

$$
\|x_n-z\|\le\max\{\|x_0-z\|,\|u-z\|\}.
$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce immediately that $\{(F+T)x_n\}$, $\{z_n\},\{y_n\},\{Sy_n\}$ and $\{u_n\}$ are bounded, where $u_n=T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)$ for all $n\ge 0$.

Step 2. $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Indeed, define $x_{n+1}=\beta_nx_n+(1-\beta_n)w_n$ for all $n\ge 0$. It follows that

$$
\begin{aligned}
&w_{n+1}-w_n\\
&=\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}\\
&=\frac{\alpha_{n+1}u+\gamma_{n+1}y_{n+1}+\delta_{n+1}Sy_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_nu+\gamma_ny_n+\delta_nSy_n}{1-\beta_n}\\
&=\frac{\alpha_{n+1}u}{1-\beta_{n+1}}-\frac{\alpha_nu}{1-\beta_n}+\frac{\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)}{1-\beta_{n+1}}\\
&\quad+(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n})y_n+(\frac{\delta_{n+1}}{1-\beta_{n+1}}-\frac{\delta_n}{1-\beta_n})Sy_n.
\end{aligned} \tag{3.6}
$$

Observe that

$$
\begin{aligned}
&\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)\|^2\\
&=\gamma_{n+1}^2\|y_{n+1}-y_n\|^2+\delta_{n+1}^2\|Sy_{n+1}-Sy_n\|^2\\
&\quad+2\gamma_{n+1}\delta_{n+1}\langle Sy_{n+1}-Sy_n,y_{n+1}-y_n\rangle\\
&\le\gamma_{n+1}^2\|y_{n+1}-y_n\|^2+\delta_{n+1}^2[\|y_{n+1}-y_n\|^2\\
&\quad+k\|(y_{n+1}-Sy_{n+1})-(y_n-Sy_n)\|^2]
\end{aligned}
$$

$$
\begin{aligned}
&+2\gamma_{n+1}\delta_{n+1}[\|y_{n+1}-y_n\|^2-\tfrac{1-k}{2}\|(y_{n+1}-Sy_{n+1})-(y_n-Sy_n)\|^2] \\
&=(\gamma_{n+1}+\delta_{n+1})^2\|y_{n+1}-y_n\|^2+[\delta_{n+1}^2k-(1-k)\gamma_{n+1}\delta_{n+1}] \\
&\quad\|(y_{n+1}-Sy_{n+1})-(y_n-Sy_n)\|^2 \\
&=(\gamma_{n+1}+\delta_{n+1})^2\|y_{n+1}-y_n\|^2+\delta_{n+1}[(\gamma_{n+1}+\delta_{n+1})k-\gamma_{n+1}] \\
&\quad\|(y_{n+1}-Sy_{n+1})-(y_n-Sy_n)\|^2 \\
&\le(\gamma_{n+1}+\delta_{n+1})^2\|y_{n+1}-y_n\|^2,
\end{aligned}
$$

which implies that

$$\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)\|\le(\gamma_{n+1}+\delta_{n+1})\|y_{n+1}-y_n\|. \tag{3.7}$$

Next, we estimate $\|y_{n+1}-y_n\|$. From (3.1) we have

$$
\begin{aligned}
&\|y_{n+1}-y_n\|^2 \\
&=\|T_{\mu_1}^{(G_1,\psi_1)}(u_{n+1}-\mu_1(B_1+T_1)u_{n+1}) \\
&\quad-T_{\mu_1}^{(G_1,\psi_1)}(u_n-\mu_1(B_1+T_1)u_n)\|^2 \\
&\le\|(u_{n+1}-\mu_1(B_1+T_1)u_{n+1})-(u_n-\mu_1(B_1+T_1)u_n)\|^2 \\
&=\|(u_{n+1}-u_n)-\mu_1((B_1+T_1)u_{n+1}-(B_1+T_1)u_n)\|^2 \\
&\le\tfrac{1}{2}\|(u_{n+1}-u_n)-\mu_1(B_1u_{n+1}-B_1u_n)\|^2 \\
&\quad+\tfrac{1}{2}\|(u_{n+1}-u_n)-\mu_1(T_1u_{n+1}-T_1u_n)\|^2 \\
&\le\|u_{n+1}-u_n\|^2-2\mu_1(\tilde{\beta}_1-\mu_1)\|B_1u_{n+1} \\
&\quad-B_1u_n\|^2-2\mu_1(\eta_1-\mu_1)\|T_1u_{n+1}-T_1u_n\|^2 \\
&\le\|u_{n+1}-u_n\|^2 \qquad (3.8)\\
&=\|T_{\mu_2}^{(G_2,\psi_2)}(z_{n+1}-\mu_2(B_2+T_2)z_{n+1}) \\
&\quad-T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)\|^2 \\
&\le\|(z_{n+1}-\mu_2(B_2+T_2)z_{n+1})-(z_n-\mu_2(B_2+T_2)z_n)\|^2 \\
&=\|(z_{n+1}-z_n)-\mu_2((B_2+T_2)z_{n+1}-(B_2+T_2)z_n)\|^2 \\
&\le\tfrac{1}{2}\|(z_{n+1}-z_n)-\mu_2(B_2z_{n+1}-B_2z_n)\|^2 \\
&\quad+\tfrac{1}{2}\|(z_{n+1}-z_n)-\mu_2(T_2z_{n+1}-T_2z_n)\|^2 \\
&=\|z_{n+1}-z_n\|^2-2\mu_2(\tilde{\beta}_2-\mu_2)\|B_2z_{n+1} \\
&\quad-B_2z_n\|^2-2\mu_2(\eta_2-\mu_2)\|T_2z_{n+1}-T_2z_n\|^2 \\
&\le\|z_{n+1}-z_n\|^2,
\end{aligned}
$$

$$
\begin{aligned}
&\|(x_{n+1}-\lambda_{n+1}(F+T)x_{n+1})-(x_n-\lambda_n(F+T)x_n)\| \\
&= \|x_{n+1}-x_n-\lambda_{n+1}((F+T)x_{n+1} \\
&\quad -(F+T)x_n)+(\lambda_n-\lambda_{n+1})(F+T)x_n\| \\
&\leq \|x_{n+1}-x_n-\lambda_{n+1}((F+T)x_{n+1} \\
&\quad -(F+T)x_n)\|+|\lambda_{n+1}-\lambda_n|\|(F+T)x_n\| \\
&\leq \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|(F+T)x_n\|,
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
&\|z_{n+1}-z_n\| \\
&= \|T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_{n+1}-\lambda_{n+1}(F+T)x_{n+1})-T^{(\Theta,\varphi)}_{\lambda_n}(x_n-\lambda_n(F+T)x_n)\| \\
&= \|T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_{n+1}-\lambda_{n+1}(F+T)x_{n+1})-T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(F+T)x_n) \\
&\quad +T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(F+T)x_n)-T^{(\Theta,\varphi)}_{\lambda_n}(x_n-\lambda_n(F+T)x_n)\| \\
&\leq \|T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_{n+1}-\lambda_{n+1}(F+T)x_{n+1})-T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(F+T)x_n)\| \\
&\quad +\|T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(F+T)x_n)-T^{(\Theta,\varphi)}_{\lambda_n}(x_n-\lambda_n(F+T)x_n)\| \\
&\leq \|(x_{n+1}-\lambda_{n+1}(F+T)x_{n+1})-(x_n-\lambda_n(F+T)x_n)\| \\
&\quad +\|T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(F+T)x_n)-T^{(\Theta,\varphi)}_{\lambda_n}(x_n-\lambda_n(F+T)x_n)\| \\
&\leq \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|(F+T)x_n\| \\
&\quad +\|T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(F+T)x_n)-T^{(\Theta,\varphi)}_{\lambda_n}(x_n-\lambda_n(F+T)x_n)\|.
\end{aligned} \tag{3.10}
$$

So, from (3.8) and (3.10) it follows that

$$
\begin{aligned}
&\|y_{n+1}-y_n\| \\
&\leq \|z_{n+1}-z_n\| \\
&\leq \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|(F+T)x_n\| \\
&\quad +\|T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(F+T)x_n)-T^{(\Theta,\varphi)}_{\lambda_n}(x_n-\lambda_n(F+T)x_n)\|.
\end{aligned} \tag{3.11}
$$

Hence it follows from (3.6), (3.7) and (3.11) that

$$
\begin{aligned}
&\|w_{n+1}-w_n\| \\
&\leq \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}+\frac{\alpha_n}{1-\beta_n}\right)\|u\|+\frac{\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)\|}{1-\beta_{n+1}} \\
&\quad +\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|\|y_n\|+\left|\frac{\delta_{n+1}}{1-\beta_{n+1}}-\frac{\delta_n}{1-\beta_n}\right|\|Sy_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|u\|+\|Sy_n\|)+\frac{\alpha_n}{1-\beta_n}(\|u\|+\|Sy_n\|)+\frac{\gamma_{n+1}+\delta_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\|
\end{aligned}
$$

$$
\begin{aligned}
&+|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}|(\|y_n\|+\|Sy_n\|)\\
\le\ &\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|u\|+\|Sy_n\|)+\frac{\alpha_n}{1-\beta_n}(\|u\|+\|Sy_n\|)\\
&+\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|(F+T)x_n\|\\
&+\|T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(F+T)x_n)-T^{(\Theta,\varphi)}_{\lambda_n}(x_n-\lambda_n(F+T)x_n)\|\\
&+|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}|(\|y_n\|+\|Sy_n\|).
\end{aligned}
$$

Note that $0<\liminf_{n\to\infty}\lambda_n\le\limsup_{n\to\infty}\lambda_n<\min\{\alpha,\eta\}$ and $\lim_{n\to\infty}(\lambda_n-\lambda_{n+1})=0$. Then utilizing Proposition 2.1 we have

$$
\lim_{n\to\infty}\|T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(F+T)x_n)-T^{(\Theta,\varphi)}_{\lambda_n}(x_n-\lambda_n(F+T)x_n)\|=0. \tag{3.12}
$$

Consequently, it follows from (3.12) and conditions (ii), (iv), (v) that

$$
\begin{aligned}
&\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)\\
\le\ &\limsup_{n\to\infty}\{\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|u\|+\|Sy_n\|)+\frac{\alpha_n}{1-\beta_n}(\|u\|+\|Sy_n\|)\\
&+|\lambda_{n+1}-\lambda_n|\|(F+T)x_n\|\\
&+\|T^{(\Theta,\varphi)}_{\lambda_{n+1}}(x_n-\lambda_n(F+T)x_n)-T^{(\Theta,\varphi)}_{\lambda_n}(x_n-\lambda_n(F+T)x_n)\|\\
&+|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}|(\|y_n\|+\|Sy_n\|)\}\\
=\ &0.
\end{aligned}
$$

Hence by Lemma 2.3 we get $\lim_{n\to\infty}\|w_n-x_n\|=0$. Thus,

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|w_n-x_n\|=0. \tag{3.13}
$$

Step 3. $\lim_{n\to\infty}\|(B_1+T_1)u_n-(B_1+T_1)u^*\|=0$, $\lim_{n\to\infty}\|(B_2+T_2)z_n-(B_2+T_2)z\|=0$ and $\lim_{n\to\infty}\|(F+T)x_n-(F+T)z\|=0$, where $u^*=T^{(G_2,\psi_2)}_{\mu_2}(z-\mu_2(B_2+T_2)z)$.

Indeed, from (3.1) and (3.7) we get

$$
\begin{aligned}
&\|x_{n+1}-z\|^2\\
=\ &\langle\alpha_n(u-z)+\beta_n(x_n-z)+\gamma_n(y_n-z)+\delta_n(Sy_n-z),x_{n+1}-z\rangle\\
=\ &\alpha_n\langle u-z,x_{n+1}-z\rangle+\beta_n\langle x_n-z,x_{n+1}-z\rangle\\
&+\langle\gamma_n(y_n-z)+\delta_n(Sy_n-z),x_{n+1}-z\rangle\\
\le\ &\alpha_n\langle u-z,x_{n+1}-z\rangle+\beta_n\|x_n-z\|\|x_{n+1}-z\|\\
&+\|\gamma_n(y_n-z)+\delta_n(Sy_n-z)\|\|x_{n+1}-z\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha_n\langle u-z, x_{n+1}-z\rangle + \beta_n\|x_n-z\|\|x_{n+1}-z\| \\
&\quad +(\gamma_n+\delta_n)\|y_n-z\|\|x_{n+1}-z\| \\
&\leq \alpha_n\langle u-z, x_{n+1}-z\rangle \\
&\quad +\tfrac{\beta_n}{2}(\|x_n-z\|^2+\|x_{n+1}-z\|^2)+\tfrac{\gamma_n+\delta_n}{2}(\|y_n-z\|^2+\|x_{n+1}-z\|^2),
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\|x_{n+1}-z\|^2 \\
&\leq \frac{2\alpha_n}{1+\alpha_n}\langle u-z, x_{n+1}-z\rangle+\frac{\beta_n}{1+\alpha_n}\|x_n-z\|^2+\frac{\gamma_n+\delta_n}{1+\alpha_n}\|y_n-z\|^2.
\end{aligned}
\tag{3.14}
$$

So, in terms of (3.2), (3.3) and (3.14), we have

$$
\begin{aligned}
&\|x_{n+1}-z\|^2 \\
&\leq \tfrac{2\alpha_n}{1+\alpha_n}\|u-z\|\|x_{n+1}-z\|+\tfrac{\beta_n}{1+\alpha_n}\|x_n-z\|^2 \\
&\quad +\tfrac{\gamma_n+\delta_n}{1+\alpha_n}[\|T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)-T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2 \\
&\quad -2\mu_1(\tilde{\beta}_1-\mu_1)\|B_1T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n) \\
&\quad -B_1T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2 \\
&\quad -2\mu_1(\eta_1-\mu_1)\|T_1T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n) \\
&\quad -T_1T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2] \\
&= \tfrac{2\alpha_n}{1+\alpha_n}\|u-z\|\|x_{n+1}-z\|+\tfrac{\beta_n}{1+\alpha_n}\|x_n-z\|^2 \\
&\quad +\tfrac{\gamma_n+\delta_n}{1+\alpha_n}[\|T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)-T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2 \\
&\quad -2\mu_1(\tilde{\beta}_1-\mu_1)[\|B_1u_n-B_1u^*\|^2-2\mu_1(\eta_1-\mu_1)\|T_1u_n-T_1u^*\|^2] \\
&\leq \tfrac{2\alpha_n}{1+\alpha_n}\|u-z\|\|x_{n+1}-z\|+\tfrac{\beta_n}{1+\alpha_n}\|x_n-z\|^2 \\
&\quad +\tfrac{\gamma_n+\delta_n}{1+\alpha_n}[\|z_n-z\|^2-2\mu_2(\tilde{\beta}_2-\mu_2)\|B_2z_n-B_2z\|^2-2\mu_2(\eta_2-\mu_2)\|T_2z_n-T_2z\|^2 \\
&\quad -2\mu_1(\tilde{\beta}_1-\mu_1)[\|B_1u_n-B_1u^*\|^2-2\mu_1(\eta_1-\mu_1)\|T_1u_n-T_1u^*\|^2] \\
&\leq \tfrac{2\alpha_n}{1+\alpha_n}\|u-z\|\|x_{n+1}-z\|+\tfrac{\beta_n}{1+\alpha_n}\|x_n-z\|^2+\tfrac{\gamma_n+\delta_n}{1+\alpha_n}[\|x_n-z\|^2 \\
&\quad -2\lambda_n(\alpha-\lambda_n)\|Fx_n-Fz\|^2-2\lambda_n(\eta-\lambda_n)\|Tx_n-Tz\|^2 \\
&\quad -2\mu_2(\tilde{\beta}_2-\mu_2)\|B_2z_n-B_2z\|^2-2\mu_2(\eta_2-\mu_2)\|T_2z_n-T_2z\|^2 \\
&\quad -2\mu_1(\tilde{\beta}_1-\mu_1)[\|B_1u_n-B_1u^*\|^2-2\mu_1(\eta_1-\mu_1)\|T_1u_n-T_1u^*\|^2] \\
&= \tfrac{2\alpha_n}{1+\alpha_n}\|u-z\|\|x_{n+1}-z\|+\tfrac{1-\alpha_n}{1+\alpha_n}\|x_n-z\|^2
\end{aligned}
$$

$$-\tfrac{\gamma_n+\delta_n}{1+\alpha_n}[2\lambda_n(\alpha-\lambda_n)\|Fx_n-Fz\|^2+2\lambda_n(\eta-\lambda_n)\|Tx_n-Tz\|^2$$
$$+2\mu_2(\tilde{\beta}_2-\mu_2)\|B_2z_n-B_2z\|^2+2\mu_2(\eta_2-\mu_2)\|T_2z_n-T_2z\|^2$$
$$+2\mu_1(\tilde{\beta}_1-\mu_1)\|B_1u_n-B_1u^*\|^2+2\mu_1(\eta_1-\mu_1)\|T_1u_n-T_1u^*\|^2].$$

Therefore,

$$\begin{aligned}
&2\lambda_n(\alpha-\lambda_n)\|Fx_n-Fz\|^2+2\lambda_n(\eta-\lambda_n)\|Tx_n-Tz\|^2\\
&+2\mu_2(\tilde{\beta}_2-\mu_2)\|B_2z_n-B_2z\|^2+2\mu_2(\eta_2-\mu_2)\|T_2z_n-T_2z\|^2\\
&+2\mu_1(\tilde{\beta}_1-\mu_1)\|B_1u_n-B_1u^*\|^2+2\mu_1(\eta_1-\mu_1)\|T_1u_n-T_1u^*\|^2\\
\le\ &\frac{2\alpha_n}{\gamma_n+\delta_n}\|u-z\|\|x_{n+1}-z\|+\frac{1-\alpha_n}{\gamma_n+\delta_n}(\|x_n-z\|^2-\|x_{n+1}-z\|^2)\\
\le\ &\frac{2\alpha_n}{\gamma_n+\delta_n}\|u-z\|\|x_{n+1}-z\|+\frac{1-\alpha_n}{\gamma_n+\delta_n}(\|x_n-z\|+\|x_{n+1}-z\|)\|x_n-x_{n+1}\|.
\end{aligned}$$

Since $\alpha_n\to 0$, $\|x_n-x_{n+1}\|\to 0$, $0<\liminf_{n\to\infty}\lambda_n\le\limsup_{n\to\infty}\lambda_n<\min\{\alpha,\eta\}$, and $\liminf_{n\to\infty}(\gamma_n+\delta_n)>0$, we have

$$(3.15)\qquad \begin{cases}\lim\limits_{n\to\infty}\|B_1u_n-B_1u^*\|=\lim\limits_{n\to\infty}\|T_1u_n-T_1u^*\|=0,\\ \lim\limits_{n\to\infty}\|B_2z_n-B_2z\|=\lim\limits_{n\to\infty}\|T_2z_n-T_2z\|=0,\\ \lim\limits_{n\to\infty}\|Fx_n-Fz\|=\lim\limits_{n\to\infty}\|Tx_n-Tz\|=0.\end{cases}$$

Step 4. $\lim_{n\to\infty}\|Sy_n-y_n\|=0$.

Indeed, using the firm nonexpansivity of $T_{\mu_1}^{(G_1,\psi_1)}$ and $T_{\mu_2}^{(G_2,\psi_2)}$, we get from (3.2) and (3.3)

$$\begin{aligned}
&\|u_n-u^*\|^2\\
=\ &\|T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)-T_{\mu_2}^{(G_2,\psi_2)}(z-\mu_2(B_2+T_2)z)\|^2\\
\le\ &\langle(z_n-\mu_2(B_2+T_2)z_n)-(z-\mu_2(B_2+T_2)z),u_n-u^*\rangle\\
=\ &\tfrac{1}{2}[\|(z_n-\mu_2(B_2+T_2)z_n)-(z-\mu_2(B_2+T_2)z)\|^2+\|u_n-u^*\|^2\\
&-\|(z_n-\mu_2(B_2+T_2)z_n)-(z-\mu_2(B_2+T_2)z)-(u_n-u^*)\|^2]\\
\le\ &\tfrac{1}{2}[\|z_n-z\|^2+\|u_n-u^*\|^2-\|(z_n-u_n)\\
&-\mu_2((B_2+T_2)z_n-(B_2+T_2)z)-(z-u^*)\|^2]\\
\le\ &\tfrac{1}{2}[\|x_n-z\|^2+\|u_n-u^*\|^2-\|(z_n-u_n)-(z-u^*)\|^2\\
&+2\mu_2\langle(z_n-u_n)-(z-u^*),(B_2+T_2)z_n-(B_2+T_2)z\rangle\\
&-\mu_2^2\|(B_2+T_2)z_n-(B_2+T_2)z\|^2],
\end{aligned}$$

and

$$\begin{aligned}
&\|y_n - z\|^2 \\
&= \|T_{\mu_1}^{(G_1,\psi_1)}(u_n - \mu_1(B_1+T_1)u_n) - T_{\mu_1}^{(G_1,\psi_1)}(u^* - \mu_1(B_1+T_1)u^*)\|^2 \\
&\leq \langle (u_n - \mu_1(B_1+T_1)u_n) - (u^* - \mu_1(B_1+T_1)u^*), y_n - z\rangle \\
&= \tfrac{1}{2}[\|(u_n - \mu_1(B_1+T_1)u_n) - (u^* - \mu_1(B_1+T_1)u^*)\|^2 + \|y_n - z\|^2 \\
&\quad -\|(u_n - \mu_1(B_1+T_1)u_n) - (u^* - \mu_1(B_1+T_1)u^*) - (y_n - z)\|^2] \\
&\leq \tfrac{1}{2}[\|u_n - u^*\|^2 + \|y_n - z\|^2 - \|(u_n - y_n) + (z - u^*)\|^2 \\
&\quad +2\mu_1\langle (B_1+T_1)u_n - (B_1+T_1)u^*, (u_n - y_n) + (z - u^*)\rangle \\
&\quad -\mu_1^2\|(B_1+T_1)u_n - (B_1+T_1)u^*\|^2] \\
&\leq \tfrac{1}{2}[\|x_n - z\|^2 + \|y_n - z\|^2 - \|(u_n - y_n) + (z - u^*)\|^2 \\
&\quad +2\mu_1\langle (B_1+T_1)u_n - (B_1+T_1)u^*, (u_n - y_n) + (z - u^*)\rangle].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|u_n - u^*\|^2 \leq\ & \|x_n - z\|^2 - \|(z_n - u_n) - (z - u^*)\|^2 \\
& +2\mu_2\langle (z_n - u_n) - (z - u^*), (B_2+T_2)z_n - (B_2+T_2)z\rangle \\
& -\mu_2^2\|(B_2+T_2)z_n - (B_2+T_2)z\|^2,
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
\|y_n - z\|^2 \leq\ & \|x_n - z\|^2 - \|(u_n - y_n) + (z - u^*)\|^2 \\
& +2\mu_1\|(B_1+T_1)u_n - (B_1+T_1)u^*\|\|(u_n - y_n) + (z - u^*)\|.
\end{aligned} \tag{3.17}$$

In terms of (3.3), (3.14) and (3.16), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 \leq\ & \frac{2\alpha_n}{1+\alpha_n}\langle u - z, x_{n+1} - z\rangle + \frac{\beta_n}{1+\alpha_n}\|x_n - z\|^2 \\
& + \frac{\gamma_n + \delta_n}{1+\alpha_n}[\|x_n - z\|^2 - \|(z_n - u_n) - (z - u^*)\|^2 \\
& +2\mu_2\langle (z_n - u_n) - (z - u^*), (B_2+T_2)z_n - (B_2+T_2)z\rangle].
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&\frac{\gamma_n + \delta_n}{1+\alpha_n}\|(z_n - u_n) - (z - u^*)\|^2 \\
&\leq \frac{2\alpha_n}{1+\alpha_n}\|u - z\|\|x_{n+1} - z\| + \frac{1-\alpha_n}{1+\alpha_n}\|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\quad + \frac{2\mu_2(\gamma_n + \delta_n)}{1+\alpha_n}\|(z_n - u_n) - (z - u^*)\|\|(B_2+T_2)z_n - (B_2+T_2)z\|
\end{aligned}$$

$$\le \frac{2\alpha_n}{1+\alpha_n}\|u-z\|\|x_{n+1}-z\| + (\|x_n-z\|+\|x_{n+1}-z\|)\|x_n-x_{n+1}\|$$
$$+\frac{2\mu_2(\gamma_n+\delta_n)}{1+\alpha_n}\|(z_n-u_n)-(z-u^*)\|\|(B_2+T_2)z_n-(B_2+T_2)z\|.$$

Since $\liminf_{n\to\infty}\frac{\gamma_n+\delta_n}{1+\alpha_n}>0,\ \alpha_n\to 0,\ \|x_{n+1}-x_n\|\to 0$ and $\|(B_2+T_2)z_n-(B_2+T_2)z\|\to 0$, we conclude that

$$\lim_{n\to\infty}\|(z_n-u_n)-(z-u^*)\|=0. \tag{3.18}$$

Utilizing (3.14) and (3.17), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 \le\ & \frac{2\alpha_n}{1+\alpha_n}\langle u-z, x_{n+1}-z\rangle + \frac{\beta_n}{1+\alpha_n}\|x_n-z\|^2\\ &+\frac{\gamma_n+\delta_n}{1+\alpha_n}[\|x_n-z\|^2-\|(u_n-y_n)+(z-u^*)\|^2\\ &+2\mu_1\|(B_1+T_1)u_n-(B_1+T_1)u^*\|\|(u_n-y_n)+(z-u^*)\|].\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{\gamma_n+\delta_n}{1+\alpha_n}\|(u_n-y_n)+(z-u^*)\|^2\\ &\le \frac{2\alpha_n}{1+\alpha_n}\|u-z\|\|x_{n+1}-z\| + (\|x_n-z\|+\|x_{n+1}-z\|)\|x_n-x_{n+1}\|\\ &\quad+\frac{2\mu_1(\gamma_n+\delta_n)}{1+\alpha_n}\|(B_1+T_1)u_n-(B_1+T_1)u^*\|\|(u_n-y_n)+(z-u^*)\|,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|(u_n-y_n)+(z-u^*)\|=0. \tag{3.19}$$

In addition, from the firm nonexpansivity of $T_{\lambda_n}^{(\Theta,\varphi)}$, we have

$$\begin{aligned}\|z_n-z\|^2 &= \|T_{\lambda_n}^{(\Theta,\varphi)}(x_n-\lambda_n(F+T)x_n)-T_{\lambda_n}^{(\Theta,\varphi)}(z-\lambda_n(F+T)z)\|^2\\ &\le \langle (x_n-\lambda_n(F+T)x_n)-(z-\lambda_n(F+T)z), z_n-z\rangle\\ &= \frac{1}{2}[\|(x_n-\lambda_n(F+T)x_n)-(z-\lambda_n(F+T)z)\|^2+\|z_n-z\|^2\\ &\quad -\|(x_n-\lambda_n(F+T)x_n)-(z-\lambda_n(F+T)z)-(z_n-z)\|^2]\\ &\le \frac{1}{2}[\|x_n-z\|^2+\|z_n-z\|^2-\|x_n-z_n-\lambda_n((F+T)x_n-(F+T)z)\|^2]\\ &= \frac{1}{2}[\|x_n-z\|^2+\|z_n-z\|^2-\|x_n-z_n\|^2\\ &\quad +2\lambda_n\langle (F+T)x_n-(F+T)z, x_n-z_n\rangle\\ &\quad -\lambda_n^2\|(F+T)x_n-(F+T)z\|^2],\end{aligned}$$

which implies that

$$(3.20)\quad \|z_n-z\|^2 \le \|x_n-z\|^2-\|x_n-z_n\|^2+2\lambda_n\|(F+T)x_n-(F+T)z\|\|x_n-z_n\|.$$

From (3.3), (3.14) and (3.20), we have

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
\le\ & \frac{2\alpha_n}{1+\alpha_n}\langle u-z, x_{n+1}-z\rangle + \frac{\beta_n}{1+\alpha_n}\|x_n-z\|^2\\
&+\frac{\gamma_n+\delta_n}{1+\alpha_n}\|z_n-z\|^2\\
\le\ & \frac{2\alpha_n}{1+\alpha_n}\langle u-z, x_{n+1}-z\rangle + \frac{\beta_n}{1+\alpha_n}\|x_n-z\|^2\\
&+\frac{\gamma_n+\delta_n}{1+\alpha_n}[\|x_n-z\|^2-\|x_n-z_n\|^2+2\lambda_n\|(F+T)x_n-(F+T)z\|\|x_n-z_n\|].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{\gamma_n+\delta_n}{1+\alpha_n}\|x_n-z_n\|^2\\
\le\ & \frac{2\alpha_n}{1+\alpha_n}\|u-z\|\|x_{n+1}-z\| + \frac{1-\alpha_n}{1+\alpha_n}\|x_n-z\|^2\\
&-\|x_{n+1}-z\|^2 + \frac{2\lambda_n(\gamma_n+\delta_n)}{1+\alpha_n}\|(F+T)x_n-(F+T)z\|\|x_n-z_n\|\\
\le\ & \frac{2\alpha_n}{1+\alpha_n}\|u-z\|\|x_{n+1}-z\| + (\|x_n-z\|+\|x_{n+1}-z\|)\|x_{n+1}-x_n\|\\
&+\frac{2\lambda_n(\gamma_n+\delta_n)}{1+\alpha_n}\|(F+T)x_n-(F+T)z\|\|x_n-z_n\|.
\end{aligned}$$

Hence, we deduce immediately that

$$\lim_{n\to\infty}\|x_n-z_n\|=0.$$

Thus, from (3.18), (3.19) and (3.20), we conclude that

$$(3.21)\qquad \lim_{n\to\infty}\|z_n-y_n\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|x_n-y_n\|=0.$$

So, from (3.1), (3.13) and (3.21), we have

$$\lim_{n\to\infty}\|Sy_n-x_n\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|Sy_n-y_n\|=0.$$

Step 5. $\limsup_{n\to\infty}\langle u-\bar{x}, x_n-\bar{x}\rangle \le 0$ where $\bar{x}=P_{F(S)\cap GMEP\cap\mho}u$.

Indeed, take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle u-\bar{x}, y_n-\bar{x}\rangle = \lim_{n\to\infty}\langle u-\bar{x}, y_{n_i}-\bar{x}\rangle. \tag{3.22}$$

Without loss of generality, we may assume that $y_{n_i} \rightharpoonup w$. First, it is clear from Lemma 2.4 that $w \in F(S)$. Second, let us show that $w \in \mho$. Utilizing Lemma 2.1 we have for all $x, y \in C$

$$\begin{aligned}
&\|\Gamma(x)-\Gamma(y)\|^2\\
&= \|T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(x-\mu_2(B_2+T_2)x)\\
&\quad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(x-\mu_2(B_2+T_2)x)]\\
&\quad -T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(y-\mu_2(B_2+T_2)y)\\
&\quad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(y-\mu_2(B_2+T_2)y)]\|^2\\
&\le \|T_{\mu_2}^{(G_2,\psi_2)}(x-\mu_2(B_2+T_2)x)-\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(x-\mu_2(B_2+T_2)x)\\
&\quad -[T_{\mu_2}^{(G_2,\psi_2)}(y-\mu_2(B_2+T_2)y)-\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(y-\mu_2(B_2+T_2)y)]\|^2\\
&= \|T_{\mu_2}^{(G_2,\psi_2)}(x-\mu_2(B_2+T_2)x)-T_{\mu_2}^{(G_2,\psi_2)}(y-\mu_2(B_2+T_2)y)\\
&\quad -\mu_1[(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(x-\mu_2(B_2+T_2)x)\\
&\quad -(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(y-\mu_2(B_2+T_2)y)]\|^2\\
&\le \|T_{\mu_2}^{(G_2,\psi_2)}(x-\mu_2(B_2+T_2)x)-T_{\mu_2}^{(G_2,\psi_2)}(y-\mu_2(B_2+T_2)y)\|^2\\
&\quad -2\mu_1(\tilde{\beta}_1-\mu_1)\|B_1T_{\mu_2}^{(G_2,\psi_2)}(x-\mu_2(B_2+T_2)x)\\
&\quad -B_1T_{\mu_2}^{(G_2,\psi_2)}(y-\mu_2(B_2+T_2)y)\|^2\\
&\quad -2\mu_1(\eta_1-\mu_1)\|T_1T_{\mu_2}^{(G_2,\psi_2)}(x-\mu_2(B_2+T_2)x)\\
&\quad -T_1T_{\mu_2}^{(G_2,\psi_2)}(y-\mu_2(B_2+T_2)y)\|^2\\
&\le \|T_{\mu_2}^{(G_2,\psi_2)}(x-\mu_2(B_2+T_2)x)-T_{\mu_2}^{(G_2,\psi_2)}(y-\mu_2(B_2+T_2)y)\|^2\\
&\le \|(x-\mu_2(B_2+T_2)x)-(y-\mu_2(B_2+T_2)y)\|^2\\
&\le \|x-y\|^2-2\mu_2(\tilde{\beta}_2-\mu_2)\|B_2x-B_2y\|^2-2\mu_2(\eta_2-\mu_2)\|T_2x-T_2y\|^2\\
&\le \|x-y\|^2.
\end{aligned}$$

This shows that $\Gamma : C \to C$ is nonexpansive. Note that

$$\begin{aligned}
\|y_n-\Gamma(y_n)\| &= \|T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)\\
&\qquad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z_n-\mu_2(B_2+T_2)z_n)]-\Gamma(y_n)\|\\
&= \|\Gamma(z_n)-\Gamma(y_n)\|\\
&\le \|z_n-y_n\| \to 0.
\end{aligned}$$

According to Lemma 2.4 we obtain $w \in \mho$.

Next, let us show that $w \in GMEP$. From $z_n = T_{\lambda_n}^{(\Theta,\varphi)}(x_n - \lambda_n(F+T)x_n)$, we know that

$$\Theta(z_n, y)+\varphi(y)-\varphi(z_n)+\langle (F+T)x_n, y-z_n\rangle+\frac{1}{\lambda_n}\langle y-z_n, z_n-x_n\rangle \geq 0, \quad \forall y \in C.$$

From (H2) it follows that

$$\varphi(y) - \varphi(z_n) + \langle (F+T)x_n, y-z_n\rangle + \frac{1}{\lambda_n}\langle y-z_n, z_n-x_n\rangle \geq \Theta(y, z_n), \quad \forall y \in C.$$

Replacing n by n_i, we have

$$\begin{aligned} &\varphi(y) - \varphi(z_{n_i}) + \langle (F+T)x_{n_i}, y-z_{n_i}\rangle \\ &+\langle y - z_{n_i}, \tfrac{z_{n_i}-x_{n_i}}{\lambda_{n_i}}\rangle \geq \Theta(y, z_{n_i}), \quad \forall y \in C. \end{aligned} \tag{3.23}$$

Put $z_t = ty + (1-t)w$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.23) we have

$$\begin{aligned} &\langle z_t - z_{n_i}, (F+T)z_t\rangle \\ &\geq \langle z_t - z_{n_i}, (F+T)z_t\rangle - \varphi(z_t) + \varphi(z_{n_i}) - \langle z_t - z_{n_i}, (F+T)x_{n_i}\rangle \\ &\quad -\langle z_t - z_{n_i}, \tfrac{z_{n_i}-x_{n_i}}{\lambda_{n_i}}\rangle + \Theta(z_t, z_{n_i}) \\ &= \langle z_t - z_{n_i}, (F+T)z_t - (F+T)z_{n_i}\rangle + \langle z_t - z_{n_i}, (F+T)z_{n_i} - (F+T)x_{n_i}\rangle \\ &\quad -\varphi(z_t) + \varphi(z_{n_i}) - \langle z_t - z_{n_i}, \tfrac{z_{n_i}-x_{n_i}}{\lambda_{n_i}}\rangle + \Theta(z_t, z_{n_i}). \end{aligned}$$

Since $\|z_{n_i} - x_{n_i}\| \to 0$, we have $\|(F+T)z_{n_i} - (F+T)x_{n_i}\| \to 0$. Further, from the monotonicity of $F+T$, we have $\langle z_t - z_{n_i}, (F+T)z_t - (F+T)z_{n_i}\rangle \geq 0$. So, from (H4), the weakly lower semicontinuity of φ, $\frac{z_{n_i}-x_{n_i}}{\lambda_{n_i}} \to 0$ and $z_{n_i} \rightharpoonup w$, we have

$$\langle z_t - w, (F+T)z_t\rangle \geq -\varphi(z_t) + \varphi(z_w) + \Theta(z_t, w), \tag{3.24}$$

as $i \to \infty$. From (H1), (H4) and (3.24), we also have

$$\begin{aligned} 0 &= \Theta(z_t, z_t) + \varphi(z_t) - \varphi(z_t) \\ &\leq t\Theta(z_t, y) + (1-t)\Theta(z_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(z_t) \\ &= t[\Theta(z_t, y) + \varphi(y) - \varphi(z_t)] + (1-t)[\Theta(z_t, w) + \varphi(w) - \varphi(z_t)] \\ &\geq t[\Theta(z_t, y) + \varphi(y) - \varphi(z_t)] + (1-t)\langle z_t - w, (F+T)z_t\rangle \\ &= t[\Theta(z_t, y) + \varphi(y) - \varphi(z_t)] + (1-t)t\langle y - w, (F+T)z_t\rangle, \end{aligned}$$

and hence

$$0 \leq \Theta(z_t, y) + \varphi(y) - \varphi(z_t) + (1-t)\langle y - w, (F+T)z_t \rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq \Theta(w, y) + \varphi(y) - \varphi(w) + \langle y - w, (F+T)w \rangle.$$

This implies that $w \in GMEP$. Now, we show that $w \in F(S)$. Indeed, since $t_{n_i} \rightharpoonup w$ and $\|St_{n_i} - t_{n_i}\| \to 0$ due to (3.19), utilizing Lemma 2.5 we have $(I-S)w = 0$ and hence $w \in F(S)$. Therefore, $w \in F(S) \cap GMEP \cap \mho$. This together with (3.21) and the property of metric projection, implies that

$$\begin{aligned} \limsup_{n\to\infty} \langle u - \bar{x}, x_n - \bar{x} \rangle &= \lim_{i\to\infty} \langle u - \bar{x}, x_{n_i} - \bar{x} \rangle \\ &= \langle u - \bar{x}, w - \bar{x} \rangle \leq 0. \end{aligned}$$

Step 6. $x_n \to \bar{x}$ as $n \to \infty$.

Indeed, from (3.2), (3.3) and (3.14), we have

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 \leq & \frac{2\alpha_n}{1+\alpha_n}\langle u - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{\beta_n}{1+\alpha_n}\|x_n - \bar{x}\|^2 \\ & + \frac{\gamma_n + \delta_n}{1+\alpha_n}\|x_n - \bar{x}\|^2 \\ = & (1 - \frac{2\alpha_n}{1+\alpha_n})\|x_n - \bar{x}\|^2 + \frac{2\alpha_n}{1+\alpha_n}\langle u - \bar{x}, x_{n+1} - \bar{x} \rangle. \end{aligned}$$

It is clear that $\sum_{n=0}^{\infty} \frac{2\alpha_n}{1+\alpha_n} = \infty$. Hence, applying Lemma 2.5 to the last inequality, we immediately obtain that $x_n \to \bar{x}$ as $n \to \infty$. This completes the proof. ∎

Utilizing Theorem 3.1, we obtain several strong convergence results in a Hilbert space.

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta, G_1, G_2 : C \times C \to \mathbf{R}$ be three bifunctions which satisfy assumptions (H1)-(H4) and $\varphi, \psi_1, \psi_2 : C \to \mathbf{R}$ be three lower semicontinuous and convex functions with assumption (A1) or (A2). Let the mappings $F, B_1, B_2 : C \to H$ be α-inverse-strongly monotone, $\tilde{\beta}_1$-inverse-strongly monotone and $\tilde{\beta}_2$-inverse-strongly monotone, respectively, and $T, T_1, T_2 : C \to H$ be η-inverse-strongly monotone, η_1-inverse-strongly monotone and η_2-inverse-strongly monotone, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap GMEP \cap \mho \neq \emptyset$. For fixed $u \in C$ and $x_0 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} z_n = T_{\lambda_n}^{(\Theta,\varphi)}(x_n - \lambda_n(F+T)x_n), \\ y_n = T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n) \\ \qquad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n)], \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $0 < \mu_1 < \min\{\tilde{\beta}_1, \eta_1\},\ 0 < \mu_2 < \min\{\tilde{\beta}_2, \eta_2\}$, *and* $0 \le \lambda_n \le \min\{\alpha, \eta\}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\} \subset [0,1]$ *satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \le \gamma_n$ *for all* $n \ge 0$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$*;*

(iv) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$*;*

(v) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \min\{\alpha, \eta\}$ *and* $\liminf_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Then, $\{x_n\}$ *converges strongly to* $\bar{x} = P_{F(S)\cap GMEP\cap\mho}u$ *and* $(\bar{x}, \bar{y})$ *is a solution of problem (1.4), where* $\bar{y} = T_{\mu_2}^{(G_2,\psi_2)}(\bar{x} - \mu_2(B_2 + T_2)\bar{x})$.

Corollary 3.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\Theta, G_1, G_2 : C \times C \to \mathbf{R}$ *be three bifunctions which satisfy assumptions (H1)-(H4) and* $\varphi, \psi_1, \psi_2 : C \to \mathbf{R}$ *be three lower semicontinuous and convex functions with assumption (A1) or (A2). Let the mappings* $B_1, B_2 : C \to H$ *be* $\tilde{\beta}_1$*-inverse-strongly monotone and* $\tilde{\beta}_2$*-inverse-strongly monotone, respectively, and* $T_1, T_2 : C \to H$ *be* η_1*-inverse-strongly monotone and* η_2*-inverse-strongly monotone, respectively. Let* $S : C \to C$ *be a* k*-strictly pseudocontractive mapping such that* $F(S) \cap MEP \cap \mho \ne \emptyset$. *For fixed* $u \in C$ *and* $x_0 \in C$ *arbitrary, let* $\{x_n\} \subset C$ *be a sequence generated by*

$$\begin{cases} \Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \ge 0,\ \forall y \in C, \\ y_n = T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2 + T_2)z_n) \\ \quad -\mu_1(B_1 + T_1)T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2 + T_2)z_n)], \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \ge 0, \end{cases}$$

where $0 < \mu_1 < \min\{\tilde{\beta}_1, \eta_1\},\ 0 < \mu_2 < \min\{\tilde{\beta}_2, \eta_2\}$, *and* $\{\lambda_n\} \subset (0, \infty), \{\alpha_n\}$, $\{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\} \subset [0,1]$ *satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \le \gamma_n$ *for all* $n \ge 0$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$*;*

(iv) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$*;*

(v) $0 < \lim_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \infty$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Then, $\{x_n\}$ *converges strongly to* $\bar{x} = P_{F(S)\cap MEP\cap\mho}u$ *and* $(\bar{x}, \bar{y})$ *is a solution of problem (1.4), where* $\bar{y} = T_{\mu_2}^{(G_2,\psi_2)}(\bar{x} - \mu_2(B_2 + T_2)\bar{x})$.

Proof. In Theorem 3.1, for all $n \geq 0$, $z_n = T_{\lambda_n}^{(\Theta,\varphi)}(x_n - \lambda_n(F+T)x_n)$ is equivalent to

$$\Theta(z_n, y)+\varphi(y)-\varphi(z_n)+\langle (F+T)x_n, y-z_n\rangle+\frac{1}{\lambda_n}\langle y-z_n, z_n-x_n\rangle \geq 0, \quad \forall y \in C.$$

Now, put $F = T \equiv 0$. Then it follows that

$$\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

Observe that for all $\alpha, \eta \in (0, \infty)$ and $x, y \in C$

$$\langle x - y, Fx - Fy\rangle \geq \alpha\|Fx - Fy\|^2 \quad \text{and} \quad \langle x - y, Tx - Ty\rangle \geq \eta\|Tx - Ty\|^2.$$

So, taking α and η in $(0, \infty)$ such that $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \min\{\alpha, \eta\}$, we obtain the desired result by Theorem 3.1. ∎

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G_1, G_2 : C \times C \to \mathbf{R}$ be two bifunctions which satisfy assumptions (H1)-(H4) and $\psi_1, \psi_2 : C \to \mathbf{R}$ be two lower semicontinuous and convex functions with assumption (A1) or (A2). Let the mappings $A, B_1, B_2 : C \to H$ be α-inverse-strongly monotone, $\tilde{\beta}_1$-inverse-strongly monotone and $\tilde{\beta}_2$-inverse-strongly monotone, respectively, and $T_1, T_2 : C \to H$ be η_1-inverse-strongly monotone and η_2-inverse-strongly monotone, respectively. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $VI(A, C) \cap F(S) \cap \mho \neq \emptyset$. For fixed $u \in C$ and $x_0 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} z_n = P_C(x_n - \lambda_n Ax_n), \\ y_n = T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2 + T_2)z_n) \\ \quad -\mu_1(B_1 + T_1)T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2 + T_2)z_n)], \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n Sy_n, \quad \forall n \geq 0, \end{cases}$$

where $0 < \mu_1 < \min\{\tilde{\beta}_1, \eta_1\}$, $0 < \mu_2 < \min\{\tilde{\beta}_2, \eta_2\}$, and $0 \leq \lambda_n \leq 2\alpha$, $\{\alpha_n\}$, $\{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\} \subset [0, 1]$ satisfy the following conditions:

(*i*) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$ *for all* $n \geq 0$;

(*ii*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(*iv*) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$;

(*v*) $0<\lim_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1})=0.$

Then, $\{x_n\}$ *converges strongly to* $\bar{x} = P_{VI(A,C)\cap F(S)\cap\mho}u$ *and* $(\bar{x}, \bar{y})$ *is a solution of problem (1.4), where* $\bar{y} = T_{\mu_2}^{(G_2,\psi_2)}(\bar{x} - \mu_2(B_2 + T_2)\bar{x})$.

Proof. In Theorem 3.1, for all $n \geq 0,\ z_n = T_{\lambda_n}^{(\Theta,\varphi)}(x_n - \lambda_n(F + T)x_n)$ is equivalent to

$$\Theta(z_n, y)+\varphi(y)-\varphi(z_n)+\langle(F+T)x_n, y-z_n\rangle+\frac{1}{\lambda_n}\langle y-z_n, z_n-x_n\rangle \geq 0, \quad \forall y \in C.$$

Now, put $\Theta \equiv 0,\ \varphi \equiv 0$ and $F = T = \frac{1}{2}A$. Then, we obtain that

$$\langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C,\ \forall n \geq 0.$$

This implies that

$$\langle y - z_n, x_n - \lambda_n Ax_n - z_n\rangle \leq 0, \quad \forall y \in C.$$

Hence it follows that $P_C(x_n - \lambda_n Ax_n) = z_n$ for all $n \geq 0$.

In the meantime, $F\ (= T)$ is 2α-inverse-strongly monotone since it is easy to see that

$$\langle x - y, \frac{1}{2}Ax - \frac{1}{2}Ay\rangle \geq 2\alpha\|\frac{1}{2}Ax - \frac{1}{2}Ay\|^2, \quad \forall y \in C.$$

So, taking 2α in $(0, \infty)$ such that $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$. Thus, we obtain the desired result by Theorem 3.1. ■

Let $A : C \to C$ be a $\tilde{\kappa}$-strictly pseudocontractive mapping. For recent convergence result for strictly pseudocontractive mappings, we refer to Zeng, Wong and Yao [28]. Putting $F = I - A$, we know that for all $x, y \in C$

$$\|(I - F)x - (I - F)y\|^2 \leq \|x - y\|^2 + \tilde{\kappa}\|Fx - Fy\|^2.$$

Note that

$$\|(I - F)x - (I - F)y\|^2 = \|x - y\|^2 + \|Fx - Fy\|^2 - 2\langle x - y, Fx - Fy\rangle.$$

Hence we have for all $x, y \in C$

$$\langle x - y, Fx - Fy\rangle \geq \frac{1 - \tilde{\kappa}}{2}\|Fx - Fy\|^2.$$

Consequently, if $A : C \to C$ is a $\tilde{\kappa}$-strictly pseudocontractive mapping, then the mapping $F = I - A$ is $(1 - \tilde{\kappa})/2$-inverse-strongly monotone.

Corollary 3.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\Theta, G_1, G_2 : C \times C \to \mathbf{R}$ *be three bifunctions which satisfy assumptions (H1)-(H4) and* $\varphi, \psi_1, \psi_2 : C \to \mathbf{R}$ *be three lower semicontinuous and convex* *functions with assumption (A1) or (A2). Let* $A : C \to C$ *be a* $\tilde{\kappa}$*-strictly pseudocontractive mapping,* $B_1, B_2 : C \to H$ *be* $\tilde{\beta}_1$*-inverse-strongly monotone and* $\tilde{\beta}_2$*-inverse-strongly monotone, respectively, and* $T_1, T_2 : C \to H$ *be* η_1*-inverse-strongly monotone and* η_2*-inverse-strongly monotone, respectively. Let* $S : C \to C$ *be a* k*-strictly pseudocontractive mapping such that* $F(S) \cap GMEP \cap \mho \neq \emptyset$*, where* $F = T = (I - A)/2$*. For fixed* $u \in C$ *and* $x_0 \in C$ *arbitrary, let* $\{x_n\} \subset C$ *be a sequence generated by*

$$(3.1) \qquad \begin{cases} z_n = T_{\lambda_n}^{(\Theta,\varphi)}((1-\lambda_n)x_n + \lambda_n A x_n), \\ y_n = T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n) \\ \qquad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n)], \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $0 < \mu_1 < \min\{\tilde{\beta}_1, \eta_1\},\ 0 < \mu_2 < \min\{\tilde{\beta}_2, \eta_2\}$*, and* $0 \leq \lambda_n \leq 1 - \tilde{\kappa},\ \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\},\ \{\delta_n\} \subset [0,1]$ *satisfy the following conditions:*

(*i*) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$ *for all* $n \geq 0$*;*

(*ii*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$*;*

(*iv*) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$*;*

(*v*) $0 < \lim_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1 - \tilde{\kappa}$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Then, $\{x_n\}$ *converges strongly to* $\bar{x} = P_{F(S)\cap GMEP\cap\mho}u$ *and* $(\bar{x}, \bar{y})$ *is a solution of problem* (*1.4*)*, where* $\bar{y} = T_{\mu_2}^{(G_2,\psi_2)}(\bar{x} - \mu_2(B_2+T_2)\bar{x})$.

Proof. Since A is a $\tilde{\kappa}$-strictly pseudocontractive mapping, the mapping $I - A$ is $(1-\tilde{\kappa})/2$-inverse-strongly monotone. In Theorem 3.1, put $F = T = (I - A)/2$. Then $F\ (= T)$ is $(1 - \tilde{\kappa})$-inverse-strongly monotone. Moreover, we obtain that

$$\begin{aligned} z_n &= T_{\lambda_n}^{(\Theta,\varphi)}(x_n - \lambda_n(F+T)x_n) \\ &= T_{\lambda_n}^{(\Theta,\varphi)}(x_n - \lambda_n(I-A)x_n) \\ &= T_{\lambda_n}^{(\Theta,\varphi)}((1-\lambda_n)x_n + \lambda_n A x_n). \end{aligned}$$

So, from Theorem 3.1, we obtain the desired result. ∎

In (3.1), if we set $\alpha_n = \alpha'_n,\ \beta_n = \beta'_n,\ \gamma_n = \gamma'_n k$ and $\delta_n = \gamma'_n(1-k)$ for all

$n \geq 0$, then we obtain the following algorithm

$$(3.25) \qquad \begin{cases} z_n = T_{\lambda_n}^{(\Theta,\varphi)}(x_n - \lambda_n(F+T)x_n), \\ y_n = T_{\mu_1}^{(G_1,\psi_1)}[T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n) \\ \qquad -\mu_1(B_1+T_1)T_{\mu_2}^{(G_2,\psi_2)}(z_n - \mu_2(B_2+T_2)z_n)], \\ x_{n+1} = \alpha_n' u + \beta_n' x_n + \gamma_n'[ky_n + (1-k)Sy_n], \quad \forall n \geq 0. \end{cases}$$

From Theorem 3.1 and (3.25), we have immediately the following corollary.

Corollary 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta, G_1, G_2 : C \times C \to \mathbf{R}$ be three bifunctions which satisfy assumptions (H1)-(H4) and $\varphi, \psi_1, \psi_2 : C \to \mathbf{R}$ be three lower semicontinuous and convex functions with assumption (A1) or (A2). Let the mappings $F, B_1, B_2 : C \to H$ be α-inverse-strongly monotone, $\tilde{\beta}_1$-inverse-strongly monotone and $\tilde{\beta}_2$-inverse-strongly monotone, respectively, and $T, T_1, T_2 : C \to H$ be η-inverse-strongly monotone, η_1-inverse-strongly monotone and η_2-inverse-strongly monotone, respectively. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $F(S) \cap GMEP \cap \mho \neq \emptyset$. Let $0 < \mu_1 < \min\{\tilde{\beta}_1, \eta_1\}$, $0 < \mu_2 < \min\{\tilde{\beta}_2, \eta_2\}$, and $0 \leq \lambda_n \leq \min\{\alpha, \eta\}$, $\{\alpha_n'\}, \{\beta_n'\}, \{\gamma_n'\} \subset [0,1]$ satisfy the following conditions:*

(*i*) $\alpha_n' + \beta_n' + \gamma_n' = 1$ *for all* $n \geq 0$;

(*ii*) $\lim_{n\to\infty} \alpha_n' = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n' = \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n' \leq \limsup_{n\to\infty} \beta_n' < 1$;

(*iv*) $0 < \lim_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \min\{\alpha, \eta\}$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

For fixed $u \in C$ and $x_0 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by (3.25). Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap GMEP\cap\mho}u$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.4), where $\bar{y} = T_{\mu_2}^{(G_2,\psi_2)}(\bar{x} - \mu_2(B_2+T_2)\bar{x})$.

Proof. It is easy to see that

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n = \alpha_n' + \beta_n' + \gamma_n' k + \gamma_n'(1-k) = 1$ and $(\gamma_n + \delta_n)k = [\gamma_n' k + \gamma_n'(1-k)]k = \gamma_n' k = \gamma_n$;

(ii) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha_n' = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \sum_{n=0}^{\infty} \alpha_n' = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n' = \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n = \limsup_{n\to\infty} \beta_n' < 1$ and $\liminf_{n\to\infty} \delta_n = \liminf_{n\to\infty} \gamma_n'(1-k) > 0$;

(iv) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = \lim_{n\to\infty} k(\frac{\gamma_{n+1}'}{1-\beta_{n+1}'} - \frac{\gamma_n'}{1-\beta_n'}) = \lim_{n\to\infty} k(\frac{\alpha_n'}{1-\beta_n'} - \frac{\gamma_{n+1}'}{1-\beta_{n+1}'}) = 0$.

Hence, all conditions of Theorem 3.1 are satisfied. Therefore, the desired conclusion follows. This completes the proof. ∎

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