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CONTROLLABILITY FOR NONLINEAR VARIATIONAL INEQUALITIES OF PARABOLIC TYPE

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Abstract. This paper deal with the approximate controllability for the nonlinear functional differential control problem governed by the variational inequality. Sufficient conditions for the approximate controllability of the system are discussed under the bounded condition on the controller operator independent of the the time interval. We also prove the regularity and norm estimations for solutions of the given problems.

1. Introduction

Let H and V be two complex Hilbert spaces. Assume that V is a dense subspace in H and the injection of V into H is continuous. The norms on V and H will be denoted by $||\cdot||$ and $|\cdot|$, respectively. Let A be a continuous linear operator from V into V^* which is assumed to satisfy

$$(Au,u) \geq \omega_1 ||u||^2 - \omega_2 |u|^2$$

where $\omega_1 > 0$ and ω_2 is a real number and let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Consider the following the variational inequality problem with nonlinear term:

$$\text{(VIP)} \quad \begin{cases} (x'(t)+Ax(t), x(t)-z) + \phi(x(t)) - \phi(z) \\ \quad \leq (f(t,x(t)) + k(t), x(t)-z), \text{ a.e., } 0 < t \leq T, \quad z \in V \\ x(0) = x_0. \end{cases}$$

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According to the subdifferential operator $\partial\phi$, the problem (VIP) is represented by the following nonlinear functional differential problem on H:

$$\text{(NDE)} \qquad \begin{cases} x'(t) + Ax(t) + \partial\phi(x(t)) \ni f(t, x(t)) + k(t), & 0 < t \leq T, \\ x(0) = x_0. \end{cases}$$

The existence and regularity for the parabolic variational inequality in the linear case($f \equiv 0$), which was first investigated by Brézis [2], has been developed as seen in Section 4.3.2 of Barbu [3](also see Section 4.3.1 in [4]).

First, in Section 2 we will deal with the existence for solutions of (NDE) when the nonlinear mapping f is a Lipschitz continuous from $\mathbb{R} \times V$ into H and the norm estimate of a solution of the above nonlinear equation on $L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \cap C([0,T];H)$ as seen in [8]. Consequently, in view of the monotonicity of $\partial\phi$, we show that the mapping

$$H \times L^2(0,T;V^*) \ni (x_0, k) \mapsto x \in L^2(0,T;V) \cap C([0,T];H)$$

is continuous.

Thereafter, we can obtain the approximate controllability for the nonlinear functional differential control problem governed by the variational inequality in Section 4. Let U be a complex Banach space and B be a bounded linear operator from $L^2(0,T;U)$ to $L^2(0,T;H)$. Let us consider the following control system governed by the variational inequality problem with the control term Bu instead of k:

$$\text{(NCE)} \qquad \begin{cases} x'(t) + Ax(t) + \partial\phi(x(t)) \ni f(t, x(t)) + (Bu)(t), & 0 < t \leq T, \\ x(0) = x_0. \end{cases}$$

For every $\epsilon > 0$, we define the Moreau-Yosida approximation of ϕ as

$$\phi_\epsilon(x) = \inf\{||x - y||_*^2/2\epsilon + \phi(y) : y \in H\}.$$

Then the function ϕ_ϵ is Fréchet differentiable on H. By using the facts that its Fréchet differential $\partial\phi_\epsilon$ is a single valued and Lipschitz continuous on H, we investigate the control problem of (NCE) by transforming onto the semilinear differential equation with $\partial\phi_\epsilon$ in place of $\partial\phi$ and obtain the norm estimate of a solution of the above nonlinear equation on $L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \cap C([0,T];H)$ in Section 3.

In recent years, as for the controllability for semilinear differential equations, Carrasco and Lebia [7] discussed sufficient conditions for approximate controllability of a system of parabolic equations with delay, Mahmudov [11] in case the semilinear equations with nonlocal conditions with condition on the uniform boundedness of the Frechet derivative of nonlinear term, and Sakthivel et al. [13] on impulsive and neutral functional differential equations.

In this paper, in order to show the investigate the approximate controllability problem for (NCE), we assume range conditions of the controller B, which is that for any $\varepsilon > 0$ and $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$\begin{cases} |\int_0^T S(T-s)\{p(s)-(Bu)(s)\}| < \varepsilon, \\ ||Bu||_{L^2(0,t;H)} \le q_1||p||_{L^2(0,t;H)}, \quad 0 \le t \le T, \end{cases}$$

where q_1 is a constant independent of p and $S(t)$ is an analytic semigroup generated by A.

Here, we remark that the quantity condition of the constant q_1 as seen in Zhou ([15]; (3.3)) is not necessary. Some examples to which main result can be applied are given in [12, 15].

If $D(A)$ is compactly embedded in V (or the semigroup operator $S(t)$ is compact), the following embedding

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V))$$

is compact in view of Theorem 2 of Aubin [1]. Hence, the mapping $u \mapsto x$ is compact from $L^2(0,T;U)$ to $L^2(0,T;V)$. From these results we can obtain the approximate controllability for the equation (NCE), which is the extended result of Naito [12] to the equation (NCE). Finally, a simple examples which our main result can be applied is given.

2. Preliminaries

Forming Gelfand triple $V \subset H \subset V^*$ with pivot space H, for the sake of simplicity, we may consider

$$||u||_* \le |u| \le ||u||, \quad u \in V$$

where $||\cdot||_*$ is the norm of the element of V^*. We also assume that there exists a constant C_1 such that

$$||u|| \le C_1||u||_{D(A)}^{1/2}|u|^{1/2} \tag{2.1}$$

for every $u \in D(A)$, where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$. Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Garding's inequality

$$\text{Re } a(u,u) \geq \omega_1 ||u||^2 - \omega_2 |u|^2, \quad u \in V \tag{2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number.

Let A be the operator associated with the sesquilinear form $a(\cdot,\cdot)$:

$$(Au, v) = a(u,v), \quad u, \ v \in V.$$

Then A is a bounded linear operator from V to V^* and $-A$ generates an analytic semigroup in both of H and V^* as is seen in [[14]; Theorem 3.6.1]. The realization for the operator A in H which is the restriction of A to

$$D(A) = \{u \in V; Au \in H\}$$

be also denoted by A.

The following L^2-regularity for the abstract linear parabolic equation

$$\text{(LE)} \qquad \begin{cases} x'(t) + Ax(t) = k(t), \quad 0 < t \leq T, \\ x(0) = x_0 \end{cases}$$

has a unique solution x in $[0,T]$ for each $T > 0$ if $x_0 \in (D(A),H)_{1/2,2}$ and $k \in L^2(0,T;H)$ where $(D(A),H)_{1/2,2}$ is the real interpolation space between $D(A)$ and H. Moreover, we have

$$||x||_{L^2(0,T;D(A)) \cap W^{1,2}(0,T,H)} \leq C_2(||x_0||_{(D(A),H)_{1/2,2}} + ||k||_{L^2(0,T;H)}) \tag{2.3}$$

where C_2 depends on T and M (see Theorem 2.3 of [5], [10]).

Let $0 < \theta < 1$, $1 < p < \infty$. Then by considering an intermediate method between the initial Banach space and the domain of the infinitesimal generator A of the analytic semigroup $T(t)$ is represented by

$$(V,V^*)_{\theta,p} = \{x \in V^* : \int_0^\infty (t^\theta ||Ae^{tA}x||_*)^p \frac{dt}{t} < \infty\}$$

(see Theorem 3.5.3 of [6]). If an operator A is bounded linear from V to V^* associated with the sesquilinear form $a(\cdot,\cdot)$ then it is easily seen that

$$H = \{x \in V^* : \int_0^T ||Ae^{tA}x||_*^2 dt < \infty\},$$

for the time $T > 0$. Therefore, in terms of the intermediate theory we can see that

$$(V,V^*)_{1/2,2} = H$$

and obtain the following results.

Proposition 2.1. *Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution x of (LE) belonging to*

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_2(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.4}$$

where C_2 is a constant depending on T.

Let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Then the subdifferential operator $\partial\phi$ of ϕ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \leq \phi(y) + (x^*, x - y), \quad y \in V\}.$$

First, let us concern with the following perturbation of subdifferential operator:

$$\text{(VE)} \qquad \begin{cases} x'(t) + Ax(t) + \partial\phi(x(t)) \ni k(t), \quad 0 < t \leq T, \\ x(0) = x_0. \end{cases}$$

Using the regularity for the variational inequality of parabolic type as seen in [2; Section 4.3] we have the following result on the equation (VE).

Proposition 2.2. (1) *Let $k \in L^2(0,T;V^*)$ and $x_0 \in \overline{D(\phi)}$ where $\overline{D(\phi)}$ is the closure in H of the set $D(\phi) = \{u \in V : \phi(u) < \infty\}$. Then the equation (VE) has a unique solution*

$$x \in L^2(0,T;V) \cap C([0,T];H),$$

which satisfies

$$x'(t) = (k(t) - Ax(t) - \partial\phi(x(t)))^0$$

and

$$||x||_{L^2 \cap C} \leq C_3(1 + |x_0| + ||k||_{L^2(0,T;V^*)}) \tag{2.5}$$

where C_3 is some positive constant and $L^2 \cap C = L^2(0,T;V) \cap C([0,T];H)$ and where $(\partial\phi)^0$ is the minimal segment of $\partial\phi$.

(2) *Let A be symmetric and let us assume that there exists $h \in H$ such that for every $\epsilon > 0$ and any $y \in D(\phi)$*

$$J_\epsilon(y + \epsilon h) \in D(\phi) \text{ and } \phi(J_\epsilon(y + \epsilon h)) \leq \phi(y)$$

where $J_\epsilon = (I+\epsilon A)^{-1}$. *Then for* $k \in L^2(0,T;H)$ *and* $x_0 \in \overline{D(\phi)} \cap V$ *the equation (VE) has a unique solution*

$$x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \cap C([0,T];H),$$

which satisfies

$$||x||_{L^2\cap W^{1,2}\cap C} \le C_3(1+||x_0||+||k||_{L^2(0,T;H)}). \tag{2.6}$$

Here, we remark that if $D(A)$ is compactly embedded in V and $x \in L^2(0,T;$ $D(A))$ (or the semigroup operator $S(t)$ generated by A is compact), the following embedding

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V))$$

is compact in view of Theorem 2 of Aubin [1]. Hence, the mapping $k \mapsto x$ is compact from $L^2(0,T;H)$ to $L^2(0,T;V)$, which is applicable to the control problem.

Let f be a nonlinear single valued mapping from $[0,\infty)\times V$ into H. We assume that

$$|f(t,x_1) - f(t,x_2)| \le L||x_1 - x_2||, \tag{F}$$

for every $x_1,\ x_2 \in V$.

The following result is from Jeong and Park [9].

Theorem 2.1. ([9]). *Let the assumption (F) be satisfied. Assume that* $k \in L^2(0,T;V^*)$ *and* $x_0 \in \overline{D(\phi)}$. *Then, the equation (NDE) has a unique solution*

$$x \in L^2(0,T;V) \cap C([0,T];H)$$

and there exists a constant C_4 *depending on* T *such that*

$$||x||_{L^2\cap C} \le C_4(1+|x_0|+||k||_{L^2(0,T;V^*)}). \tag{2.7}$$

Furthermore, if $k \in L^2(0,T;H)$ *then the solution* x *belongs to* $W^{1,2}(0,T;H)$ *and satisfies*

$$||x||_{W^{1,2}(0,T;H)} \le C_4(1+|x_0|+||k||_{L^2(0,T;H)}). \tag{2.8}$$

If $(x_0,k) \in H \times L^2(0,T;H)$, *then the solution* x *of the equation (NDE) belongs to* $x \in L^2(0,T;V) \cap C([0,T];H)$ *and the mapping*

$$H \times L^2(0,T;H) \ni (x_0,k) \mapsto x \in L^2(0,T;V) \cap C([0,T];H)$$

is continuous.

3. Smoothing System Corresponding to (NCE)

For every $\epsilon > 0$, define

$$\phi_\epsilon(x) = \inf\{||x-y||_*^2/2\epsilon + \phi(y) : y \in H\}.$$

Then the function ϕ_ϵ is Fréchet differentiable on H and its Frećhet differential $\partial\phi_\epsilon$ is Lipschitz continuous on H with Lipschitz constant ϵ^{-1} where $\partial\phi_\epsilon = \epsilon^{-1}(I - (I + \epsilon\partial\phi)^{-1}]$ as is seen in Corollary 2.2 in [[4]; Chapter II). It is also well known results that $\lim_{\epsilon\to 0}\phi_\epsilon = \phi$ and $\lim_{\epsilon\to 0}\partial\phi_\epsilon(x) = (\partial\phi)^0(x)$ for every $x \in D(\partial\phi)$.

Now, we introduce the smoothing system corresponding to (NCE) as follows.

$$\text{(SCE)} \quad \begin{cases} x'(t) + Ax(t) + \partial\phi_\epsilon(x(t)) = f(t, x(t)) + (Bu)(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases}$$

Since $-A$ generates a semigroup $S(t)$ on H, the mild solution of (SCE) can be represented by

$$x_\epsilon(t) = S(t)x_0 + \int_0^t S(t-s)\{f(s, x_\epsilon(s)) + (Bu)(s) - \partial\phi_\epsilon(x_\epsilon(s))\}ds.$$

In virtue of Theorem 2.1 we know that if the assumption (F) is satisfied then for every $x_0 \in H$ and every $u \in L^2(0,T;U)$ the equation (SCE) has a unique solution

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \cap C([0,T];H)$$

and there exists a constant C_4 depending on T such that

$$||x||_{L^2\cap W^{1,2}\cap C} \le C_4(1 + |x_0| + ||u||_{L^2(0,T;U)}). \tag{3.1}$$

Now, we assume the hypothesis that $(\partial\phi)^0$ is uniformly bounded, i.e.,

$$|(\partial\phi)^0 x| \le M_1, \quad x \in H. \tag{A}$$

Lemma 3.1. *Let x_ϵ and x_λ be the solutions of (SCE) with same control u. Then there exists a constant C independent of ϵ and λ such that*

$$||x_\epsilon - x_\lambda||_{C([0,T];H)\cap L^2(0,T;V)} \le C(\epsilon + \lambda), \quad 0 < T.$$

Proof. For given ϵ, $\lambda > 0$, let x_ϵ and x_λ be the solutions of (SCE) corresponding to ϵ and λ, respectively. Then from the equation (SCE) we have

$$\begin{aligned} &x_\epsilon'(t) - x_\lambda'(t) + A(x_\epsilon(t) - x_\lambda(t)) + \partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)) \\ &= f(t, x_\epsilon(t)) - f(t, x_\lambda(t)), \end{aligned}$$

and hence, from (2.2) and multiplying by $x_\epsilon(t) - x_\lambda(t)$, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|x_\epsilon(t) - x_\lambda(t)|^2 + \omega_1||x_\epsilon(t) - x_\lambda(t)||^2 \\
&+(\partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)), x_\epsilon(t) - x_\lambda(t)) \qquad (3.2)\\
\le\ &(f(t, x_\epsilon(t)) - f(t, x_\lambda(t)), x_\epsilon(t) - x_\lambda(t)) + \omega_2|x_\epsilon(t) - x_\lambda(t)|^2.
\end{aligned}
$$

Let us choose a constant $c > 0$ such that $2\omega_1 - cL > 0$. Noting that

$$
\begin{aligned}
&(f(t, x_\epsilon(t)) - f(t, x_\lambda(t)), x_\epsilon(t) - x_\lambda(t)) \\
&\le |f(t, x_\epsilon(t)) - f(t, x_\lambda(t))|\,|x_\epsilon(t) - x_\lambda(t)| \\
&\le \frac{cL}{2}||x_\epsilon(t) - x_\lambda(t)||^2 + \frac{1}{2c}|x_\epsilon(t) - x_\lambda(t)|^2,
\end{aligned}
$$

by integrating (3.2) over $[0, T]$ and using the monotonicity of $\partial\phi$ we have

$$
\begin{aligned}
&\frac{1}{2}|x_\epsilon(t) - x_\lambda(t)|^2 + (\omega_1 - \frac{cL}{2})\int_0^T ||x_\epsilon(t) - x_\lambda(t)||^2 dt \\
\le &\int_0^T (\partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)), \lambda\partial\phi_\lambda(x_\lambda(t) - \epsilon\partial\phi_\epsilon(x_\epsilon(t))dt \\
&+ \Big(\frac{1}{2c} + \omega_2\Big)\int_0^T |x_\epsilon(t) - x_\lambda(t)|^2 dt.
\end{aligned}
$$

Here, we used that

$$\partial\phi_\epsilon(x_\epsilon(t)) = \epsilon^{-1}(x_\epsilon(t) - (I + \epsilon\partial\phi)^{-1}x_\epsilon(t)).$$

Since $|\partial\phi_\epsilon(x)| \le |(\partial\phi)^0 x|$ for every $x \in D(\partial\phi)$ it follows from (A) and using Gronwall's inequality that

$$||x_\epsilon - x_\lambda||_{C([0,T];H)\cap L^2(0,T;V)} \le C(\epsilon + \lambda), \quad 0 < T. \qquad \blacksquare$$

Theorem 3.1. *Let the assumptions* (*F*) *and* (*A*) *be satisfied. Then* $x = \lim_{\epsilon\to} x_\epsilon$ *in* $L^2(0, T; V) \cap C([0, T]; H)$ *is a solution of the equation* (*NCE*) *where* x_ϵ *is the solution of* (*SCE*).

Proof. In virtue of Lemma 3.1, there exists $x(\cdot) \in L^2(0, T; V)$ such that

$$x_\epsilon(\cdot) \to x(\cdot) \quad \text{in} \quad L^2(0, T; V) \cap C([0, T]; H).$$

From (F) it follows that

$$f(\cdot, x_\epsilon) \to f(\cdot, x), \quad \text{strongly in } L^2(0, T; H) \qquad (3.3)$$

and

$$Ax_n \to Ax, \quad \text{strongly in } L^2(0,T;V^*). \tag{3.4}$$

Since $\partial\phi_\epsilon(x_\epsilon)$ are uniformly bounded by assumption (A), from (3.3), (3.4) we have that

$$\frac{d}{dt}x_\epsilon \to \frac{d}{dt}x, \quad \text{weakly in } L^2(0,T;V^*),$$

therefore

$$\partial\phi_\epsilon(x_\epsilon) \to f(\cdot,x) + k - x' - Ax, \quad \text{weakly in } L^2(0,T;V^*),$$

Note that $\partial\phi_\epsilon(x_\epsilon) = \partial\phi((I+\epsilon\partial\phi)^{-1}x_\epsilon)$. Since $(I+\epsilon\partial\phi)^{-1}x_\epsilon \to x$ strongly and $\partial\phi$ is demiclosed, we have that

$$f(\cdot,x) + k - x' - Ax \in \partial\phi(x) \text{ in } L^2(0,T;V^*).$$

Thus we have proved that $x(t)$ satisfies a.e. on $(0,T)$ the equation (NCE). ∎

4. Approximate Controllability

In this section we show the approximate controllability for the equation (NCE) with the more general condition for the range of the control operator, which is the extended result of Zhou [[15]; Section 3] and Naito [12] to the equation (SCE).

For the sake of simplicity we assume that the solution semigroup $S(t)$ is uniformed bounded:

$$|S(t)| \le M \quad t \ge 0.$$

Lemma 4.1. *Let $u_i \in L^2(0,T;U)$ and $x_{\epsilon i}$ be the solution of (SCE) with u_i in place of u for $i = 1,\ 2$. Then there exists a constant $C > 0$ such that*

$$|x_{\epsilon 1}(t) - x_{\epsilon 2}(t)| \le M\sqrt{t}\{C(\epsilon^{-1} + L) + 1\}||Bu_1 - Bu_2||_{L^2(0,t;H)}$$

for $0 < t \le T$.

Proof. In virtue of Theorem 2.1 it holds that there exists a constant $C > 0$ such that

$$||x_{\epsilon 1} - x_{\epsilon 2}||_{L^2(0,t;V)} \le C||Bu_1 - Bu_2||_{L^2(0,t;H)}, \quad t > 0.$$

The proof of Lemma 3.1 is a consequence of the estimate

$$
\begin{aligned}
|x_{\epsilon 1}(t) - x_{\epsilon 2}(t)| = \Big| \int_0^t & S(t-s)[\{f(s, x_{\epsilon 1}(s)) - f(s, x_{\epsilon 2}(t))\} \\
& + \{\partial\phi_\epsilon(x_{\epsilon 1}(s) - \partial\phi_\epsilon(x_{\epsilon 2}(s)))\} + \{(Bu_1)(s) - (Bu_2)(s)\}]ds \Big| \\
\le & M\sqrt{t}(\epsilon^{-1} + L)||x_{\epsilon 1} - x_{\epsilon 2}||_{L^2(0,t;V)} + M\sqrt{t}||Bu_1 - Bu_2||_{L^2(0,t;H)} \\
\le & M\sqrt{t}\{C(\epsilon^{-1} + L) + 1\}||Bu_1 - Bu_2||_{L^2(0,t;H)}.
\end{aligned}
$$ ■

We denote the linear operator $\hat{S}$ from $L^2(0, T; H)$ to H by

$$\hat{S}p = \int_0^T S(T-s)p(s)ds$$

for $p \in L^2(0, T; H)$. The system (SCE) is approximately controllable on $[0, T]$ if for any $\varepsilon > 0$ and $\xi_T \in H$ there exists a control $u \in L^2(0, T; U)$ such that

$$|\xi_T - S(T)x_0 - \hat{S}\{f(\cdot, x_\epsilon(\cdot; g)) - \partial\phi_\epsilon(x_\epsilon(\cdot)\} - \hat{S}Bu| < \varepsilon.$$

We need the following hypothesis:

For any $\varepsilon > 0$ and $p \in L^2(0, T; H)$ there exists a $u \in L^2(0, T; U)$ such that

$$\text{(B)} \qquad \begin{cases} |\hat{S}p - \hat{S}Bu| < \varepsilon, \\ ||Bu||_{L^2(0,t;H)} \le q_1||p||_{L^2(0,t;H)}, \quad 0 \le t \le T. \end{cases}$$

where q_1 is a constant independent of p.

Remark. If the range of B is dense in $L^2(0, T; H)$ then Hypothesis (B) is satisfied (Theorem 3.3 of [12]). Some examples to which main result can be applied are given in [12, 15]. Those examples will be given which show that even if the range of B is not dense in $L^2(0, T; H)$. In [15], Zhou proved that such a system is approximately controllable under Hypothesis (B) dependent of the time T. In this paper, sufficient conditions for the approximate controllability of the system (SCE) are no need to assume the condition on the length T of the time interval, which has a simple form and can be easily checked in many examples. So this sufficient condition is more general than previous ones. It is suitable not only for a nonlinear abstract control system in Hilbert space, but also for the finite dimensional ordinary differential equations by using the spectral projection operator with finite rank associated with the generalized eigenspace.

The solutions of (NCE) and (SCE) are denoted by $x(t; \phi, f, u)$ and $x_\epsilon(t; \phi_\epsilon, f, u)$, respectively.

Theorem 4.1. *Under the assumptions* (F) *and* (B)*, the system* (SCE) *is approximately controllable on* $[0, T]$.

Proof. We shall show that

$$D(A) \subset \mathrm{Cl}\{x_\epsilon(T; \phi_\epsilon, f, u) : u \in L^2(0, T; U)\}$$

where Cl denotes the closure in H, i.e., for given $\varepsilon > 0$ and $\xi_T \in D(A)$ there exists $u \in L^2(0, T; U)$ such that

$$|\xi_T - x_\epsilon(T; \phi_\epsilon, f, u)| < \varepsilon,$$

where

$$\begin{aligned} x_\epsilon(t; \phi_\epsilon, f, u) =& S(T)x_0 + \int_0^T S(T-s)\{f(s, x_\epsilon(s; \phi_\epsilon, f, u)) \\ &- \partial\phi_\epsilon(x_\epsilon(s; \phi_\epsilon, f, u) + (Bu)(s)\}ds. \end{aligned}$$

As $\xi_T \in D(A)$ there exists a $p \in L^2(0, T; H)$ such that

$$\hat{S}p = \xi_T - S(T)x_0,$$

for instance, take $p(s) = (\xi_T + sA\xi_T - S(s)x_0)/T$.

Set

$$F(x_\epsilon(s; \phi_\epsilon, f, u)) = f(s, x_\epsilon(s; \phi_\epsilon, f, u)) - \partial\phi_\epsilon(x_\epsilon(s; \phi_\epsilon, f, u)).$$

Then

$$\begin{aligned} &|F(x_\epsilon(s; \phi_\epsilon, f, u_1)) - F(x_\epsilon(s; \phi_\epsilon, f, u_2))| \\ &\leq (\epsilon^{-1} + L)||x_\epsilon(s; \phi_\epsilon, f, u_1) - x_\epsilon(s; \phi_\epsilon, f, u_2)||. \end{aligned}$$

Let $u_1 \in L^2(0, T; U)$ be arbitrary fixed. Since by the assumption (B) there exists $u_2 \in L^2(0, T; U)$ such that

$$|\hat{S}(p - F(x_\epsilon(\cdot; \phi_\epsilon, f, u_1))) - \hat{S}Bu_2| < \frac{\varepsilon}{4},$$

it follows

$$|\xi_T - S(T)x_0 - \hat{S}F(x_\epsilon(\cdot; \phi_\epsilon, f, u_1)) - \hat{S}Bu_2| < \frac{\varepsilon}{4}. \tag{4.1}$$

We can also choose $w_2 \in L^2(0, T; U)$ by the assumption (B) such that

$$|\hat{S}(F(x_\epsilon(\cdot; \phi_\epsilon, f, u_2)) - F(x_\epsilon(\cdot; \phi_\epsilon, f, u_1))) - \hat{S}Bw_2| < \frac{\varepsilon}{8} \tag{4.2}$$

and

$$\|Bw_2\|_{L^2(0,t;H)} \le q_1 \|F(x_\epsilon(\cdot;\phi_\epsilon,f,u_1)) - F(x_\epsilon(\cdot;\phi_\epsilon,f,u_2))\|_{L^2(0,t;H)}$$

for $0 \le t \le T$. Therefore, in view of Lemma 4.1 and the assumption (B)

$$\begin{aligned}
||Bw_2||_{L^2(0,t;H)} &\le q_1 \Big\{ \int_0^t |F(x_\epsilon(\tau;\phi_\epsilon,f,u_2)) - F(x_\epsilon(\tau;\phi_\epsilon,f,u_1))|^2 d\tau \Big\}^{\frac{1}{2}} \\
&\le q_1(\epsilon^{-1}+L) \Big\{ \int_0^t ||x_\epsilon(\tau;\phi_\epsilon,f,u_1) - x_\epsilon(\tau;\phi_\epsilon,f,u_2))||^2 d\tau \Big\}^{\frac{1}{2}} \\
&\le q_1(\epsilon^{-1}+L) \Big[\int_0^t M^2 \{C(\epsilon^{-1}+L)+1\}^2 \tau ||Bu_2 - Bu_1||^2_{L^2(0,\tau;H)} d\tau \Big]^{\frac{1}{2}} \\
&\le q_1 M(\epsilon^{-1}+L)\{C(\epsilon^{-1}+L)+1\} \Big(\int_0^t \tau d\tau \Big)^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;H)} \\
&= q_1 M(\epsilon^{-1}+L)\{C(\epsilon^{-1}+L)+1\} \Big(\frac{t^2}{2} \Big)^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;H)}.
\end{aligned}$$

Put $u_3 = u_2 - w_2$. We determine w_3 such that

$$|\hat{S}(F(x_\epsilon(\cdot;\phi_\epsilon,f,u_3))) - F(x_\epsilon(\cdot;\phi_\epsilon,f,u_2))) - \hat{S}Bw_3| < \frac{\varepsilon}{8},$$

$$||Bw_3||_{L^2(0,t;H)} \le q_1 ||F(x_\epsilon(\cdot;\phi_\epsilon,f,u_3)) - F(x_\epsilon(\cdot;\phi_\epsilon,f,u_2))||_{L^2(0,t;H)}$$

for $0 \le t \le T$. Hence, we have

$$\begin{aligned}
&||Bw_3||_{L^2(0,t;H)} \\
&\le q_1 \Big\{ \int_0^t |F(x_\epsilon(\tau;\phi_\epsilon,f,u_3)) - F(x_\epsilon(\tau;\phi_\epsilon,f,u_2))|^2 d\tau \Big\}^{\frac{1}{2}} \\
&\le q_1(\epsilon^{-1}+L) \Big\{ \int_0^t ||x_\epsilon(\tau;\phi_\epsilon,f,u_3) - x_\epsilon(\tau;\phi_\epsilon,f,u_2)||^2 d\tau \Big\}^{\frac{1}{2}} \\
&\le q_1 M(\epsilon^{-1}+L)\{C(\epsilon^{-1}+L)+1\} \Big\{ \int_0^t \tau ||Bu_3 - Bu_2||^2_{L^2(0,\tau:H)} d\tau \Big\}^{\frac{1}{2}} \\
&\le q_1 M(\epsilon^{-1}+L)\{C(\epsilon^{-1}+L)+1\} \Big\{ \int_0^t \tau ||Bw_2||^2_{L^2(0,\tau;H)} d\tau \Big\}^{\frac{1}{2}} \\
&\le q_1 M(\epsilon^{-1}+L)\{C(\epsilon^{-1}+L)+1\} \\
&\quad \Big\{ \int_0^t \tau [q_1 M(\epsilon^{-1}+L)\{C(\epsilon^{-1}+L)+1\}]^2 \frac{\tau^2}{2} ||Bu_2 - Bu_1||^2_{L^2(0,\tau;H)} d\tau \Big\}^{\frac{1}{2}}
\end{aligned}$$

$$\leq [q_1 M(\epsilon^{-1}+L)\{C(\epsilon^{-1}+L)+1\}]^2\Big(\int_0^t \frac{\tau^3}{2}d\tau\Big)^{\frac{1}{2}}||Bu_2 - Bu_1||_{L^2(0,t;H)}$$
$$= [q_1 M(\epsilon^{-1}+L)\{C(\epsilon^{-1}+L)+1\}]^2\Big(\frac{t^4}{2\cdot 4}\Big)^{\frac{1}{2}}||Bu_2 - Bu_1||_{L^2(0,t;H)}.$$

By proceeding this process for $u_{n+1} = u_n - w_n (n = 1, 2, ...)$, and from that

$$||B(u_n - u_{n+1})||_{L^2(0,t;H)} = ||Bw_n||_{L^2(0,t;H)}$$
$$\leq [q_1 M(\epsilon^{-1}+L)\{C(\epsilon^{-1}+L)+1\}]^{n-1}$$
$$\Big(\frac{t^{2n-2}}{2\cdot 4\cdots(2n-2)}\Big)^{\frac{1}{2}}||Bu_2 - Bu_1||_{L^2(0,t;H)}$$
$$= [\frac{q_1 TM(\epsilon^{-1}+L)\{C(\epsilon^{-1}+L)+1\}}{\sqrt{2}}]^{n-1}\frac{1}{\sqrt{(n-1)!}}||Bu_2 - Bu_1||_{L^2(0,t;H)},$$

it follows that

$$\sum_{n=1}^{\infty}||Bu_{n+1} - Bu_n||_{L^2(0,T;H)}$$
$$\leq \sum_{n=0}^{\infty}\Big[\frac{q_1 TM(\epsilon^{-1}+L)\{C(\epsilon^{-1}+L)+1\}}{\sqrt{2}}\Big]^n\frac{1}{\sqrt{n!}}||Bu_2 - Bu_1||_{L^2(0,T;H)}$$
$$< \infty.$$

Therefore, there exists $u^* \in L^2(0,T;H)$ such that

$$\lim_{n\to\infty} Bu_n = u^* \quad \text{in} \quad L^2(0,T;H). \tag{4.3}$$

From (4.1), (4.2) it follows that

$$|\xi_T - S(T)x_0 - \hat{S}F(x_\epsilon(\cdot;\phi_\epsilon,f,u_2)) - \hat{S}Bu_3|$$
$$= |\xi_T - S(T)x_0 - \hat{S}F(x_\epsilon(\cdot;\phi_\epsilon,f,u_1)) - \hat{S}Bu_2 + \hat{S}Bw_2$$
$$- \hat{S}[F(x_\epsilon(\cdot;\phi_\epsilon,f,u_2)) - F(x_\epsilon(\cdot;\phi_\epsilon,f,u_1))]|$$
$$< (\frac{1}{2^2}+\frac{1}{2^3})\varepsilon.$$

By choosing choose $w_n \in L^2(0,T;U)$ by the assumption (B) such that

$$|\hat{S}(F(x_\epsilon(\cdot;\phi_\epsilon,f,u_n)) - F(x_\epsilon(\cdot;\phi_\epsilon,f,u_{n-1})) - \hat{S}Bw_n| < \frac{\varepsilon}{2^{n+1}},$$

since $u_{n+1} = u_n - w_n$, we have

$$|\xi_T - S(T)x_0 - \hat{S}F(x_\epsilon(\cdot;\phi_\epsilon, f, u_n)) - \hat{S}Bu_{n+1}| < \Big(\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}}\Big)\varepsilon, \quad n = 1\ 2,\ \cdots.$$

According to (4.3) for $\varepsilon > 0$ there exists integer N such that

$$|\hat{S}Bu_{N+1} - \hat{S}Bu_N| < \frac{\varepsilon}{2}$$

and

$$\begin{aligned} &|\xi_T - S(T)x_0 - \hat{S}F(x_\epsilon(\cdot;\phi_\epsilon, f, u_N)) - \hat{S}Bu_N| \\ &\le |\xi_T - S(T)x_0 - \hat{S}F(x_\epsilon(\cdot;\phi_\epsilon, f, u_N)) - \hat{S}Bu_{N+1}| \\ &\quad + |\hat{S}Bu_{N+1} - \hat{S}\Phi u_N| \\ &< \Big(\frac{1}{2^2} + \cdots + \frac{1}{2^{N+1}}\Big)\varepsilon + \frac{\varepsilon}{2} \le \varepsilon. \end{aligned}$$

Thus the system (SCE) is approximately controllable on $[0, T]$ as N tends to infinity. ■

From Theorem 3.1 and Theorem 4.1 we obtain the following results.

Theorem 4.2. *Under the assumptions (A), (F) and (B), the system (NCE) is approximately controllable on* $[0, T]$.

Example. Let Ω be a region in an n-dimensional Euclidean space $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and closure $\overline{\Omega}$. $C^m(\Omega)$ is the set of all m-times continuously differential functions on Ω. $C_0^m(\Omega)$ will denote the subspace of $C^m(\Omega)$ consisting of these functions which have compact support in Ω.

For $1 \le p \le \infty$, $W^{m,p}(\Omega)$ is the set of all functions $f = f(x)$ whose derivative $D^\alpha f$ up to degree m in distribution sense belong to $L^p(\Omega)$. As usual, the norm is then given by

$$||f||_{m,p} = \Big(\sum_{\alpha \le m} ||D^\alpha f||_p^p\Big)^{\frac{1}{p}}, \quad 1 \le p < \infty, \quad ||f||_{m,\infty} = \max_{\alpha \le m} ||D^\alpha u||_\infty,$$

where $D^0 f = f$. In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ with the norm $||\cdot||_p$. $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. For $p = 2$ we denote $W^{m,2}(\Omega) = H^m(\Omega)$ (simply, $W^{1,2}(\Omega) = H(\Omega)$), $W_0^{m,2}(\Omega) = H_0^m(\Omega)$. $H^{-1}(\Omega)$ stands for the dual space $W_0^{1,2}(\Omega)^*$ whose norm is denoted by $||\cdot||_{-1}$.

From now on, we consider a Gelfand triple as $V = H_0(\Omega)$, $H = L^2(\Omega)$ and $V = H^{-1}(\Omega)$ to discuss our problems given in Section 2.

We consider the control problem of the following variational inequality problem:

$$
\text{(4.4)} \quad \begin{cases} (\partial/\partial t)u(x,t) + \mathcal{A}(x,D_x)u(x,t), u(x,t)-z) \\ \quad + \int_\Omega |\mathrm{grad}\, u(t,x)|^2 dx - \int_\Omega |\mathrm{grad}\, z(t,x)|^2 dx \\ \leq \left(\int_0^t k(t-s)g(s,x(s))ds + (B_\alpha w(t))(x)), u(x,t) - z(x,t) \right), \\ \qquad (x,t) \in \Omega \times (0,T], \quad z(\cdot,t) \in H_0(\Omega), \\ u(x,t) = 0, \qquad x \in \partial\Omega,\ t \in (0,T], \end{cases}
$$

Here, $\mathcal{A}(x,D_x)$ is a second order linear differential operator with smooth coefficients in $\overline{\Omega}$, and $\mathcal{A}(x,D_x)$ is elliptic. If we put that $Au = \mathcal{A}(x,D_x)u$ then it is known that $-A$ generates an analytic semigroup in $H^{-1}(\Omega)$ as is seen in [9].

We denote the realization of $\mathcal{A}$ in $L^2(\Omega)$ under the Dirichlet boundary condition by $\hat{A}$:

$$
D(\hat{A}) = H^2(\Omega) \cap H_0(\Omega),
$$

$$
\hat{A}u = \mathcal{A}u \quad \text{for} \quad u \in D(\hat{A}).
$$

The operator $-\hat{A}$ generates analytic semigroup in $L^2(\Omega)$. From now on, both A and $\hat{A}$ are denoted simply by A. So, we may consider that $-A$ generates an analytic semigroup in both of $H = L^p(\Omega)$ and $V^* = H^{-1}(\Omega)$ as seen in Section 2.

For every $u \in H_0(\Omega)$ define

$$
\phi(u) = \begin{cases} \int_\Omega |\mathrm{grad}\, u)|^2 dx, & \text{if } u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ +\infty, & \text{otherwise.} \end{cases}
$$

It is easy to check if ϕ is proper and lower semicontinuous on V to $(-\infty, +\infty]$(see in Section 2.3 of [4]).

Let $g : [0,T] \times V \longrightarrow H$ be a nonlinear mapping such that $t \mapsto g(t,x)$ is measurable and

$$
\text{(4.5)} \qquad |g(t,x) - g(t,y)| \leq L||x-y||,
$$

for a positive constant L. We assume that $g(t,0) = 0$ for the sake of simplicity.

For $x \in L^2(0,T;V)$ we set

$$
f(t,x) = \int_0^t k(t-s)g(s,x(s))ds
$$

where k belongs to $L^2(0,T)$. By (4.5) it is easily seen that the nonlinear term f satisfies hypothesis (F) in Section 2.

Let $U = H$, $0 < \alpha < T$ and define the intercept controller operator B_α on $L^2(0,T;H)$ by

$$B_\alpha u(t) = \begin{cases} 0, & 0 \le t < \alpha, \\ u(t), & \alpha \le t \le T \end{cases}$$

for $u \in L^2(0,T;H)$. For a given $p \in L^2(0,T;H)$ let us choose a control function u satisfying

$$u(t) = \begin{cases} 0, \quad 0 \le t < \alpha, \\ p(t) + \dfrac{\alpha}{T-\alpha} S\Big(t - \dfrac{\alpha}{T-\alpha}(t-\alpha)\Big) p\Big(\dfrac{\alpha}{T-\alpha}(t-\alpha)\Big), \ \alpha \le t \le T. \end{cases}$$

Then $u \in L^2(0,T;H)$ and $\hat{S}p = \hat{S}B_\alpha u$. From the following:

$$\begin{aligned} ||B_\alpha u||_{L^2(0,T;H)} &= ||u||_{L^2(\alpha,T;H)} \\ &\le ||p||_{L^2(\alpha,T;H)} + \frac{\alpha M}{T-\alpha}\Big\| p\Big(\frac{\alpha}{T-\alpha}(\cdot - \alpha)\Big)\Big\|_{L^2(\alpha,T;H)} \\ &\le \Big(1 + M\sqrt{\frac{\alpha}{T-\alpha}}\Big)||p||_{L^2(0,T;H)} \end{aligned}$$

it follows that the controller B_α satisfies hypothesis (B). Hence from Theorem 4.1, 4.2, it follows that the system (4.4) is approximately controllable on $[0,T]$.

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