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EMBEDDED WAVEFORM RELAXATION METHODS FOR PARABOLIC PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Waveform relaxation methods are decoupling or splitting methods for large scale ordinary differential equations. In this paper, we apply the methods directly to semi-linear parabolic partial functional differential equations. Taking into consideration of the complicated forms of these parabolic equations, we propose a kind of embedded waveform relaxation methods, which are in fact two-level waveform relaxation methods and which can also be applied to some other systems. We provide explicit iterative expressions of the embedded methods and exhibit the superlinear rate of convergence on finite time intervals. We also apply the two-level idea to the functional differential equations derived from semi-discretization of the original system. The windowing technique is employed for the situation of long time intervals. Finally, two numerical experiments are performed to confirm our theory.

1. Introduction

Waveform relaxations are a kind of dynamic iteration methods used to solve large systems of time dependent equations in parallel. This kind of methods are totally different from some classical numerical methods, such as Runge-Kutta method and multi-step method. If we regard a waveform as an approximate curve of the original system, the dynamic iteration method will produce a series of waveforms, each of which is defined on the whole time interval.

Waveform relaxation methods were originally proposed in [11] to simulate large circuits, and well developed in the past several years. Most of them are proposed for ordinary differential equations [15, 13] and differential algebra equations [8, 9]. In

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these applications, systems are decoupled and subsystems can be solved numerically in parallel with different time steps. Therefore, Waveform relaxation methods are quite suitable for stiff systems. Another advantage of waveform relaxation methods is to convert the originally complicated systems into a series of simpler systems which are also more easily to be solved.

To apply waveform relaxations for solving partial differential equations (PDEs), one traditionally discretizes the PDEs in space to get a large system of ODEs, and then apply waveform relaxation methods to solve the resulted ODEs. There are two concerns with this approach: firstly, information about relaxation processes may be lost during the processes of discretization; secondly, the convergence results derived in this fashion depend on the mesh parameters and convergence rates would deteriorate when the meshes are refined. A kind of Schwarz waveform relaxation methods is shown in [5].

In this paper we study approximate solutions to the initial boundary value problem of the following parabolic partial differential equation with a functional term,

$$\frac{\partial}{\partial t}u(\mathbf{x},t) = a\Delta u + g(\mathbf{x},t,u(\mathbf{x},t),u_{(\mathbf{x},t)}(\tau)), \quad \mathbf{x}\in\Omega, \tag{1}$$

where $a > 0$, $\tau \in [0, \tau_0]$, with initial boundary conditions

$$\begin{cases} u(\mathbf{x},t) = \varphi(\mathbf{x},t), & t\in[-\tau_0,0], & \mathbf{x}\in\Omega, \\ u(\mathbf{x},t) = \phi(\mathbf{x},t), & t\in[0,T], & \mathbf{x}\in\partial\Omega. \end{cases}$$

The function $u_{(\mathbf{x},t)}(\cdot)$ is defined by

$$u_{(\mathbf{x},t)}(\tau) = u(\mathbf{x},t+\tau), \quad \tau\in[-\tau_0,0].$$

The nonlinear function g has different representations, which means that Eq. (1) indeed covers many different kinds of equations. As shown in [12], if the function

$$g(\mathbf{x},t,u,v) = f(\mathbf{x},t,u,v(-\tau_0)),$$

then system (1) has the form

$$\frac{\partial}{\partial t}u(\mathbf{x},t) = a\Delta u + f(\mathbf{x},t,u(\mathbf{x},t),u(\mathbf{x},t-\tau_0));$$

while if the function

$$g(\mathbf{x},t,u,v) = f\left(\mathbf{x},t,u,\int_{-\tau_0}^{0} v(\tau)d\tau\right),$$

system (1) becomes

$$\frac{\partial}{\partial t}u(\mathbf{x},t) = a\Delta u + f\left(\mathbf{x},t,u(\mathbf{x},t),\int_{-\tau_0}^{0} u(\mathbf{x},t+\tau)d\tau\right).$$

Functional equations like (1) are used to model cancer cells in human tumors and heat conduction in materials with memory viscoelasticity. Problem (1) is also popular in population dynamics. A set of examples which illustrate the wide range of such models can be found in [16]. Several different methods have been used to solve the problem, such as iterated pseudospectral method [12], traveling waves [2], adaptive higher order method [4], and so on.

Waveform relaxation methods apply to similar systems as system (1) at the ODE level [7, 17], where the nonlinear delay partial differential equations are spatially discretized. We also want to apply waveform relaxation algorithms to system (1), but at the PDE level directly. Differently from the Schwarz waveform relaxation methods, we do not decompose the spatial domain into subdomains, but keep the spatial domain in one piece, and construct an iteration scheme from the original systems. Therefore, we do not need to consider the information transmission among subdomains and the convergence under different transmission conditions.

The outline of our paper is as follows. In Section 2, we propose a kind of embedded waveform relaxation methods for partial functional equations with fixed delays, present the iteration expressions, and analyze the corresponding convergence theory. In order to improve behavior of the approximate waveforms on long time interval, we bring windowing technique into the embedded waveform relaxations. In Section 3, we discuss applications of the idea in parabolic functional differential equations with mixed delays. The convergence analysis of semi-discrete situations is given in Section 4. Numerical experiments are carried out in Section 5.

2. Equations with Fixed Delays

In this section, we take a simple parabolic partial functional differential equation in one dimensional space as follows to show the form of embedded waveform relaxation methods and the corresponding convergence. Consider the system

$$(2)\qquad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} = a\dfrac{\partial^2 u(x,t)}{\partial x^2} + g(x,t,u(x,t),u(x,t-\tau)), \\ \qquad 0 < x < l, \quad 0 < t < T, \\ u(0,t) = u(l,t) = 0, \quad 0 \le t \le T, \\ u(x,t) = h(x,t), \quad 0 \le x \le l, \quad -\tau_0 \le t \le 0, \end{cases}$$

where g is a given nonlinear function. A natural kind of waveform relaxation method for system (2) is to generate a sequence of approximate solutions $\{u^{(k)}\}$ by the following algorithm:

$$\begin{cases} \dfrac{\partial u^{(k+1)}(x,t)}{\partial t} = a\dfrac{\partial^2 u^{(k+1)}(x,t)}{\partial x^2} \\ \qquad +G(x,t,u^{(k+1)}(x,t),u^{(k)}(x,t),u^{(k)}(x,t-\tau)), \\ u^{(k+1)}(0,t) = u^{(k+1)}(l,t) = 0, \quad 0 \le t \le T, \\ u^{(k+1)}(x,0) = h(x,0), \quad 0 \le x \le l, \quad k = 0,1,\cdots, \end{cases}$$

where, for any fixed k, $u^{(k)}(x,t) = h(x,t)$, $-\tau_0 \le t \le 0$. We usually choose the initial guess $u^{(0)}(x,t) = h(x,0)$, $0 \le t \le T$. For any fixed k, such system is a nonlinear reaction diffusion equation without functional term.

In fact, what we care about is the limit of the series of the waveforms, rather than every individual waveform in the series. Therefore, we would better make sure that the waveforms in the series could be solved easily. Considering the nonlinear reaction diffusion equations is still expensive to solve, we embed an internal waveform relaxation to approximate the nonlinear reaction diffusion equations, just like the two-level scheme. This idea leads to the following embedded waveform relaxation process:

$$(3) \quad \begin{cases} \dfrac{\partial u^{(k+\frac{i+1}{m})}(x,t)}{\partial t} = a\dfrac{\partial^2 u^{(k+\frac{i+1}{m})}(x,t)}{\partial x^2} \\ \qquad +G(x,t,u^{(k+\frac{i}{m})}(x,t),u^{(k)}(x,t),u^{(k)}(x,t-\tau)), \\ u^{(k+\frac{i+1}{m})}(0,t) = u^{(k+\frac{i+1}{m})}(l,t) = 0, \quad 0 \le t \le T, \\ u^{(k+\frac{i+1}{m})}(x,0)=h(x,0), \quad 0\le x\le l,\ i=0,1,\cdots,m-1,\ k=0,1,\cdots. \end{cases}$$

We take the same initial function on $[0,l]\times[-\tau_0,0]$ and the same initial guess as before. We assume that the splitting function G satisfies

$$G(x,t,u,u,v) = g(x,t,u,v), \quad \forall u \in C([0,l]\times[0,T]), \quad v \in C([0,l]\times[-\tau_0,T]).$$

It is easy to find that, for any fixed i and k, system (3) is a linear diffusion equation. We also regard the corresponding solution as a "waveform". In the following discussion we will prove convergence of the relaxation process.

We know easily that the solution of equation (3) for any fixed k is

$$(4) \quad \begin{aligned} & u^{(k+\frac{i+1}{m})}(x,t) \\ &= \frac{2}{l}\sum_{n=1}^{\infty}\int_0^l h(\xi,0)\sin\frac{n\pi\xi}{l}d\xi e^{-a\left(\frac{n\pi}{l}\right)^2 t}\sin\frac{n\pi x}{l} \\ &+\int_0^t\left(\sum_{n=1}^{\infty}\frac{2}{l}\int_0^l G(\xi,s,u^{(k+\frac{i}{m})}(\xi,s),u^{(k)}(\xi,s),u^{(k)}(\xi,s-\tau)\right)\sin\frac{n\pi\xi}{l}d\xi \\ &\times e^{-a(\frac{n\pi}{l})^2(t-s)}\sin\tfrac{n\pi x}{l})ds. \end{aligned}$$

We define a function sequence

$$
\begin{aligned}
a_n^{(k+\frac{i+1}{m})}(t) \; :=& \; \frac{2}{l}\int_0^l h(\xi,0)\sin\frac{n\pi\xi}{l}d\xi e^{-a\left(\frac{n\pi}{l}\right)^2 t} \\
&+\int_0^t \left(\frac{2}{l}\int_0^l G(\xi,s,u^{(k+\frac{i}{m})}(\xi,s),u^{(k)}(\xi,s),u^{(k)}(\xi,s-\tau)) \right. \\
&\left. \sin\frac{n\pi\xi}{l}d\xi e^{-a(\frac{n\pi}{l})^2(t-s)}\right) ds.
\end{aligned} \tag{5}
$$

The solution of equation (3) has the expression

$$
u^{(k+\frac{i+1}{m})}(x,t)=\sum_{n=1}^{\infty} a_n^{(k+\frac{i+1}{m})}(t)\sin\frac{n\pi x}{l}.
$$

We also define the function

$$
\begin{aligned}
a_n(t) \; :=& \; \frac{2}{l}\int_0^l h(\xi,0)\sin\frac{n\pi\xi}{l}d\xi e^{-a\left(\frac{n\pi}{l}\right)^2 t} \\
&+\int_0^t \left(\frac{2}{l}\int_0^l g(\xi,s,u(\xi,s),u(\xi,s-\tau))\sin\frac{n\pi\xi}{l}d\xi e^{-a(\frac{n\pi}{l})^2(t-s)}\right) ds.
\end{aligned} \tag{6}
$$

and it is not hard to verify that the function $\sum\limits_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{l}$ satisfies equation (2).

Next, we show the convergence of the relaxation series (4) and the rate of convergence.

Theorem 2.1. *Assume that the partial derivative of nonlinear function G with respect to its j-th argument is bounded by constant M_j, where $j = 3, 4, 5$. Then the approximate solution $u^{(k+\frac{i}{m})}(x,t)$ of the embedded waveform relaxation method (3) converges to the exact solution of system (2), and satisfies*

$$
\begin{aligned}
&\max_{\substack{0\le x\le l\\ 0\le t\le T}}\left|u^{(k)}(x,t)-\sum_{n=1}^{\infty}a_n(t)\sin\frac{n\pi x}{l}\right| \\
&\le \frac{(CT)^k}{k!}\max_{\substack{0\le x\le l\\ 0\le t\le T}}\left|u^{(0)}(x,t)-\sum_{n=1}^{\infty}a_n(t)\sin\frac{n\pi x}{l}\right|,
\end{aligned}
$$

where m is a positive integer, $i = 0, 1, \cdots, m-1$, $k = 1, 2, \cdots$, and

$$
C=\left(\frac{2MM_3}{l}\right)^m\frac{T^{m-1}}{(m-1)!}+\frac{M_4+M_5}{M_3}\sum_{j=1}^{m-1}\left(\frac{2MM_3}{l}\right)^j\frac{T^j}{j!}+\frac{2M(M_4+M_5)}{l}
$$

is a constant. Moreover, the rate of convergence of the embedded waveform relaxation method is superlinear.

Proof. Define

$$\varepsilon^{(k+\frac{i}{m})}(x,t) := u^{(k+\frac{i}{m})}(x,t) - \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{l}, \quad 0 \le x \le l, \quad -\tau_0 \le t \le T,$$

where, for $-\tau_0 \le t \le 0$ and $0 \le x \le l$, $\varepsilon^{(k+\frac{i}{m})}(x,t) \equiv 0$. From the expressions (4) and (6), the error function $\varepsilon^{(k+\frac{i+1}{m})}(x,t)$ can be written directly as

$$\begin{aligned}
&\varepsilon^{(k+\frac{i+1}{m})}(x,t)\\
&= \int_0^t \Bigg(\sum_{j=1}^{+\infty} \frac{2}{l} \int_0^l \big[G(\xi, s, u^{(k+\frac{i}{m})}(\xi,s), u^{(k)}(\xi,s), u^{(k)}(\xi, s-\tau))\\
&\quad - G(\xi, s, u(\xi,s), u(\xi,s), u(\xi,s-\tau)) \big] \sin \frac{j\pi\xi}{l} d\xi \cdot e^{-a(\frac{j\pi}{l})^2(t-s)} \sin \frac{j\pi x}{l} \Bigg) ds.
\end{aligned}$$

We notice that

$$\begin{aligned}
&G(\xi, s, u^{(k+\frac{i}{m})}(\xi,s), u^{(k)}(\xi,s), u^{(k)}(\xi,s-\tau)) - G(\xi, s, u(\xi,s), u(\xi,s), u(\xi,s-\tau))\\
&= G'_3 \varepsilon^{(k+\frac{i}{m})}(\xi,s) + G'_4 \varepsilon^{(k)}(\xi,s) + G'_5 \varepsilon^{(k)}(\xi, s-\tau),
\end{aligned}$$

where

$$G'_j = G'_j \left(\xi, s, u_*^{(k+\frac{i}{m})}(\xi,s), u_*^{(k)}(\xi,s), u_*^{(k)}(\xi,s-\tau) \right)$$

denotes the j-th partial derivative of the nonlinear function G with respect to its j-th argument, $j = 3, 4, 5$, and

$$u_*^{(k+\frac{i}{m})}(\xi,s) \in \left[\min(u^{(k+\frac{i}{m})}(\xi,s), u(\xi,s)), \max(u^{(k+\frac{i}{m})}(\xi,s), u(\xi,s)) \right],$$

so do $u_*^{(k)}(\xi,s)$ and $u_*^{(k)}(\xi,s-\tau)$).

Then,

$$\begin{aligned}
&\varepsilon^{(k+\frac{i+1}{m})}(x,t)\\
&= \int_0^t \Bigg(\sum_{j=1}^{+\infty} \frac{2}{l} \int_0^l \big[G'_3 \varepsilon^{(k+\frac{i}{m})}(\xi,s) + G'_4 \varepsilon^{(k)}(\xi,s) + G'_5 \varepsilon^{(k)}(\xi,s-\tau) \big] \sin \frac{j\pi\xi}{l} d\xi\\
&\quad \times e^{-a(\frac{j\pi}{l})^2(t-s)} \sin \frac{j\pi x}{l} \Bigg) ds\\
&= \int_0^t \Bigg(\sum_{j=1}^{+\infty} \frac{2}{l} \Big[G'_3(\tilde{\xi},s) \varepsilon^{(k+\frac{i}{m})}(\tilde{\xi},s) + G'_4(\tilde{\xi},s) \varepsilon^{(k)}(\tilde{\xi},s) + G'_5(\tilde{\xi},s) \varepsilon^{(k)}(\tilde{\xi},s-\tau) \Big]
\end{aligned}$$

$$\times\int_0^l \sin\frac{j\pi\xi}{l}d\xi\cdot e^{-a(\frac{j\pi}{l})^2(t-s)}\sin\frac{j\pi x}{l}\Bigg)ds$$

$$= \int_0^t \Bigg(\sum_{j=1}^{+\infty}\frac{2}{l}\Big[G_3'(\tilde\xi,s)\varepsilon^{(k+\frac{i}{m})}(\tilde\xi,s)+G_4'(\tilde\xi,s)\varepsilon^{(k)}(\tilde\xi,s)+G_5'(\tilde\xi,s)\varepsilon^{(k)}(\tilde\xi,s-\tau)\Big]$$

$$\times\frac{l}{j\pi}[1-(-1)^j]e^{-a(\frac{j\pi}{l})^2(t-s)}\sin\frac{j\pi x}{l}\Bigg)ds$$

$$= \int_0^t \frac{2}{l}\Big[G_3'(\tilde\xi,s)\varepsilon^{(k+\frac{i}{m})}(\tilde\xi,s)+G_4'(\tilde\xi,s)\varepsilon^{(k)}(\tilde\xi,s)+G_5'(\tilde\xi,s)\varepsilon^{(k)}(\tilde\xi,s-\tau)\Big]$$

$$\times\left(\sum_{j=1}^{+\infty}\frac{2l}{(2j-1)\pi}e^{-a(\frac{2j\pi-\pi}{l})^2(t-s)}\sin\frac{(2j-1)\pi x}{l}\right)ds,$$

where $\tilde\xi\in[0,l]$. Let us take a look at the series of functions in the expression above. For any $0<x<l$, we have

$$\begin{aligned}\left|\sum_{j=1}^{N}\sin\frac{(2j-1)\pi x}{l}\right| &= \left|\frac{1}{\sin\frac{\pi x}{l}}\sum_{j=1}^{N}\sin\frac{(2j-1)\pi x}{l}\sin\frac{\pi x}{l}\right|\\ &= \left|\frac{1}{\sin\frac{\pi x}{l}}\sum_{j=1}^{N}\frac{1}{2}\left(\cos\frac{(2j-2)\pi x}{l}-\cos\frac{2j\pi x}{l}\right)\right|\\ &= \left|\frac{1}{\sin\frac{\pi x}{l}}\frac{1}{2}\left(1-\cos\frac{2N\pi x}{l}\right)\right|\\ &\le \left|\frac{1}{\sin\frac{\pi x}{l}}\right|.\end{aligned}$$

This means that the partial sum of the series $\sum\sin\frac{(2m-1)\pi x}{l}$ is bounded. On the other hand, for any fixed $0\le s\le t$, $\frac{2l}{(2j-1)\pi}e^{-a(\frac{2j\pi-\pi}{l})^2(t-s)}$ is a monotone decreasing function with respect to index j on $[0,t]$ uniformly. According to the Dirichlet rule, the series

$$\sum_{j=1}^{\infty}\frac{2l}{(2j-1)\pi}e^{-a(\frac{2j\pi-\pi}{l})^2(t-s)}\sin\frac{(2j-1)\pi x}{l}$$

converge uniformly on $[0,t]$, and assume M is a uniform bound. Since we assume that the three partial derivatives of the nonlinear function G are bounded by M_3, M_4 and M_5 respectively, we obtain that,

$$|\varepsilon^{(k+\frac{i+1}{m})}(x,t)|\le\frac{2M}{l}\int_0^t\Big(M_3|\varepsilon^{(k+\frac{i}{m})}(\tilde\xi,s)|+M_4|\varepsilon^{(k)}(\tilde\xi,s)|+M_5|\varepsilon^{(k)}(\tilde\xi,s-\tau)|\Big)ds.$$

Notice that, for $-\tau_0 \le t \le 0$, $\varepsilon^{(k)}(x,t) \equiv 0$, and

$$\int_0^t |\varepsilon^{(k)}(\tilde{\xi}, s-\tau)|ds = \int_{-\tau}^{t-\tau} |\varepsilon^{(k)}(\tilde{\xi}, s)|ds \le \int_0^t |\varepsilon^{(k)}(\tilde{\xi}, s)|ds.$$

If we define the norm $\|\varepsilon^{(k)}\|(t) = \max_{0\le x\le l} |\varepsilon^{(k)}(x,t)|$, then we obtain,

$$\|\varepsilon^{(k+\frac{i+1}{m})}\|(t) \le \frac{2M}{l}\int_0^t \left[M_3\|\varepsilon^{(k+\frac{i}{m})}\|(s) + (M_4+M_5)\|\varepsilon^{(k)}\|(s)\right] ds.$$

By induction we get

$$\begin{aligned} \|\varepsilon^{(k+\frac{m-1}{m})}\|(t) \le & \left(\frac{2MM_3}{l}\right)^{m-1} \frac{t^{m-1}}{(m-1)!} \max_{0\le s\le t} \|\varepsilon^{(k)}\|(s) \\ & + \frac{M_4+M_5}{M_3}\left[\sum_{j=1}^{m-1}\left(\frac{2MM_3}{l}\right)^j \frac{t^j}{j!}\right] \max_{0\le s\le t}\|\varepsilon^{(k)}\|(s) \end{aligned}$$

and

$$\begin{aligned} \|\varepsilon^{(k+1)}\|(t) \le & \frac{2M}{l}\int_0^t \left[M_3\|\varepsilon^{(k+\frac{m-1}{m})}\|(s) + (M_4+M_5)\|\varepsilon^{(k)}\|(s)\right] ds \\ = & \int_0^t \left[\left(\frac{2MM_3}{l}\right)^m \frac{s^{m-1}}{(m-1)!} + \frac{M_4+M_5}{M_3}\sum_{j=1}^{m-1}\left(\frac{2MM_3}{l}\right)^{j+1}\frac{s^j}{j!} \right. \\ & \left. + \frac{2M(M_4+M_5)}{l}\right] \max_{0\le \mu\le s}\|\varepsilon^{(k)}\|(\mu) ds. \end{aligned}$$

Set

$$C = \left(\frac{2MM_3}{l}\right)^m \frac{T^{m-1}}{(m-1)!} + \frac{M_4+M_5}{M_3}\sum_{j=1}^{m-1}\left(\frac{2MM_3}{l}\right)^{j+1}\frac{T^j}{j!} + \frac{2M(M_4+M_5)}{l}.$$

It then follows that

$$\|\varepsilon^{(k+1)}\|(t) \le C\int_0^t \max_{0\le\mu\le s}\|\varepsilon^{(k)}\|(\mu)ds \le C\int_0^t \max_{0\le s\le t}\|\varepsilon^{(k)}\|(s)ds.$$

Suppose that the k-th error function has the estimation

$$\|\varepsilon^{(k)}\|(t) \le \frac{(Ct)^k}{k!}\max_{0\le s\le t}\|\varepsilon^{(0)}\|(s). \tag{7}$$

Then the $(k+1)$-th error function has the following estimation

$$\begin{aligned}\|\varepsilon^{(k+1)}\|(t) &\leq C\int_0^t \max_{0\leq\mu\leq s}\frac{(C\mu)^k}{k!}\max_{0\leq r\leq\mu}\|\varepsilon^{(0)}\|(r)ds\\ &\leq C\int_0^t \frac{(Cs)^k}{k!}\max_{0\leq r\leq s}\|\varepsilon^{(0)}\|(r)ds\\ &\leq \frac{(Ct)^{k+1}}{(k+1)!}\max_{0\leq s\leq t}\|\varepsilon^{(0)}\|(s).\end{aligned}$$

Therefore, we see by induction that the error estimation (7) is valid for all k. Taking the maximum norm on both sides of (7) over the time interval $[0,T]$ yields the estimation:

(8) $$\|\varepsilon^{(k)}\|_T \leq \frac{(CT)^k}{k!}\|\varepsilon^{(0)}\|_T.$$

Since as the iteration number k approaches to infinity, $\frac{(CT)^k}{k!}$ converges to 0 superlinearly, the results stated in Theorem 2.1 follow. ■

Remark 2.2. In the special case where $m = 1$, the embedded waveform relaxation method (3) degenerates to a kind of general waveform relaxations, which seems similar to the Jacobi waveform relaxation method. The corresponding error estimation is then

$$\|\varepsilon^{(k)}\|_T \leq \left(\frac{2M(M_3+M_4+M_5)T}{l}\right)^k \frac{1}{k!}\|\varepsilon^{(0)}\|_T.$$

The rate of convergence is also superlinear.

As far as we know, waveform relaxation methods have obvious disadvantage when performing simulation for systems defined on long time intervals. For general ODEs on finite time intervals, waveform relaxation methods have superlinear convergence rate [15]. However, the waveform relaxation methods converge linearly for linear ODEs on long time intervals if some dissipation condition on the splitting is assumed [13]. Besides, divergent examples of the waveform relaxation methods for nonlinear systems on long time intervals can be found in [15].

In this situation, we should turn to a windowing waveform relaxation method, which is a modified waveform relaxation method, and also regarded as an accelerated method. For system (2), we first divide the time domain into several windows $[T_j, T_{j+1}]$ with time points $T_j\,(0 < T_1 < \ldots)$, and let all lengthes of windows be identically H. For the windowing strategy, the series of relaxed systems are separately and successively solved on windows $[T_j, T_{j+1}]$. The embedded waveform relaxation method with windowing for (2) is

$$(9)\quad \begin{cases} \dfrac{\partial u_j^{(k+\frac{i+1}{m})}(x,t)}{\partial t} = a\dfrac{\partial^2 u_j^{(k+\frac{i+1}{m})}(x,t)}{\partial x^2} \\ \qquad +G(x,t,u_j^{(k+\frac{i}{m})}(x,t),u_j^{(k)}(x,t),u_j^{(k)}(x,t-\tau)), \\ \qquad\quad T_j < t < T_{j+1},\ 0<x<l, \\ u_j^{(k+\frac{i+1}{m})}(0,t) = u_j^{(k+\frac{i+1}{m})}(l,t) = 0,\ T_j \le t \le T_{j+1}, \\ u_j^{(k+\frac{i+1}{m})}(x,T_j) = u_{j-1}^{(k_0)}(x,T_j), \\ \qquad 0 \le x \le l,\ i=0,1,\cdots,m-1,\ k=0,1,\cdots,k_0-1, \end{cases}$$

where $u_j^{(k+\frac{i+1}{m})}(x,t)$ denotes the waveform on the time window $[T_j, T_{j+1}]$, and k_0 is a fixed iteration number on every window. When applying windowing technique, we choose $u_{-1}^{(k_0)}(x,0) = h(x,0)$, and for all j, the initial guess $u_j^{(0)}(x,t) \equiv u_{j-1}^{(k_0)}(x,T_j)$, $T_j \le t \le T_{j+1}$.

For the windowing waveform relaxation method for ODEs, the continuous iteration solution at any iteration can not, in general, be a semigroup [1, 10]. The mappings between the approximations at the end points of all time windows generate a discrete semigroup if the following three conditions are satisfied:

(a) the system is autonomous;
(b) all the windows have the same length;
(c) waveform relaxation is carried out same times on every window.

We know more from [1] that the discrete semigroup generated from windowing waveform relaxation preserves a variety of invariant sets of the original ODEs on long time intervals. We assume that the results still work for the windowing embedded waveform relaxation (9) without derivation. We notice that k_0-th waveform on every time window provides a continuous connection among the end points of every time window. Therefore, the windowing waveform relaxation method (9) preserves the continuous behavior of system (2) on long time intervals.

In the next section, we will investigate the embedded waveform relaxation method for another type of partial functional differential equations.

3. Equations with Fixed and Distributed Delays

We consider the following parabolic partial functional differential equations with both fixed and distributed delays,

$$(10)\quad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} = a\dfrac{\partial^2 u(x,t)}{\partial x^2} + g\left(x,t,u(x,t-\tau),\displaystyle\int_0^t u(x,s)ds\right), \\ 0 < x < l,\ 0 < t < T, \\ u(0,t) = u(l,t) = 0, \quad 0 \le t \le T, \\ u(x,t) = h(x,t), \quad 0 \le x \le l,\ -\tau_0 \le t \le 0, \end{cases}$$

where g is a given nonlinear function.

For simplicity, we construct the similar embedded waveform relaxation method as before,

$$(11)\quad \begin{cases} \dfrac{\partial u^{(k+\frac{i+1}{m})}(x,t)}{\partial t} = a\dfrac{\partial^2 u^{(k+\frac{i+1}{m})}(x,t)}{\partial x^2} \\ \quad +g\left(x,t,u^{(k+\frac{i}{m})}(x,t-\tau), \int_0^t u^{(k)}(x,s)ds\right), \\ u^{(k+\frac{i+1}{m})}(0,t) = u^{(k+\frac{i+1}{m})}(l,t) = 0, \quad 0 \le t \le T, \\ u^{(k+\frac{i+1}{m})}(x,0) = h(x,0),\ 0 \le x \le l,\ i=0,1,\cdots,m-1,\ k=0,1,\cdots. \end{cases}$$

We take the same initial function on $[0,l] \times [-\tau_0, 0]$ and the same initial guess as before. Such a relaxation scheme is proposed to make sure that both fixed and distributed delays are relaxed.

It is easy to find that, for any fixed i and k, system (11) is a linear diffusion equation. Similarly to system (3), the solution of this system is

$$(12)\quad \begin{aligned} &u^{(k+\frac{i+1}{m})}(x,t) \\ &= \frac{2}{l}\sum_{n=1}^{\infty}\int_0^l h(\xi,0)\sin\frac{n\pi\xi}{l}d\xi e^{-a\left(\frac{n\pi}{l}\right)^2 t}\sin\frac{n\pi x}{l} \\ &+\int_0^t\Bigg(\sum_{n=1}^{\infty}\frac{2}{l}\int_0^l g(\xi,s,u^{(k+\frac{i}{m})}(\xi,s-\tau),\int_0^s u^{(k)}(\xi,\theta)d\theta)\sin\frac{n\pi\xi}{l}d\xi \\ &\times e^{-a(\frac{n\pi}{l})^2(t-s)}\sin\tfrac{n\pi x}{l}\Bigg)ds. \end{aligned}$$

We define a function sequence by

$$(13)\quad \begin{aligned} b_n^{(k+\frac{i+1}{m})}(t) \ := &\ \frac{2}{l}\int_0^l h(\xi,0)\sin\frac{n\pi\xi}{l}d\xi e^{-a\left(\frac{n\pi}{l}\right)^2 t} \\ &+\int_0^t\left(\frac{2}{l}\int_0^l g(\xi,s,u^{(k+\frac{i}{m})}(\xi,s-\tau),\right. \\ &\left.\int_0^s u^{(k)}(\xi,\theta)d\theta)\sin\frac{n\pi\xi}{l}d\xi e^{-a(\frac{n\pi}{l})^2(t-s)}\right)ds. \end{aligned}$$

The solution of equation (11) has the expression

$$u^{(k+\frac{i+1}{m})}(x,t) = \sum_{n=1}^{\infty} b_n^{(k+\frac{i+1}{m})}(t)\sin\frac{n\pi x}{l}.$$

We also define the function

$$
\begin{aligned}
&b_n(t)\\
(14)\quad &:= \frac{2}{l}\int_0^l h(\xi,0)\sin\frac{n\pi\xi}{l}d\xi e^{-a(\frac{n\pi}{l})^2 t}\\
&+\int_0^t\left(\frac{2}{l}\int_0^l g(\xi,s,u(\xi,s-\tau),\int_0^s u(\xi,\theta)d\theta)\sin\frac{n\pi\xi}{l}d\xi e^{-a(\frac{n\pi}{l})^2(t-s)}\right)ds.
\end{aligned}
$$

It is not hard to verify that the function $\sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{l}$ satisfies equation (10).

Next, we present convergence of the relaxation series (12) and the corresponding rate of convergence.

Theorem 3.1. *Assume that the partial derivative of the nonlinear function g with respect to its j-th argument is bounded by constant $\tilde{M}_j$, where $j = 3, 4$. Then the approximation solution $u^{(k+\frac{i}{m})}(x,t)$ of the embedded waveform relaxation method (11) converges to the exact solution of system (10), and satisfies*

$$
\max_{\substack{0\le x\le l\\0\le t\le T}}\left|u^{(k)}(x,t)-\sum_{n=1}^{\infty}b_n(t)\sin\frac{n\pi x}{l}\right|\le\frac{(\tilde{C}T)^k}{k!}\max_{\substack{0\le x\le l\\0\le t\le T}}\left|u^{(0)}(x,t)-\sum_{n=1}^{\infty}b_n(t)\sin\frac{n\pi x}{l}\right|,
$$

where m (usually) is a small positive integer, $i = 0, 1, \cdots, m-1$, $k = 1, 2, \cdots$, and

$$
\tilde{C} = \left(\frac{2M\tilde{M}_3}{l}\right)^m \frac{T^{m-1}}{(m-1)!} + \left(\frac{2M\tilde{M}_4}{l}\right)\sum_{j=0}^{m-1}\left(\frac{2M\tilde{M}_3}{l}\right)^j \frac{T^{j+1}}{(j+1)!}
$$

is a constant. The rate of convergence of this embedded waveform relaxation method is superlinear.

Proof. Firstly, define the error function

$$
\varepsilon^{(k+\frac{i}{m})}(x,t) := u^{(k+\frac{i}{m})}(x,t) - \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{l},\quad 0\le x\le l,\quad -\tau_0\le t\le T,
$$

where, for $-\tau_0 \le t \le 0$ and $0 \le x \le l$, $\varepsilon^{(k+\frac{i}{m})}(x,t) \equiv 0$. From the expressions (12) and (14), the error function $\varepsilon^{(k+\frac{i+1}{m})}(x,t)$ can be written directly as

$$
\begin{aligned}
&\varepsilon^{(k+\frac{i+1}{m})}(x,t)\\
&= \int_0^t\left(\sum_{j=1}^{+\infty}\frac{2}{l}\int_0^l\left[g(\xi,s,u^{(k+\frac{i}{m})}(\xi,s-\tau),\int_0^s u^{(k)}(\xi,\theta)d\theta)\right.\right.\\
&\left.\left.\quad -g(\xi,s,u(\xi,s-\tau),\int_0^s u(\xi,\theta)d\theta)\right]\sin\frac{j\pi\xi}{l}d\xi\cdot e^{-a(\frac{j\pi}{l})^2(t-s)}\sin\frac{j\pi x}{l}\right)ds.
\end{aligned}
$$

Notice that

$$g\left(\xi, s, u^{(k+\frac{i}{m})}(\xi, s-\tau), \int_0^s u^{(k)}(\xi,\theta)d\theta) - g(\xi, s, u(\xi, s-\tau), \int_0^s u(\xi,\theta)d\theta\right)$$
$$= g_3' \varepsilon^{(k+\frac{i}{m})}(\xi, s-\tau) + g_4' \int_0^s \varepsilon(\xi,\theta)d\theta,$$

where

$$g_j' = g_j'\left(\xi, s, u_*^{(k+\frac{i}{m})}(\xi, s-\tau), \int_0^s u_*^{(k)}(\xi,\theta)d\theta\right), \quad j = 3, 4$$

denotes the j-th partial derivative of the nonlinear function g with respect to its j-th argument (sometimes we denote the expression as $g_i'(\xi, s)$ for short), and

$$u_*^{(k+\frac{i}{m})}(\xi, s) \in \left[\min\{u^{(k+\frac{i}{m})}(\xi, s), u(\xi, s)\}, \max\{u^{(k+\frac{i}{m})}(\xi, s), u(\xi, s)\}\right].$$

It then follows that

$$\begin{aligned}
&\varepsilon^{(k+\frac{i+1}{m})}(x,t) \\
= &\int_0^t \left(\sum_{j=1}^{+\infty} \frac{2}{l} \int_0^l \left[g_3' \varepsilon^{(k+\frac{i}{m})}(\xi, s-\tau)\right.\right. \\
&\left.\left. + g_4' \int_0^s \varepsilon(\xi,\theta)d\theta\right] \sin\frac{j\pi\xi}{l} d\xi e^{-a(\frac{j\pi}{l})^2(t-s)} \sin\frac{j\pi x}{l}\right) ds \\
= &\int_0^t \left(\sum_{j=1}^{+\infty} \frac{2}{l} \left[g_3'(\tilde{\xi}, s)\varepsilon^{(k+\frac{i}{m})}(\tilde{\xi}, s-\tau) + g_4'(\tilde{\xi}, s) \int_0^s \varepsilon^{(k)}(\tilde{\xi},\theta)d\theta\right] \int_0^l \sin\frac{j\pi\xi}{l} d\xi\right. \\
&\left. \times e^{-a(\frac{j\pi}{l})^2(t-s)} \sin\frac{j\pi x}{l}\right) ds \\
= &\int_0^t \left(\sum_{j=1}^{+\infty} \frac{2}{l} \left[g_3'(\tilde{\xi}, s)\varepsilon^{(k+\frac{i}{m})}(\tilde{\xi}, s-\tau) + g_4'(\tilde{\xi}, s) \int_0^s \varepsilon^{(k)}(\tilde{\xi},\theta)d\theta\right] \frac{l}{j\pi}[1-(-1)^j]\right. \\
&\left. \times e^{-a(\frac{j\pi}{l})^2(t-s)} \sin\frac{j\pi x}{l}\right) ds \\
= &\int_0^t \frac{2}{l} \left[g_3'(\tilde{\xi}, s)\varepsilon^{(k+\frac{i}{m})}(\tilde{\xi}, s-\tau) + g_4'(\tilde{\xi}, s) \int_0^s \varepsilon^{(k)}(\tilde{\xi},\theta)d\theta\right] \\
&\times \left(\sum_{j=1}^{+\infty} \frac{2l}{(2j-1)\pi} e^{-a(\frac{2j\pi-\pi}{l})^2(t-s)} \sin\frac{(2j-1)\pi x}{l}\right) ds,
\end{aligned}$$

where $\tilde{\xi} \in [0, l]$. By the same argument as in the proof of Theorem 2.1, we find that the series in the expression above is uniformly bounded by a constant M.

Therefore, we have the following estimation for the error,

$$\begin{aligned} |\varepsilon^{(k+\frac{i+1}{m})}(x,t)| &\le \frac{2M}{l}\int_0^t \left(\tilde{M}_3|\varepsilon^{(k+\frac{i}{m})}(\tilde{\xi}, s-\tau)| + \tilde{M}_4\int_0^s |\varepsilon^{(k)}(\tilde{\xi},\theta)|d\theta\right) ds \\ &\le \frac{2M}{l}\int_0^t \left(\tilde{M}_3|\varepsilon^{(k+\frac{i}{m})}(\tilde{\xi}, s)| + \tilde{M}_4|\varepsilon^{(k)}(\tilde{\xi},\tilde{\theta}(s))|s\right) ds, \end{aligned}$$

where $0 < \tilde{\theta}(s) < s < t$.

If we use the sup norm $\|\varepsilon^{(k)}\|(t) := \max\limits_{0\le x\le l} |\varepsilon^{(k)}(x,t)|$, then we obtain

$$\|\varepsilon^{(k+\frac{i+1}{m})}\|(t) \le \frac{2M}{l}\int_0^t \left[\tilde{M}_3\|\varepsilon^{(k+\frac{i}{m})}\|(s) + \tilde{M}_4 s\|\varepsilon^{(k)}\|(\tilde{\theta}(s))\right] ds.$$

By induction we further get that

$$\begin{aligned} &\|\varepsilon^{(k+\frac{i}{m})}\|(t) \\ &\le \left[\left(\frac{2M\tilde{M}_3}{l}\right)^i \frac{t^i}{i!} + \left(\frac{2M\tilde{M}_4}{l}\right)\sum_{j=1}^{i}\left(\frac{2M\tilde{M}_3}{l}\right)^{j-1}\frac{t^{j+1}}{(j+1)!}\right]\max_{0\le s\le t}\|\varepsilon^{(k)}\|(s) \end{aligned}$$

and

$$\begin{aligned} \|\varepsilon^{(k+1)}\|(t) &\le \frac{2M}{l}\int_0^t \tilde{M}_3\|\varepsilon^{(k+\frac{m-1}{m})}\|(s)ds + \frac{2M}{l}\int_0^t \tilde{M}_4 s\|\varepsilon^{(k)}\|(\tilde{\theta}(s))ds \\ &= \int_0^t \left[\left(\frac{2M\tilde{M}_3}{l}\right)^m \frac{s^{m-1}}{(m-1)!} + \left(\frac{2M\tilde{M}_4}{l}\right)\sum_{j=0}^{m-1}\left(\frac{2M\tilde{M}_3}{l}\right)^j \frac{s^{j+1}}{(j+1)!}\right] \\ &\quad \times \max_{0\le\mu\le s}\|\varepsilon^{(k)}\|(\mu)ds. \end{aligned}$$

Introducing the constant

$$\tilde{C} = \left(\frac{2M\tilde{M}_3}{l}\right)^m \frac{T^{m-1}}{(m-1)!} + \left(\frac{2M\tilde{M}_4}{l}\right)\sum_{j=0}^{m-1}\left(\frac{2M\tilde{M}_3}{l}\right)^j \frac{T^{j+1}}{(j+1)!},$$

we then get

$$\|\varepsilon^{(k+1)}\|(t) \le \tilde{C}\int_0^t \max_{0\le\mu\le s}\|\varepsilon^{(k)}\|(\mu)ds \le \tilde{C}\int_0^t \max_{0\le s\le t}\|\varepsilon^{(k)}\|(s)ds.$$

Consequently,

$$\|\varepsilon^{(k)}\|(t) \le \frac{(\tilde{C}t)^k}{k!}\max_{0\le s\le t}\|\varepsilon^{(0)}\|(s)$$

and the results stated in Theorem 3.1 follow. ∎

Remark 3.2. In the special case of $m = 1$, the error estimation for the corresponding Jacobi waveform relaxation method is

$$\|\varepsilon^{(k)}\|_T \leq \left(\frac{2MT(\tilde{M}_3 + \tilde{M}_4 T)}{l}\right)^k \frac{1}{k!}\|\varepsilon^{(0)}\|_T.$$

The rate of convergence is also superlinear.

We also divide the time domain into several windows with equal length, and on the time window $[T_j, T_{j+1}]$ the corresponding windowing embedded waveform relaxation scheme for system (10) is designed as follows,

$$\begin{cases} \frac{\partial u_j^{(k+\frac{i+1}{m})}(x,t)}{\partial t} = a\frac{\partial^2 u_j^{(k+\frac{i+1}{m})}(x,t)}{\partial x^2} \\ \quad +g\left(x, t, u_j^{(k+\frac{i}{m})}(x, t-\tau), \int_0^t u_j^{(k)}(x,s)ds\right), \quad T_j < t < T_{j+1},\ 0 < x < l, \\ u_j^{(k+\frac{i+1}{m})}(0,t) = u_j^{(k+\frac{i+1}{m})}(l,t) = 0, \quad T_j \leq t \leq T_{j+1}, \\ u_j^{(k+\frac{i+1}{m})}(x, T_j) = u_{j-1}^{(k_0)}(x, T_j), \\ \quad 0 \leq x \leq l,\ i = 0, 1, \cdots, m-1,\ k = 0, 1, \cdots, k_0 - 1. \end{cases}$$

where $u_{-1}^{(k_0)}(x,0) = h(x,0)$, and for all j, the initial guess $u_j^{(0)}(x,t) \equiv u_{j-1}^{(k_0)}(x, T_j)$, $T_j \leq t \leq T_{j+1}$.

In the next section, we will investigate the semi-discrete situation of the embedded waveform relaxation method.

4. Equations after Spatial Discretization

Waveform relaxation methods are first carried out for ordinary differential equations of large scale. We will extend the embedded waveform relaxation methods to the ordinary differential equations which are obtained from spatial discretization of parabolic partial functional differential equations.

4.1. Fixed delays

We first investigate the parabolic partial functional differential equations (2). We divide the space domain $[0, l]$ into J segments with equal length Δx, and take $U_j(t)$ as the approximations to the solution $u(x,t)$, where $j = 0, 1, \ldots, J$. One of the most common approximation schemes is

$$\frac{dU_j(t)}{dt} = a\frac{U_{j+1}(t) - 2U_j(t) + U_{j-1}(t)}{\Delta x^2} + g(x_j, t, U_j(t), U_j(t-\tau)).$$

If we set $U(t) = [U_1(t), U_2(t), \ldots, U_{J-1}(t)]^{\mathrm{T}}$, we then obtain the functional differential equations

$$\begin{cases} \dfrac{dU(t)}{dt} = \dfrac{a}{\Delta x^2} AU(t) + \tilde{g}(t, U(t), U(t-\tau)), \ 0 < t \le T, \\ U(t) = [u(x_1, t), u(x_2, t), \ldots, u(x_{J-1}, t)]^{\mathrm{T}}, \ -\tau_0 \le t \le 0, \end{cases} \tag{15}$$

where,

$$A = \begin{pmatrix} -2 & 1 & 0 & & 0 \\ 1 & -2 & 1 & \ddots & \\ 0 & 1 & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots & 1 \\ 0 & & 0 & 1 & -2 \end{pmatrix} \in \mathbb{R}^{(J-1)\times(J-1)}, \tag{16}$$

$$\tilde{g}(t, U(t), U(t-\tau)) = \begin{pmatrix} g(x_1, t, U_1(t), U_1(t-\tau)) \\ g(x_2, t, U_2(t), U_2(t-\tau)) \\ \vdots \\ g(x_{J-1}, t, U_{J-1}(t), U_{J-1}(t-\tau)) \end{pmatrix} \in \mathbb{R}^{(J-1)}. \tag{17}$$

When solving equations (15) with waveform relaxation methods, one usually splits the matrix A into A_1 and A_2, where the matrix A_1 has special structure, such as diagonal or lower triangular, to make sure that each system in the relaxation series can be solved easily or in parallel. Such splitting methods lead to Jacobi and Gauss-Seidel waveform relaxation methods. However, the corresponding convergence results derived in these two fashions depend on mesh parameters and convergence rates would deteriorate when the meshes are refined.

In order to reduce the dependence of the convergence on mesh parameters, we take $A_1 = A$ and $A_2 = 0$, which means that the matrix A is not split. We only relax the nonlinear term and the functional term, which leads to the following relaxation scheme,

$$\begin{cases} \dfrac{dU^{(k+\frac{i+1}{m})}(t)}{dt} = \dfrac{a}{\Delta x^2} AU^{(k+\frac{i+1}{m})}(t) \\ \quad +\tilde{G}(t, U^{(k+\frac{i}{m})}(t), U^{(k)}(t), U^{(k)}(t-\tau)), \\ U^{(k+\frac{i+1}{m})}(0) = [u(x_1, 0), u(x_2, 0), \ldots, u(x_{J-1}, 0)]^{\mathrm{T}}, \end{cases} \tag{18}$$

where, the splitting function $\tilde{G}$ satisfies $\tilde{G}(t, u_1, u_1, u_2) = \tilde{g}(t, u_1, u_2)$, $u_1, u_2 \in \mathbb{R}^{(J-1)}$. The initial guess is taken as

$$U^{(0)}(t) = \begin{cases} [u(x_1,t), u(x_2,t), \ldots, u(x_{J-1},t)]^{\mathrm{T}}, & -\tau_0 \le t \le 0, \\ [u(x_1,0), u(x_2,0), \ldots, u(x_{J-1},0)]^{\mathrm{T}}, & 0 < t \le T. \end{cases}$$

For any fixed k and i, the relaxed system (18) consists of linear ordinary differential equations with no functional terms, which is more easily dealt with numerically or by other ways. Below is a convergence analysis of the embedded waveform relaxation method (18).

Theorem 4.1. *Assume that the Jacobian of the nonlinear splitting function $\tilde{G}$ with respect to its j-th argument is uniformly bounded by constant M_j, $j = 2, 3, 4$. Then the approximation solution $U^{(k+\frac{i}{m})}(t)$ of the embedded waveform relaxation method (18) converges to the solution of system (15), and satisfies*

$$\max_{0\le t\le T} \|U^{(k)}(t) - U(t)\| \le \frac{(M_2C_\Delta + M_3 + M_4)^k T^k}{k!} \max_{0\le t\le T} \|U^{(0)}(t) - U(t)\|, \tag{19}$$

where $k = 1, 2, \cdots,$

$$C_\Delta = M_2^{m-1}\left(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a}\right)^{m-1} + \sum_{j=1}^{m-1} M_2^{j-1}(M_3+M_4)\left(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a}\right)^j$$

is a constant, m is a positive integer, and $\lambda_{\max}$ is a negative real number. The rate of convergence of the embedded waveform relaxation method is superlinear.

Proof. First, we define the error function $\varepsilon^{(k+\frac{i}{m})}(t)$ by

$$\varepsilon^{(k+\frac{i}{m})}(t) = U^{(k+\frac{i}{m})}(t) - U(t), \quad -\tau_0 \le t \le T.$$

Obviously, $\varepsilon^{(k+\frac{i}{m})}(t) \equiv 0$ for $-\tau_0 \le t \le 0$. On the time interval $[0, T]$ we have,

$$\begin{aligned} \frac{d\varepsilon^{(k+\frac{i+1}{m})}(t)}{dt} &= \frac{a}{\Delta x^2} A \varepsilon^{(k+\frac{i+1}{m})}(t) \\ &\quad + \tilde{G}(t, U^{(k+\frac{i}{m})}(t), U^{(k)}(t), U^{(k)}(t-\tau)) - \tilde{g}(t, U(t), U(t-\tau)). \end{aligned}$$

Its solution can be written directly as

$$\begin{aligned} \varepsilon^{(k+\frac{i+1}{m})}(t) = \int_0^t & e^{\frac{a}{\Delta x^2}A(t-s)} \big[\tilde{G}(s, U^{(k+\frac{i}{m})}(s), U^{(k)}(s), U^{(k)}(s-\tau)) \\ & - \tilde{G}(s, U(s), U(s), U(s-\tau))\big] ds. \end{aligned}$$

We notice that

$$\begin{aligned}
&\tilde{G}(s, U^{(k+\frac{i}{m})}(s), U^{(k)}(s), U^{(k)}(s-\tau)) - \tilde{G}(s, U(s), U(s), U(s-\tau)) \\
&= \tilde{G}_2'(s)\varepsilon^{(k+\frac{i}{m})}(s) + \tilde{G}_3'(s)\varepsilon^{(k)}(s) + \tilde{G}_4'(s)\varepsilon^{(k)}(s-\tau),
\end{aligned}$$

where $\tilde{G}_j'(s) = \tilde{G}_j'(s, U_*^{(k+\frac{i}{m})}(s), U_*^{(k)}(s), U_*^{(k)}(s-\tau))$ denotes the j-th Jacobian of the function $\tilde{G}$ with respect to its j-th argument ($j = 2, 3, 4$), and

$$U_*^{(k+\frac{i}{m})}(s) \in \left[\min\{U^{(k+\frac{i}{m})}(s), U(s)\}, \max\{U^{(k+\frac{i}{m})}(s), U(s)\}\right].$$

On the other hand, the matrix A is symmetrical and negative definite. Let $\lambda_{\max}$ be the maximum eigenvalue of the matrix A. We then obtain

$$\|e^{\frac{a}{\Delta x^2}A(t-s)}\| \le e^{\frac{a}{\Delta x^2}\lambda_{\max}(t-s)},$$

where $\lambda_{\max} < 0$, and $\|\cdot\|$ is the 2-norm.

Then we have

$$\begin{aligned}
\left\|\varepsilon^{(k+\frac{i+1}{m})}\right\|(t) &= \Big\| \int_0^t e^{\frac{a}{\Delta x^2}A(t-s)} \big[\tilde{G}_2'(s)\varepsilon^{(k+\frac{i}{m})}(s) + \tilde{G}_3'(s)\varepsilon^{(k)}(s) \\
&\quad + \tilde{G}_4'(s)\varepsilon^{(k)}(s-\tau)\big] ds \Big\| \\
&\le \int_0^t e^{\frac{a}{\Delta x^2}\lambda_{\max}(t-s)} \Big[\|\tilde{G}_2'(s)\|\|\varepsilon^{(k+\frac{i}{m})}(s)\| + \|\tilde{G}_3'(s)\|\|\varepsilon^{(k)}(s)\| \\
&\quad + \|\tilde{G}_4'(s)\|\|\varepsilon^{(k)}(s-\tau)\|\Big] ds \\
&\le \int_0^t e^{\frac{a}{\Delta x^2}\lambda_{\max}(t-s)} \Big[M_2\|\varepsilon^{(k+\frac{i}{m})}(s)\| \\
&\quad + M_3\|\varepsilon^{(k)}(s)\| + M_4\|\varepsilon^{(k)}(s-\tau)\|\Big] ds.
\end{aligned}$$

For convenience we define another norm $\|\cdot\|_t$ by $\|x\|_t = \max_{0\le s\le t} \|x\|(s)$. Obviously we have $\|\varepsilon^{(k)}\|_{t-\tau} \le \|\varepsilon^{(k)}\|_t$. By induction we find that

$$\begin{aligned}
\|\varepsilon^{(k+\frac{m-1}{m})}\|(t) \le \Bigg[& M_2^{m-1}(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a})^{m-1} \\
&+ \sum_{j=1}^{m-1} M_2^{j-1}(M_3+M_4)(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a})^j\Bigg] \|\varepsilon^{(k)}\|_t.
\end{aligned}$$

Put

$$C_\Delta - M_2^{m-1}(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a})^{m-1} + \sum_{j=1}^{m-1} M_2^{j-1}(M_3+M_4)(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a})^j,$$

where the subscript Δ means that the constant C_Δ depends on the mesh parameter. Since $\lambda_{\max} < 0$, we have

$$\begin{aligned}\|\varepsilon^{(k+1)}\|(t) \le& \int_0^t e^{\frac{a}{\Delta x^2}\lambda_{\max}(t-s)} \Big[M_2\|\varepsilon^{(k+\frac{m-1}{m})}(s)\| \\ &+M_3\|\varepsilon^{(k)}(s)\| + M_4\|\varepsilon^{(k)}(s-\tau)\|\Big]\, ds \\ \le& \int_0^t (M_2C_\Delta + M_3 + M_4)\|\varepsilon^{(k)}\|_s ds.\end{aligned} \tag{20}$$

Similarly to the induction argument used in the proof of Theorem 2.1, we obtain

$$\|\varepsilon^{(k)}\|(t) \le (M_2C_\Delta + M_3 + M_4)^k \frac{t^k}{k!}\|\varepsilon^{(0)}\|_t.$$

This implies that the rate of convergence is superlinear. Taking the maximum on both sides of the inequality (20) with respect to t over $[0, T]$ gets the required result (19). ■

To conclude this subsection, we point out that if we employ different inequalities in the process of induction in the proof of Theorem 4.1, such as

$$\begin{aligned}\|\varepsilon^{(k+1)}\|(t) \le& (M_2C_\Delta + M_3 + M_4)\int_0^t e^{\frac{a}{\Delta x^2}\lambda_{\max}(t-s)} ds \max_{0\le t\le T}\|\varepsilon^{(k)}\|(t) \\ \le& (M_2C_\Delta + M_3 + M_4)(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a}) \max_{0\le t\le T}\|\varepsilon^{(k)}\|(t),\end{aligned}$$

we will get another error bound which converges linearly. We notice that the convergence factor is a quadratic function with respect to the mesh parameter; so the convergence factor can usually be very small. Although such an error estimation presents better convergence property of the embedded waveform relaxation methods within finite iterations, the convergence results depend on the choice of mesh parameter, while the superlinear convergence results in Theorem 4.1 is unconditional.

4.2. Fixed and distributed delays

For the parabolic partial functional differential equations (10), the corresponding equation after spatial discretization is

$$\begin{cases} \dfrac{dU(t)}{dt} = \dfrac{a}{\Delta x^2}AU(t) + \bar{g}\left(t, U(t-\tau), \displaystyle\int_0^t U(s)ds\right), & 0 < t \le T, \\ U(t) = [u(x_1,t), u(x_2,t), \ldots, u(x_{J-1},t)]^{\mathrm{T}}, & -\tau_0 \le t \le 0, \end{cases} \tag{21}$$

where A is the same symmetrical and negative definite matrix as given in (16), and $\bar{g}$ is a vector of nonlinear functions.

System (21) seems more suitable to be solved by the embedded waveform relaxation method. In fact, when solved numerically with Runge-Kutta or some other methods, system (21) is troublesome to handle because of the distributed delay. Just like the analysis in [3], the computation is much bigger than that of the system without distributed delays.

With the help of ideas of this paper, we propose the embedded waveform relaxation method for system (21) as follows,

$$(22)\quad \begin{cases} \dfrac{dU^{(k+\frac{i+1}{m})}(t)}{dt} = \dfrac{a}{\Delta x^2} A U^{(k+\frac{i+1}{m})}(t) + \bar{g}\left(t, U^{(k+\frac{i}{m})}(t-\tau), \displaystyle\int_0^t U^{(k)}(s)ds\right), \\ U^{(k+\frac{i+1}{m})}(0) = [u(x_1,0), u(x_2,0), \ldots, u(x_{J-1},0)]^{\mathrm{T}}, \end{cases}$$

with the same initial guess as given for the case of fixed delays. Note that we no longer need to concentrate on the approximation of the distributed delay after relaxation.

We also define the error function

$$\varepsilon^{(k+\frac{i}{m})}(t) := U^{(k+\frac{i}{m})}(t) - U(t), \quad -\tau_0 \le t \le T.$$

The following theorem shows the convergence results of the embedded waveform relaxation methods (22).

Theorem 4.2. *Assume that the Jacobian of the nonlinear splitting function $\bar{g}$ with respect to its j-th argument is uniformly bounded by constant $\bar{M}_j$, where $j = 2, 3$. Then the approximate solution $U^{(k+\frac{i}{m})}(t)$ of the embedded waveform relaxation method (22) converges to the solution of system (21), and satisfies*

$$\max_{0\le t\le T} \|U^{(k)}(t) - U(t)\| \le \min\left\{ \frac{T^k}{k!}, \left(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a}\right)^k \right\} \times (M_2\bar{C}_\Delta + M_3 T)^k \max_{0\le t\le T} \|U^{(0)}(t) - U(t)\|,$$

where $k = 1, 2, \cdots$, and the constant

$$\bar{C}_\Delta = \bar{M}_2^{m-1}\left(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a}\right)^{m-1} + \sum_{j=1}^{m-1} \bar{M}_2^{j-1}\left(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a}\right)^{j} \bar{M}_3 T,$$

m is a positive integer, and $\lambda_{\max}$ is the maximum eigenvalue of the negative definite matrix A. The rate of convergence of the embedded waveform relaxation method is superlinear.

Proof. We only investigate the series of the error functions $\varepsilon^{(k+\frac{i}{m})}(t)$ on the time interval $[0,T]$. Similar to the case of fixed delays, the error function can be written as

$$\varepsilon^{(k+\frac{i+1}{m})}(t) = \int_0^t e^{\frac{a}{\Delta x^2}A(t-s)} \Big[\bar{g}(s, U^{(k+\frac{i}{m})}(s-\tau), \int_0^s U^{(k)}(\theta)d\theta) - \bar{g}(s, U(s-\tau), \int_0^s U(\theta)d\theta)\Big] ds,$$

and

$$\begin{aligned}\|\varepsilon^{(k+\frac{i+1}{m})}\|(t) &= \| \int_0^t e^{\frac{a}{\Delta x^2}A(t-s)} \Big[\bar{g}_2'(s)\varepsilon^{(k+\frac{i}{m})}(s-\tau) + \bar{g}_3'(s)\int_0^s \varepsilon^{(k)}(\theta)d\theta\Big] ds\| \\ &\le \int_0^t e^{\frac{a}{\Delta x^2}\lambda_{\max}(t-s)} \Big[\bar{M}_2\|\varepsilon^{(k+\frac{i}{m})}(s-\tau)\| + \bar{M}_3\|\varepsilon^{(k)}(\bar{\theta})\|s\Big] ds,\end{aligned}$$

where $0 < \bar{\theta} < s$.

By induction we get

$$\|\varepsilon^{(k+\frac{m-1}{m})}\|(t) \le \big[\bar{M}_2^{m-1}(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a})^{m-1} + \sum_{j=1}^{m-1}\bar{M}_2^{j-1}(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a})^j\bar{M}_3 t\big]\|\varepsilon^{(k)}\|_t.$$

Set

$$\bar{C}_\Delta = \bar{M}_2^{m-1}(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a})^{m-1} + \sum_{j=1}^{m-1}\bar{M}_2^{j-1}(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a})^j\bar{M}_3 T.$$

Firstly, noticing the inequality $e^{\frac{a}{\Delta x^2}\lambda_{\max}(t-s)} \le 1$, we obtain the estimation

$$\begin{aligned}\|\varepsilon^{(k+1)}\|(t) &\le \int_0^t \Big[\bar{M}_2\|\varepsilon^{(k+\frac{m-1}{m})}(s-\tau)\| + \bar{M}_3\|\varepsilon^{(k)}(\bar{\theta})\|s\Big] ds \\ &\le (\bar{M}_2\bar{C}_\Delta + \bar{M}_3 T)\int_0^t \|\varepsilon^{(k)}\|_s ds.\end{aligned}$$

Similar to the induction technique used in the proof of Theorem 2.1, we have

$$\|\varepsilon^{(k)}\|(t) \le \frac{(M_2\bar{C}_\Delta + M_3 T)^k t^k}{k!} \max_{0\le s\le t}\|\varepsilon^{(0)}\|(s).$$

This clearly says that the rate of convergence is superlinear.

However, the constant M_3 is independent of mesh parameter when the mesh is refined. This error estimation seems not sharp enough, at least for the first several iterations. Therefore, we resort another error estimation,

$$\begin{aligned}\|\varepsilon^{(k+1)}\|(t) &\leq \int_0^t e^{\frac{a}{\Delta x^2}\lambda_{\max}(t-s)}\left[\bar{M}_2\|\varepsilon^{(k+\frac{m-1}{m})}(s-\tau)\|+\bar{M}_3\|\varepsilon^{(k)}(\bar{\theta})\|s\right]ds \\ &\leq (\bar{M}_2\bar{C}_\Delta+\bar{M}_3 t)(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a})\|\varepsilon^{(k)}\|_t.\end{aligned}$$

Therefore,

$$\max_{0\leq t\leq T}\|\varepsilon^{(k)}\|(t)\leq(\bar{M}_2\bar{C}_\Delta+\bar{M}_3 T)^k(-\frac{1}{\lambda_{\max}}\frac{\Delta x^2}{a})^k\max_{0\leq t\leq T}\|\varepsilon^{(0)}\|(t).$$

The mesh parameter could be taken small enough, to make sure that the convergence factor is also small enough. The convergence results stated in the theorem follow. ■

5. Numerical Experiments

In this section we present two numerical examples to show the behavior of the embedded waveform relaxations.

Example 5.1. Consider the normalized Hutchinson's equation in one-space dimension with Neumann boundary conditions:

$$(23)\qquad \begin{cases}\dfrac{\partial}{\partial t}u(x,t) \\ = d\dfrac{\partial^2}{\partial x^2}u(x,t)-(\dfrac{\pi}{2}+\mu)u(x,t-1)[1+u(x,t)],\ \ 0<x<\pi,\ \ t>0 \\ \dfrac{\partial}{\partial x}u(x,t)=0,\quad x=0,\pi,\end{cases}$$

where the constants $d=0.01$ and $\mu=0.2$. In this example, we choose the initial function $u(x,t)=0.5\cos^2(x)$, $0\leq x\leq\pi$, $-1\leq t\leq 0$.

The Hutchinson equation is used to be a rough model for the evolution of population in mathematical ecology. Some derivations of the equation are able to describe an assemblage of particles, e.g., cells, chemicals, bacteria and so on. More details can be found in [14].

As is well known, the waveform relaxation method has its inherent disadvantages when processing time-dependent systems which are defined on long time intervals. In order to overcome such disadvantages, we first investigate the behavior of the iteration method on short time interval $[0,2]$. The following kind of embedded waveform relaxation methods is employed,

$$\begin{cases} \dfrac{\partial}{\partial t}u^{(k+\frac{i+1}{m})}(x,t) = d\dfrac{\partial^2}{\partial x^2}u^{(k+\frac{i+1}{m})}(x,t) \\ \quad -(\dfrac{\pi}{2}+\mu)u^{(k)}(x,t-1)[1+u^{(k+\frac{i}{m})}(x,t)], \quad 0 < x < \pi, \\ \dfrac{\partial}{\partial x}u^{(k+\frac{i+1}{m})}(x,t) = 0, \quad x = 0, \pi, \end{cases}$$

and the initial guess is $u^{(0)}(x,t) \equiv 0.5\cos^2(x)$ for $t \in [-1,2]$ and $x \in [0,\pi]$. When carrying out in computer, we assume the numerical solution with space step $\Delta x = \pi/100$ and time step $\Delta t = 0.01$ to be the true solution. The computing error is shown in Table 1.

Table 1. Error of the embedded waveform relaxation method for the system in Example 5.1 on short time interval $[0,2]$, and k is the number of iterations

k	k=1	k=2	k=3	k=4
m=2	5.04e-001	2.82e-002	1.01e-003	1.40e-005
m=3	7.69e-001	6.05e-003	9.85e-006	4.83e-009
k	k=5	k=6	k=7	k=8
m=2	1.08e-007	5.35e-010	1.81e-012	4.72e-015
m=3	9.84e-013	1.67e-016	-	-

As for the behavior of the Hutchinson's equation (23) on a long time interval, e.g., $[0,100]$, which maybe receive more concern, we employ the windowing technique to handle it. In fact, carrying out embedded waveform relaxation method on the long time interval directly will usually make the error blow up, while windowing technique will change the situation into several systems on short time intervals. We divide the time interval $[0,100]$ into 100 time windows. The error functions of the windowing embedded waveform relaxation method with respect to time and space variables are shown in Figure 1 and the maximums of the error functions can be found in Table 2.

Example 5.2. Consider the single-species population models:

$$(24) \qquad \begin{cases} \dfrac{\partial}{\partial t}u(x,t) = d\dfrac{\partial^2}{\partial x^2}u(x,t) \\ \quad +ru(x,t)\left(1-\int_{-\tau}^{0}u(x,t+s)ds\right), \ 0 < x < \pi, \ t > 0, \\ u(x,t) = 0.3\sin(x), \quad -\tau \le t \le 0, \quad 0 \le x \le \pi, \\ u(0,t) = 0, \quad u(\pi,t) = 0, \quad t \ge 0. \end{cases}$$

where the constants $d = 0.01$, $r = 0.2$ and $\tau = 1$.

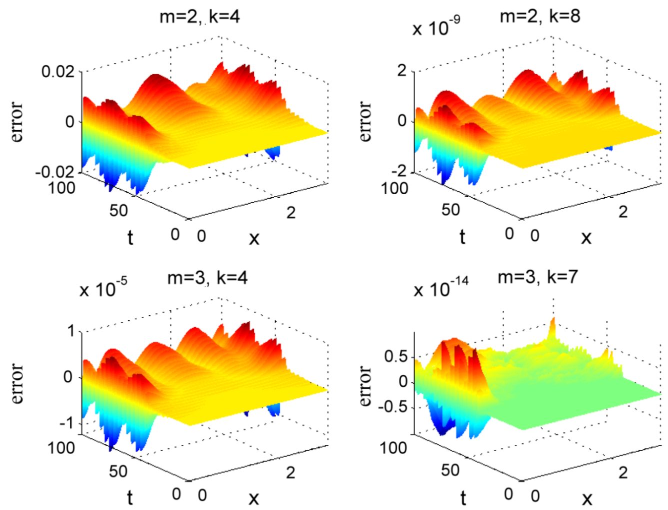

Fig. 1. The error functions of the embedded waveform relaxation method at different iterations.

Table 2. Error of the windowing embedded waveform relaxation method for the system in Example 5.1 on long time interval $[0, 100]$

k	k=1	k=2	k=3	k=4
m=2	1.945	2.133	5.04e-001	2.31e-002
m=3	inf	5.04e-001	4.10e-003	1.32e-005
m=4	2.1329	2.31e-002	1.32e-005	2.06e-009
k	k=5	k=6	k=7	k=8
m=2	6.65e-004	1.32e-005	1.91e-007	2.06e-009
m=3	2.05e-008	1.70e-011	1.07e-014	1.11e-016
m=4	1.10e-013	1.11e-016	-	-

System (24) is proposed by Green and Stech [6] for a class of single species population models with diffusion. We take the relaxation scheme

$$\begin{cases} \frac{\partial}{\partial t}u^{(k+\frac{i+1}{m})}(x,t)=d\frac{\partial^2}{\partial x^2}u^{(k+\frac{i+1}{m})}(x,t)+ru^{(k+\frac{i}{m})}(x,t)\left(1-\int_{-\tau}^{0}u^{(k)}(x,t+s)ds\right), \\ u^{(k+\frac{i+1}{m})}(x,t) = 0.3\sin(x), \quad -\tau \le t \le 0, \quad 0 \le x \le \pi, \\ u^{(k+\frac{i+1}{m})}(0,t) = 0, \quad u^{(k+\frac{i+1}{m})}(\pi,t) = 0, \quad t \ge 0. \end{cases}$$

We first consider the short time interval $[0, 5]$ and take the same discrete time step and space step as Example 5.1. The maximum of the errors of some waveforms are shown in Table 3.

Table 3. Error of the embedded waveform relaxation method for the system in Example 5.2 on short time interval $[0, 5]$

k	k=1	k=2	k=3	k=4
m=1	4.62e-002	8.22e-003	1.30e-003	1.82e-004
m=2	3.02e-002	2.44e-003	1.13e-004	3.20e-006
k	k=5	k=6	k=7	k=8
m=1	2.23e-005	2.44e-006	2.40e-007	2.14e-008
m=2	5.90e-008	7.41e-010	6.66e-012	4.50e-014

Table 4. Error of the windowing embedded waveform relaxation method for the system in Example 5.2 on long time interval $[0, 100]$

k	k=1	k=2	k=3	k=4
m=1	2.26e-002	9.26e-004	3.20e-005	9.23e-007
m=2	6.62e-003	4.24e-005	1.45e-007	3.15e-010
k	k=5	k=6	k=7	k=8
m=1	2.27e-008	4.82e-010	9.06e-012	1.52e-013
m=2	4.68e-013	1.89e-015	-	-

The windowing technique is also applied to deal with system (24) on the long time interval $[0, 100]$ which is divided into 100 windows with the same length. The corresponding error functions of the windowing embedded waveform relaxation are shown in Figure 2 and the maximums of the error functions can be found in Table 4.

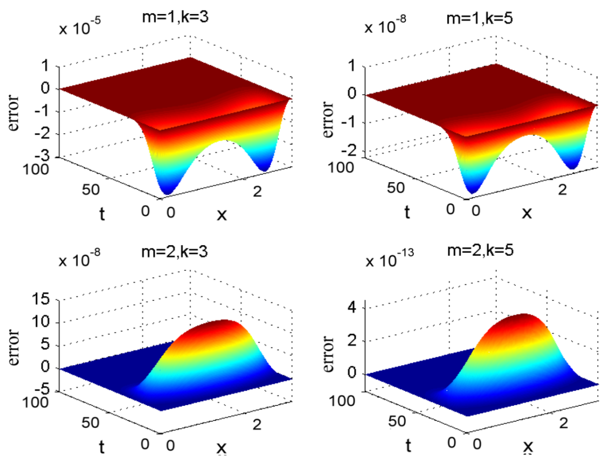

Fig. 2. The error functions of the embedded waveform relaxation method at different iterations.

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