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EQUIVALENCY BETWEEN THE GENERALIZED CARLESON MEASURE SPACES AND TRIEBEL-LIZORKIN-TYPE SPACES

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Abstract. In this note, we show that the homogeneous Triebel-Lizorkin-type spaces $\dot{F}^{\alpha,\tau}_{p,q}$ with four parameters defined in [7, 8] is essentially same as the generalized Carleson measure space $CMO^{\alpha,q}_r$ introduced in [6] with equivalent norms.

A sequence $\{z_n\}$ of points on the upper half complex plane is called an interpolating sequence if, given any bounded sequence $\{c_n\}$, there is a bounded analytic function F defined on the upper half complex plane such that

$$F(z_n) = c_n, \qquad n = 1, 2, 3, \cdots .$$

In order to answer a famous question whether it is possible to determine all interpolating sequences in terms of a simple geometric characterization, Carleson [1] proved that the necessary and sufficient condition for $\{z_n\}$ to be an interpolating sequence is the following condition

$$\prod_{k\neq n} \left| \frac{z_n - c_n}{z_n - \bar{z}_k} \right| \geq \delta > 0.$$

This condition is equivalent to the measure $d\mu(z) = \sum_n (\operatorname{Im} z_n)\, d\delta_{z_n}(z)$ to be the Carleson measure on the upper half plane, where $d\delta_z$ is the Dirac measure at the point z. And μ is a Carleson measure if and only if, for all $x \in \mathbb{R}$ and all $h > 0$,

$$\mu\big((x, x+h) \times (0, h)\big) \leq Ch$$

with a constant C independent of x and h.

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In 1972, Fefferman and Stein [3] proved the famous result that the dual of the Hardy space H^1 is the BMO space. Indeed, the key step to come out the result is the characterization of BMO space in terms of the Carleson measure. In 1990, Frazier and Jawerth [2] generalized the above duality to homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$.

We say that a cube $Q \subset \mathbb{R}^n$ is *dyadic* if $Q = Q_{j\mathbf{k}} = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : 2^{-j}k_i \le x_i < 2^{-j}(k_i + 1),\ i = 1, 2, \cdots, n\}$ for some $j \in \mathbb{Z}$ and $\mathbf{k} = (k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n$. Denote by $\ell(Q) = 2^{-j}$ the side length of Q and $x_Q = 2^{-j}\mathbf{k}$ the "left lower corner" of Q when $Q = Q_{j\mathbf{k}}$. Also we use $\sup_P$ to express the supremum taken over all dyadic cubes P, and denote the summation taken over all dyadic cubes Q contained in a dyadic cube P by $\sum_{Q\subset P}$. For any dyadic cubes P and Q, either P and Q are nonoverlapping or one contains the other.

Choose a fixed function φ in Schwartz class $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, the collection of rapidly decreasing C^∞ functions on $\mathbb{R}^n$, satisfying

$$\begin{cases} \operatorname{supp}(\widehat{\varphi}) \subset \{\xi \in \mathrm{R}^\mathrm{n} : 1/2 \le |\xi| \le 2\}; \\ |\widehat{\varphi}(\xi)| \ge c > 0 \qquad \text{if } \ 3/5 \le |\xi| \le 5/3. \end{cases} \tag{1}$$

Frazier and Jawerth introduced the space $\dot{F}_\infty^{\alpha,q}$, $\alpha \in \mathbb{R}$, $q \in (0, \infty]$, by the generalized Carleson measure, namely $f \in \mathcal{S}'/\mathcal{P}$ (the tempered distribution modulo polynomials) and satisfies

$$\|f\|_{\dot{F}_\infty^{\alpha,q}} := \sup_P \left\{ |P|^{-1} \int_P \sum_{Q\subset P} \Big(|Q|^{-\alpha/n-1/2} |\langle f, \varphi_Q\rangle| \chi_Q(x) \Big)^q dx \right\}^{1/q} < \infty,$$

where χ_Q denotes the characteristic function of Q and $\varphi_Q(x) = |Q|^{-1/2}\varphi((x - x_Q)/\ell(Q))$. (In case $q = \infty$, the above norm is understood as supremum norms and the same remark applies to similar places later on.) They showed [2, Theorem 5.13] that the dual of $\dot{F}_1^{\alpha,q}$ is $\dot{F}_\infty^{-\alpha,q'}$ for $\alpha \in \mathbb{R}$ and $0 < q < \infty$, where q' is the conjugate index of q.

In 2006, Han and Lu [4] introduced the generalized multiparameter Carleson measure space CMO^p, $p \le 1$, which is, for one parameter case, $f \in \mathcal{S}'/\mathcal{P}$ and satisfies

$$\|f\|_{CMO^p} := \sup_P \left\{ |P|^{1-\frac{2}{p}} \int_P \sum_{Q\subset P} \Big(|\langle f, \varphi_Q\rangle| \chi_Q(x) \Big)^2 dx \right\}^{1/2} < \infty.$$

It was proved in [5] that the dual space of the multiparameter product Hardy space H^p is the space CMO^p.

Almost at the same time, we [6] introduced the *generalized Carleson measure space* $CMO_r^{\alpha,q}$ by

$$\|f\|_{CMO_r^{\alpha,\infty}} := \sup_P |P|^{-r} \sup_{Q\subset P} |Q|^{-\alpha/n-1/2} |\langle f, \varphi_Q\rangle| < \infty$$

and

$$\|f\|_{CMO_r^{\alpha,q}} := \sup_P \left\{ |P|^{-r} \int_P \sum_{Q\subset P} \Big(|Q|^{-\alpha/n-1/2} |\langle f,\varphi_Q\rangle| \chi_Q(x) \Big)^q dx \right\}^{1/q} < \infty$$

for $\alpha, r \in \mathbb{R}$, $q \in (0,\infty)$ and $f \in \mathcal{S}'/\mathcal{P}$. We proved that, for $\alpha \in \mathbb{R}$ and $0 < p \le 1$, the dual spaces of $\dot F_p^{\alpha,q}$ for $1 < q < \infty$ and $0 < q \le 1$ can be characterized by $CMO^{-\alpha,q'}_{\frac{q'}{p}-\frac{q'}{q}}$ and $CMO^{-\alpha,\infty}_{\frac{1}{p}-1}$, respectively. Our preprint [6] were requested by several people in 2006, including the authors in [7].

In 2008, Yang and Yuan [7] (also in [8] later) introduced the so-called "unified and generalized" *Triebel-Lizorkin-type spaces* $\dot F_{p,q}^{\alpha,\tau}$ with four parameters by

$$\|f\|_{\dot F_{p,q}^{\alpha,\tau}} := \sup_P |P|^{-\tau} \left\{ \int_P \left[\sum_{Q\subset P} \Big(|Q|^{-\alpha/n-1/2} |\langle f,\varphi_Q\rangle| \chi_Q(x) \Big)^q \right]^{p/q} dx \right\}^{1/p} < \infty,$$

for $\alpha, \tau \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$ and $f \in \mathcal{S}'/\mathcal{P}$. Note that in [7] the space $\dot F_{p,q}^{\alpha,\tau}$ was defined for $\tau \in [0,\infty)$, $p \in (1,\infty)$ and $q \in (1,\infty]$.

It is clear that $CMO_r^{\alpha,q} = \dot F_{q,q}^{\alpha,r/q}$ for $0 < q < \infty$, and hence $CMO_r^{\alpha,q}$ "looks like" a special case of $\dot F_{p,q}^{\alpha,\tau}$. In fact, in this note, we prove that basically, the space $\dot F_{p,q}^{\alpha,\tau}$ is "same" as the space $CMO_r^{\alpha,q}$. This means that four parameters are totally unnecessary. The following is the first main result of this note.

Theorem 1. *Let* $\alpha, \tau \in \mathbb{R}$ *and* $p, q \in (0,\infty)$. *Then*

$$\|f\|_{\dot F_{p,q}^{\alpha,\tau}} \approx \|f\|_{\dot F_{q,q}^{\alpha,\tau+1/q-1/p}}.$$

First, as in [2], we define the sequence space $\dot f_{p,q}^{\alpha,\tau}$ by saying $\mathbf{t} = \{t_Q\}_Q \in \dot f_{p,q}^{\alpha,\tau}$ with $\alpha, \tau \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$, if

$$\|\mathbf{t}\|_{\dot f_{p,q}^{\alpha,\tau}} := \sup_P |P|^{-\tau} \left\{ \int_P \left[\sum_{Q\subset P} \Big(|Q|^{-\alpha/n-1/2} |t_Q| \chi_Q(x) \Big)^q \right]^{p/q} dx \right\}^{1/p} < \infty.$$

Note that the space $\dot f_{q,q}^{\alpha,1/q}$ coincides with $\dot f_\infty^{\alpha,q}$ defined by Frazier and Jawerth in [2] with the same norm. To show Theorem 1, as in [2], we only need to prove the following

Theorem 2. *Let* $\alpha, \tau \in \mathbb{R}$ *and* $p, q \in (0,\infty)$. *Then* $\|\mathbf{t}\|_{\dot f_{p,q}^{\alpha,\tau}} \approx \|\mathbf{t}\|_{\dot f_{q,q}^{\alpha,\tau+1/q-1/p}}$.

Assuming Theorem 2 for the moment, by the relationship between the Triebel-Lizorkin spaces and the sequence spaces (see [2] for more details), we immediately obtain $\|f\|_{\dot F_{p,q}^{\alpha,\tau}} \approx \|f\|_{\dot F_{q,q}^{\alpha,\tau+1/q-1/p}}$. Therefore, it suffices to show Theorem 2. We would also like to point out that indeed [2, Corollary 5.7] shows $\|\mathbf{t}\|_{\dot{f}^{\alpha,1/p}_{p,q}} \approx \|\mathbf{t}\|_{\dot{f}^{\alpha,1/q}_{q,q}}$. This means that [2, Corollary 5.7] shows Theorem 2 for the special case $\tau = 1/p$. To prove Theorem 2, we need the following two lemmas.

Lemma 3. *Suppose $\alpha, \tau \in \mathbb{R}$ and $p, q \in (0, \infty)$. Let $\varepsilon > 0$ be fixed. For each dyadic cube Q, if there is a set $E_Q \subset Q$ satisfying $|E_Q|/|Q| > \varepsilon$, then*

$$\|\{t_Q\}_Q\|_{\dot{f}^{\alpha,\tau}_{q,q}} \approx \sup_P |P|^{-\tau}\left(\int_P \sum_{Q\subset P}\left(|Q|^{-\alpha/n-1/2}|t_Q|\chi_{E_Q}(x)\right)^q dx\right)^{1/q}.$$

Proof. The result immediately follows by

$$\|\{t_Q\}_Q\|_{\dot{f}^{\alpha,\tau}_{q,q}} = \sup_P |P|^{-\tau}\left(\sum_{Q\subset P}\left(|Q|^{-\alpha/n-1/2}|t_Q|\right)^q |Q|\right)^{1/q} \tag{2}$$

and the equivalence $|E_Q| \approx |Q|$. ∎

Similar to [2], for a sequence $\mathbf{s} = \{s_Q\}_Q$, we define

$$G^{\alpha,\tau,q}_P(\mathbf{s})(x) = |P|^{-\tau+1/q}\left(\sum_{Q\subset P}\left(|Q|^{-\alpha/n-1/2}|s_Q|\chi_Q(x)\right)^q\right)^{1/q}.$$

Let

$$m^{\alpha,\tau,q}_P(\mathbf{s}) := \inf\left\{\varepsilon : \left|\left\{x\in P : G^{\alpha,\tau,q}_P(\mathbf{s})(x) > \varepsilon\right\}\right| < \frac{1}{4}|P|\right\} \tag{3}$$

and

$$m^{\alpha,\tau,q}(\mathbf{s})(x) := \sup_P m^{\alpha,\tau,q}_P(\mathbf{s})\chi_P(x).$$

Lemma 4. *Let $\alpha, \tau \in \mathbb{R}$ and $q \in (0, \infty)$. Then*

$$\|\mathbf{s}\|_{\dot{f}^{\alpha,\tau}_{q,q}} \approx \|m^{\alpha,\tau,q}(\mathbf{s})\|_{L^\infty}.$$

Proof. By Tchebyshev's inequality, we see that

$$\left|\left\{x\in P : G^{\alpha,\tau,q}_P(\mathbf{s})(x) > \varepsilon\right\}\right| \le \frac{1}{\varepsilon^q}\int_P \left(G^{\alpha,\tau,q}_P(\mathbf{s})(x)\right)^q dx \le \frac{|P|}{\varepsilon^q}\|\mathbf{s}\|^q_{\dot{f}^{\alpha,\tau}_{q,q}} < \frac{1}{4}|P| \tag{4}$$

if $\varepsilon > 4^{1/q}\|\mathbf{s}\|_{\dot{f}^{\alpha,\tau}_{q,q}}$. Hence, $\|m^{\alpha,\tau,q}(\mathbf{s})\|_{L^\infty} \le C\|\mathbf{s}\|_{\dot{f}^{\alpha,\tau}_{q,q}}$.

Following [2, Proposition 5.5], we define the extended integer-valued stopping time $v(x)$, $x \in \mathbb{R}^n$, by

$$v(x) = \inf\left\{j\in\mathbb{Z} : G^{\alpha,\tau,q}_P(\mathbf{s})(x) \le m^{\alpha,\tau,q}(\mathbf{s})(x),\ \ell(P) = 2^{-j}\right\}. \tag{5}$$

Also, set

$$E_Q := \Big\{x \in Q : 2^{-v(x)} \geq \ell(Q)\Big\} = \Big\{x \in Q : G_Q^{\alpha,\tau,q}(\mathbf{s})(x) \leq m^{\alpha,\tau,q}(\mathbf{s})(x)\Big\},$$

for each Q. By (3), $|E_Q|/|Q| \geq \frac{3}{4}$, and

$$\begin{aligned} &|P|^{-\tau+1/q}\bigg(\sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|s_Q|\chi_{E_Q}(x)\Big)^q\bigg)^{1/q} \\ &\quad\leq G_P^{\alpha,\tau,q}(\mathbf{s})(x) \leq m^{\alpha,\tau,q}(\mathbf{s})(x), \end{aligned} \tag{6}$$

for each $x \in \mathbb{R}^n$. Take q power of both sides of (6) and then integrate over P to obtain

$$|P|^{-\tau q}\int_P \sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|s_Q|\chi_{E_Q}(x)\Big)^q dx \leq \|m^{\alpha,\tau,q}(\mathbf{s})\|_{L^\infty}^q.$$

The last inequality and Lemma 3 yield $\|\mathbf{s}\|_{\dot{f}_{q,q}^{\alpha,\tau}} \leq C\|m^{\alpha,\tau,q}(\mathbf{s})\|_{L^\infty}$. ∎

The following corollary is similar to [2, Corollary 5.6].

Corollary 5. *Let $\alpha, \tau \in \mathbb{R}$ and $q \in (0,\infty)$. Then $\mathbf{s} = \{s_Q\}_Q \in \dot{f}_{q,q}^{\alpha,\tau}$ if and only if for each Q there is a subset $E_Q \subset Q$ with $|E_Q|/|Q| > \frac{1}{2}$ such that*

$$\bigg\|\,|P|^{-\tau+1/q}\bigg[\sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|s_Q|\chi_{E_Q}(x)\Big)^q\bigg]^{1/q}\bigg\|_{L^\infty} < \infty. \tag{7}$$

Moreover, the infimum of this expression over all such collections $\{E_Q\}_Q$ is equivalent to $\|\mathbf{s}\|_{\dot{f}_{q,q}^{\alpha,\tau}}$.

Proof. If $\mathbf{s} \in \dot{f}_{q,q}^{\alpha,\tau}$, the E_Q chosen in the proof of Lemma 4 above yields (7). The converse follows from Lemma 3. ∎

We are ready to prove Theorem 2.

Proof of Theorem 2. By the definition, it is equivalent to prove

$$\begin{aligned} &\sup_P |P|^{-\tau}\bigg\{\int_P \bigg[\sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|t_Q|\chi_Q(x)\Big)^q\bigg]^{p/q} dx\bigg\}^{1/p} \\ &\approx \sup_P |P|^{-\tau-1/q+1/p}\bigg\{\int_P \sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|t_Q|\chi_Q(x)\Big)^q dx\bigg\}^{1/q}. \end{aligned} \tag{8}$$

Let us consider the case $p > q$ first. By Hölder's inequality,

$$
\begin{aligned}
&|P|^{-\tau-1/q+1/p}\left\{\int_P \sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|t_Q|\chi_Q(x)\Big)^q dx\right\}^{1/q}\\
&\quad\le |P|^{-\tau-1/q+1/p}\left(\left\{\int_P\Big[\sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|t_Q|\chi_Q(x)\Big)^q\Big]^{p/q}dx\right\}^{q/p}|P|^{1-q/p}\right)^{1/q}\\
&\quad= |P|^{-\tau}\left\{\int_P\Big[\sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|t_Q|\chi_Q(x)\Big)^q\Big]^{p/q}dx\right\}^{1/p}.
\end{aligned}
$$

On the other hand, if P is a fixed dyadic cube and E_Q's are the subsets chosen in Corollary 5, then, by the facts $\chi_Q(x) \le CM(\chi_{E_Q})(x)$ and $p > q$,

$$
\begin{aligned}
&|P|^{-\tau p}\int_P\Big[\sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|t_Q|\chi_Q(x)\Big)^q\Big]^{p/q}dx\\
&\quad\le C|P|^{-\tau p}\int_{\mathbb{R}^n}\Big[M\sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|t_Q|\chi_{E_Q}(\cdot)\Big)^q(x)\Big]^{p/q}dx\\
&\quad\le C|P|^{-\tau p}\int_P\Big[\sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|t_Q|\chi_{E_Q}(x)\Big)^q\Big]^{p/q}dx\\
&\quad= C|P|^{-1}\int_P\left\{|P|^{-\tau+1/p}\Big[\sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|t_Q|\chi_{E_Q}(x)\Big)^q\Big]^{1/q}\right\}^p dx,
\end{aligned}
$$

where M is the Hardy-Littlewood maximal function. Now by (6) the right-hand side of the last inequality is clearly less than or equal to

$$C\|m^{\alpha,\tau+1/q-1/p,q}(\mathbf{t})\|_{L^\infty}^p,$$

and by Lemma 4 this is dominated by $\|\mathbf{t}\|^p_{\dot{f}_{q,q}^{\alpha,\tau+1/q-1/p}}$.

If $p \le q$, using Hölder's inequality again, we have

$$
\begin{aligned}
&|P|^{-\tau}\left\{\int_P\Big[\sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|t_Q|\chi_Q(x)\Big)^q\Big]^{p/q}dx\right\}^{1/p}\\
&\quad\le |P|^{-\tau}\left(\left\{\int_P\sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|t_Q|\chi_Q(x)\Big)^q dx\right\}^{p/q}|P|^{1-p/q}\right)^{1/p}\\
&\quad= |P|^{-\tau-1/q+1/p}\left\{\int_P\sum_{Q\subset P}\Big(|Q|^{-\alpha/n-1/2}|t_Q|\chi_Q(x)\Big)^q dx\right\}^{1/q},
\end{aligned}
$$

which shows $\|\mathbf{t}\|_{\dot{f}_{p,q}^{\alpha,\tau}} \leq \|\mathbf{t}\|_{\dot{f}_{q,q}^{\alpha,\tau+1/q-1/p}}$ and $\dot{f}_{q,q}^{\alpha,\tau+1/q-1/p} \subset \dot{f}_{p,q}^{\alpha,\tau}$. To verify the converse inequality, let $H(x) = \sum_{Q\subset P}\left(|Q|^{-\alpha/n-1/2}|t_Q|\chi_{E_Q}(x)\right)^q$. By (5) and Lemma 4,

$$H(x) \leq |P|^{\tau q-q/p}\|m^{\alpha,\tau+1/q-1/p,q}(\mathbf{t})\|_{L^\infty}^q \leq C|P|^{\tau q-q/p}\|\mathbf{t}\|_{\dot{f}_{q,q}^{\alpha,\tau+1/q-1/p}}^q,$$

and we have

$$H(x) \leq C[H(x)]^{p/q}\big(|P|^{\tau-1/p}\|\mathbf{t}\|_{\dot{f}_{q,q}^{\alpha,\tau+1/q-1/p}}\big)^{q-p}.$$

This has already yielded the result when $\mathbf{t} \in \dot{f}_{q,q}^{\alpha,\tau+1/q-1/p}$. In fact, if $\mathbf{t} \in \dot{f}_{q,q}^{\alpha,\tau+1/q-1/p}$, then

$$\begin{aligned}|P|^{-\tau q-1+q/p}\int_P H(x)dx &\leq C|P|^{-\tau p}\left(\int_P [H(x)]^{p/q}dx\right)\|\mathbf{t}\|_{\dot{f}_{q,q}^{\alpha,\tau+1/q-1/p}}^{q-p}\\ &\leq C\|\mathbf{t}\|_{\dot{f}_{p,q}^{\alpha,\tau}}^p\|\mathbf{t}\|_{\dot{f}_{q,q}^{\alpha,\tau+1/q-1/p}}^{q-p},\end{aligned}$$

which gives $\|\mathbf{t}\|_{\dot{f}_{q,q}^{\alpha,\tau+1/q-1/p}} \leq C\|\mathbf{t}\|_{\dot{f}_{p,q}^{\alpha,\tau}}$. The general case is followed by using the monotone convergence theorem. That is, given $\mathbf{t} = \{t_Q\}_Q \in \dot{f}_{p,q}^{\alpha,\tau}$, let $\mathcal{Q}(\mathbf{t})$ denote the collection of all dyadic cubes Q so that $t_Q \neq 0$. Since the collection of all dyadic cubes in $\mathbb{R}^n$ is countable, the set $\mathcal{Q}(\mathbf{t})$ is countable and enumerated as $\{P_1, P_2, P_3, \cdots\}$. For $n \in \mathbb{N}$, define $\mathbf{t}_n = \{(t_n)_Q\}_Q$ by

$$(t_n)_Q = \begin{cases} t_Q & \text{if } \ Q \in \{P_1, P_2, \cdots, P_n\} \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, $\mathbf{t}_n$ converges to $\mathbf{t}$ in $\dot{f}_{p,q}^{\alpha,\tau}$ as n tends to infinity. Moreover, $\mathbf{t}_n \in \dot{f}_{q,q}^{\alpha,\tau+1/q-1/p}$ and $\|\mathbf{t}_n\|_{\dot{f}_{q,q}^{\alpha,\tau+1/q-1/p}}$ is a monotone sequence uniformly bounded by a multiple of $\|\mathbf{t}\|_{\dot{f}_{p,q}^{\alpha,\tau}}$. Therefore, $\|\mathbf{t}\|_{\dot{f}_{q,q}^{\alpha,\tau+1/q-1/p}} \leq C\|\mathbf{t}\|_{\dot{f}_{p,q}^{\alpha,\tau}}$. ∎

The proof of Theorem 2 is completed, and hence Theorem 1 follows. As a consequence of Theorem 1, we obtain

Corollary 6. *Let* $\alpha, \tau \in \mathbb{R}$ *and* $p, q \in (0,\infty)$. *Then*

$$\|f\|_{\dot{F}_{p,q}^{\alpha,\tau}} \approx \|f\|_{CMO_{\tau q+1-q/p}^{\alpha,q}}.$$

References

1. L. Carleson, An interpolation problem for bounded analytic functions, *Amer. J. Math.*, **80** (1958), 921-930.

2. M. Frazier and B. Jawerth, A discrete transform and decomposition of distribution spaces, *J. Funct. Anal.*, **93** (1990), 34-170.

3. C. Fefferman and E. M. Stein, H^p spaces of several variables, *Acta Math.*, **129** (1972), 137-193.

4. Y. Han and G. Lu, Discrete Littlewood-Paley-Stein theory and multi-parameter Hardy spaces associated with the flag singular integrals, available at http://arXiv.org/abs/0801.1701.

5. Y. Han, J. Li and G. Lu, Duality of multiparameter Hardy space H^p on product spaces of homogeneous type, *Ann. Scuola Norm. Sup. Pisa CI. Sci.*, **9** (2010), 645-685.

6. C.-C. Lin and K. Wang, Generalized Carleson measure spaces and their applications, preprint.

7. D. Yang and W. Yuan, A new class of function spaces connecting Triebel-Lizorkin spaces and Q spaces, *J. Funct. Anal.*, **255** (2008), 2760-2809.

8. D. Yang and W. Yuan, New Besov-type spaces and Triebel-Lizorkin-type spaces including Q spaces, *Math. Z.*, **265** (2010), 451-480.

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