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ON THE SECOND EQUATION OF OBATA

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Abstract. In this paper we prove some results related to a certain vector field satisfying the second equation of Obata [8] on vector fields.

1. Introduction

In this paper we prove some results related to a non-zero vector field Z on an n-dimensional Riemannian manifold (M, g) satisfying $(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Y, Z)X + g(X, Y)Z] = 0$ for all $X, Y \in \Gamma(TM)$ and for $\lambda(> 0) \in \mathbb{R}$. In fact, the idea underlying this paper is to characterize (or represent) Riemannian manifolds analytically by a differential equation on certain class of Riemannian manifolds determined by mild geometric/topological assumptions.

2. Preliminaries

Here, we briefly state the main concepts and definitions used throughout this paper.

Let Z be a vector field on (M, g), a Riemannian manifold of dimension n, ∇ the Levi-Civita connection and

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

the curvature tensor, where $X, Y \in \Gamma(TM)$. We write also $< X, Y >$ if this is convenient. The Ricci curvature (tensor) is the trace of $R : trace(X \to R(X, Y)Z)$ and denoted by $Ric(Y, Z)$. If $\{X_1, \cdots, X_n\}$ is a local orthonormal frame for TM, then

$$Ric(Y, Z) = \Sigma_{i=1}^n g(R(X_i, Y)Z, X_i) = \Sigma_{i=1}^n g(R(Y, X_i)X_i, Z).$$

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Thus Ric is a symmetric bilinear form. It could also be defined as a symmetric (1,1) tensor

$$Ric(Z) = \Sigma_{i=1}^{n} R(Z, X_i)X_i.$$

The scalar curvature is defined by $Sc = trRic$. Let Z be a vector field on this n-dimensional Riemannian manifold (M, g) with Levi-Civita connection ∇. The second covariant differential $\nabla^2 Z$ of Z is defined by

$$(\nabla^2 Z)(X, Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z,$$

where $X, Y \in \Gamma(TM)$. We define the Laplacian ΔZ of Z on (M, g) to be the trace of $\nabla^2 Z$ with respect to g, that is,

$$\Delta Z = trace\, \nabla^2 Z = \sum_{i=1}^{n} (\nabla^2 Z)(X_i, X_i),$$

where $\{X_1, \cdots, X_n\}$ is a local orthonormal frame for TM.

Also, the affinity tensor $L_Z \nabla$ of Z is defined by

$$(L_Z \nabla)(X, Y) = L_Z \nabla_X Y - \nabla_{L_Z X} Y - \nabla_X L_Z Y,$$

where L_Z is the Lie derivative with respect to Z and $X, Y \in \Gamma(TM)$. (See, for example page 109 of [9]). We define the tension field $\Box Z$ of Z on (M, g) to be the trace of $L_Z \nabla$ with respect to g that is,

$$\Box Z = trace\, L_Z \nabla = \sum_{i=1}^{n} (L_Z \nabla)(X_i, X_i),$$

where $\{X_1, \cdots, X_n\}$ is a local orthonormal frame for TM.

By a straightforward computation, it can be shown by using the torsion-free property of ∇ that

$$(L_Z \nabla)(X, Y) = (\nabla^2 Z)(X, Y) + R(Z, X)Y,$$

(see page 110 of [9]) and hence

$$\Box Z = \Delta Z + Ric(Z),$$

where $X, Y \in \Gamma(TM)$. (Also see page 40 of [11]).

The divergence of a vector field Z, $divZ$, on (M, g) is defined as

$$divZ = tr(\nabla Z) = \sum_{i=1}^{n} g(\nabla_{X_i} Z, X_i)$$

if $\{X_i\}$ is an orthonormal basis of TM.

3. The Second Equation of Obata

The elementary results of this chapter could also be collected from [2]. First, we state a differential equation, which is a slight generalization of an equation given by Obata [8], characterizing Euclidian spheres. It is shown in [10] that, a necessary and a sufficient condition for a connected, simply connected, complete $n(\geq 2)$-dimensional Riemannian manifold (M, g) to be isometric with the Euclidian sphere of radius $\frac{1}{\sqrt{\lambda}}, \lambda > 0$ is the existence of a nonconstant function f on M satisfying the equation

$$(\nabla^2 \nabla f)(X, Y) + \lambda[2g(\nabla f, X)Y + g(Y, \nabla f)X + g(X, Y)\nabla f] = 0,$$

for all $X, Y \in \Gamma(TM)$. In fact, we can replace ∇f with a nonzero vector field in the above equation.

Lemma 3.1. *Let (M, g) be an n-dimensional Riemannian manifold and $\lambda \in \mathbb{R}$. If Z is a vector field on (M, g) satisfying the equation*

$$(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Y, Z)X + g(X, Y)Z] = 0,$$

for all $X, Y \in \Gamma(TM)$, then

$$\Delta Z = -(n + 3)\lambda Z.$$

Proof. If we take the trace of the equation

$$(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Y, Z)X + g(X, Y)Z] = 0,$$

with respect to g on (M, g) we obtain another differential equation

$$\begin{aligned}
\Delta Z &= tr(\nabla^2 Z) \\
&= \sum_{i=1}^{n} (\nabla^2 Z)(X_i, X_i) \\
&= \sum_{i=1}^{n} (-\lambda[2g(Z, X_i)X_i + g(X_i, Z)X_i + g(X_i, X_i)Z]) \\
&= -\lambda \sum_{i=1}^{n} [3g(Z, X_i)X_i + g(X_i, X_i)Z] \\
&= -\lambda(3Z + nZ) \\
&= -(n + 3)\lambda Z,
\end{aligned}$$

here $\{X_i\}$ is an orthonormal frame of TM, in fact an eigenvalue equation. ∎

Remark 3.2. Note that, on a connected, compact Riemannian manifold (M, g) the Laplacian Δ is negative semi-definite on spaces of vector fields. Thus, if (M, g) is compact, eigenvalues of Δ are non-positive on vector fields. The case Z is an eigen vector field corresponding to the 0 eigen value occurs if and only if Z is a parallel vector field on (M, g) (see Theorem 3.2 in [4]) .

In conclusion, we can say that on a compact Riemannian manifold (M, g), the eigenspace corresponding to the zero eigenvalue of Δ consist of parallel vector fields on (M, g). Also note here that, since $Ric(Z, Z) = 0$ for a parallel vector field Z, the eigenspace corresponding to the zero eigenvalue of Δ does not exist if $Ric(x, x) \neq 0$ for all $x(\neq 0) \in TpM$ for some $p \in M$.

Remark 3.3. Note also that, on a compact Riemannian manifold (M, g) the Laplacian is an elliptic operator. Thus, by the spectral theorem, the eigenvalues λ_i of Δ are of the form

$$-\infty \leftarrow \cdots < \lambda_i < \cdots < \lambda_1 < \lambda_0 = 0.$$

Thus, if $Ric(x, x) \neq 0$ for all $x(\neq 0) \in TpM$ for some $p \in M$, then the largest eigenvalue of Δ on the vector space of vector fields on (M, g) is negative.

Lemma 3.4. *Let (M, g) be an n-dimensional Riemannian manifold and $\lambda \in \mathbb{R}$. If Z is a vector field on (M, g) satisfying the equation*

$$(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Y, Z)X + g(X, Y)Z] = 0,$$

for all $X, Y \in \Gamma(TM)$, then

(*i*) $R(X, Y)Z = \lambda[g(Z, Y)X - g(X, Z)Y]$,
for all $X, Y \in \Gamma(TM)$, and hence
$Ric(Z) = \lambda(n - 1)Z$,

(*ii*) $\nabla div\, Z = -2\lambda(n + 1)Z$,
and hence
$\nabla^2 div\, Z = -2\lambda(n + 1)\nabla Z$,
where $\nabla^2 div\, Z$ is the Hessian tensor of $div\, Z$.

Proof.

(i) Let $X, Y \in \Gamma(TM)$. Then,

$$\begin{aligned} R(X, Y)Z &= \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z \\ &= -\lambda[2g(Z, X)Y + g(Y, Z)X + g(X, Y)Z] - (-\lambda)[2g(Z, Y)X \\ &\quad + g(X, Z)Y + g(Y, X)Z] \\ &= \lambda[2g(Z, Y)X - 2g(Z, X)Y + g(X, Z)Y - g(Y, Z)X] \\ &= \lambda[g(Z, Y)X - g(Z, X)Y]. \end{aligned}$$

Hence

$$
\begin{aligned}
g(Ric(Z), X) &= g\Big(\sum_{i=1}^{n} R(Z, X_i)X_i, X\Big) \\
&= \sum_{i=1}^{n} g(R(Z, X_i)X_i, X) \\
&= \sum_{i=1}^{n} R(Z, X_i, X_i, X) \\
&= \sum_{i=1}^{n} R(X_i, X, Z, X_i) \\
&= \sum_{i=1}^{n} g(R(X_i, X)Z, X_i) \\
&= \sum_{i=1}^{n} g(\lambda[g(Z, X)X_i - g(Z, X_i)X], X_i) \\
&= \lambda g(Z, X)\sum_{i=1}^{n} g(X_i, X_i) - \lambda \sum_{i=1}^{n} g(Z, X_i)g(X, X_i) \\
&= \lambda n g(Z, X) - \lambda g(Z, X) \\
&= \lambda(n-1)g(Z, X),
\end{aligned}
$$

here $\{X_1, \cdots, X_n\}$ is an orthonormal frame for TM near $p \in M$.

(ii) Let $\{X_1, \cdots, X_n\}$ be an adapted orthonormal frame near $p \in M$, that is, $\{X_1, \cdots, X_n\}$ is an orthonormal frame in TM with $(\nabla X_i)_p = 0$ for $i = 1, \ldots, n$, and let $X \in \Gamma(TM)$. Then at $p \in M$,

$$
\begin{aligned}
g(\nabla div\, Z, X) &= X(div\, Z) \\
&= \sum_{i=1}^{n} Xg(\nabla_{X_i} Z, X_i) \\
&= \sum_{i=1}^{n} [g(\nabla_X \nabla_{X_i} Z, X_i) + g(\nabla_{X_i} Z, \nabla_X X_i)] \\
&= \sum_{i=1}^{n} [g((\nabla^2 Z)(X, X_i), X_i) - g(\nabla_{\nabla_X X_i} Z, X_i)]
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{i=1}^{n} g(-\lambda\{2g(Z,X)X_i + g(X_i,Z)X_i + g(X,X_i)Z\}, X_i) \\
&= -2\lambda g(Z,X)\sum_{i=1}^{n} g(X_i,X_i) - \lambda\sum_{i=1}^{n} g(Z,X_i)g(X,X_i) \\
&\quad -\lambda\sum_{i=1}^{n} g(X,X_i)g(Z,X_i) \\
&= -2n\lambda g(Z,X) - 2\lambda g(Z,X) \\
&= -2(n+1)\lambda g(Z,X) \\
&= g(-2(n+1)\lambda Z, X).
\end{aligned}$$

Hence, it follows that $\nabla div\, Z = -2(n+1)\lambda Z$ and hence $\nabla^2 div\, Z = -2(n+1)\lambda\nabla Z$. ■

Definition 3.5. Let (M,g) be a Riemannian manifold and $\lambda \in \mathbb{R}$. A vector field Z on M satisfying

$$R(X,Y)Z = \lambda[g(Z,Y)X - g(X,Z)Y],$$

for all $X, Y \in \Gamma(TM)$, is called a λ-nullity vector field on (M,g).

That is, Z is a nullity vector field with respect to the curvature-like tensor field

$$F(X,Y)W = R(X,Y)W - \lambda[g(W,Y)X - g(X,W)Y],$$

on (M,g). (See Sections 2 and 4 of [10]).

In particular, if there exist a nonzero $\lambda(\neq 0)$-nullity vector field Z on a Riemannian manifold (M,g) then (M,g) is irreducible. (see [1], [5], [10] and the references therein for details).

Remark 3.6. Let (M,g) be an n-dimensional Riemannian manifold and $\lambda \in \mathbb{R}$. If Z is a vector field on (M,g) satisfying the equation

$$(\nabla^2 Z)(X,Y) + \lambda[2g(Z,X)Y + g(Y,Z)X + g(X,Y)Z] = 0,$$

for all $X, Y \in \Gamma(TM)$ then, Z is a λ-nullity vector field by Lemma 3.4. That is, Z is a nullity vector field with respect to the curvature-like tensor field $F(X,Y)W = R(X,Y)W - \lambda[g(W,Y)X - g(X,W)Y]$ on (M,g). If, in addition, Z is nonzero and $\lambda \neq 0$, then (M,g) is irreducible.

Definition 3.7. A vector field Z on (M,g) is projective if it satisfies

$$(L_Z\nabla)(X,Y) = \pi(X)Y - \pi(Y)X,$$

for any vector fields Y and Z, π being a certain 1-form.

Corollary 3.8. *Let (M,g) be an n-dimensional Riemannian manifold and $\lambda \in \mathbb{R}$. If Z is a vector field on (M,g) satisfying the equation*

$$(\nabla^2 Z)(X,Y) + \lambda[2g(Z,X)Y + g(Y,Z)X + g(X,Y)Z] = 0,$$

for all $X,Y \in \Gamma(TM)$, then Z is a projective vector field.

Proof. Let $X,Y \in \Gamma(TM)$. Then,

$$\begin{aligned}(L_Z\nabla)(X,Y) &= (\nabla^2 Z)(X,Y) + R(Z,X)Y \\ &= -\lambda[2g(Z,X)Y + g(Y,Z)X + g(X,Y)Z] + \lambda[g(Y,X)Z \\ &\quad -g(Z,Y)X] \\ &= -2\lambda g(Z,X)Y - 2\lambda g(Z,Y)X.\end{aligned}$$

∎

In fact, if (M,g) is compact, then this can be obtained differently (see Corollary 3.15 below).

Corollary 3.9. *Let (M,g) be an n-dimensional Riemannian manifold and $\lambda \in \mathbb{R}$. If Z is a vector field on (M,g) satisfying the equation*

$$(\nabla^2 Z)(X,Y) + \lambda[2g(Z,X)Y + g(Y,Z)X + g(X,Y)Z] = 0,$$

for all $X,Y \in \Gamma(TM)$, then

$$\Delta(div\, Z) = -2(n+1)\lambda div\, Z.$$

Proof. If we take the trace of the equation

$$\nabla^2 div\, Z = -2(n+1)\lambda\nabla Z$$

by Lemma 3.11, we obtain another differential equation

$$\begin{aligned}\Delta(div\, Z) &= tr(\nabla^2 div\, Z) \\ &= tr(-2(n+1)\lambda\nabla Z) \\ &= -2(n+1)\lambda tr(\nabla Z) \\ &= -2(n+1)\lambda div\, Z,\end{aligned}$$

in fact an eigenvalue equation. ∎

Remark 3.10. Considering the differential equations

$$(\nabla^2 Z)(X,Y) + \lambda g(Z,X)Y = 0,$$

and

$$(\nabla^2 Z)(X,Y) + \lambda[2g(Z,X)Y + g(Y,Z)X + g(X,Y)Z] = 0,$$

for $\lambda > 0$ on the n-dimensional Euclidian sphere of radius $\frac{1}{\sqrt{\lambda}}$, intuitively, the first differential equation corresponds to the first eigenvalue of the Laplacian (that is, $\Delta div\, Z = -n\lambda div\, Z$) and the latter differential equation corresponds to the second eigenvalue of the Laplacian (that is, $\Delta div\, Z = -2(n+1)\lambda div\, Z$) on the Euclidian sphere of radius $\frac{1}{\sqrt{\lambda}}$. Also, a vector field satisfying the first equation is necessarily a conformal vector field (see Remark 3.5 in [6]). A vector field satisfying the latter differential equation is necessarily a projective vector field by Corollary 3.8 (see also Corollary 3.16).

Lemma 3.11. *Let (M,g) be an n-dimensional Riemannian manifold. If Z is a non-zero vector field on (M,g) satisfying the equation*

$$(\nabla^2 Z)(X,Y) + \lambda[2g(Z,X)Y + g(Y,Z)X + g(X,Y)Z = 0,$$

for all $X, Y \in \Gamma(TM)$ then, $\nabla div\, Z$ also satisfies the same equation.

Proof. Since Z is non-zero, it follows from Lemma 3.4 that $div\, Z$ is non-constant and $\nabla^2 div\, Z = -2(n+1)\lambda \nabla Z$. Hence,$\nabla Z$ is self-adjoint and can be written as $\nabla Z = \frac{div\, Z}{n} id + \sigma$, where σ is the traceless self-adjoint part of ∇Z. Let $X, Y \in \Gamma(TM)$. Then, by Lemma 3.4,

$$\begin{aligned}
(\nabla\sigma)(X,Y) &= (\nabla(\nabla Z - \frac{div\, Z}{n} id)(X,Y) \\
&= (\nabla(\nabla Z)) - \nabla(\frac{div\, Z}{n} id)(X,Y) \\
&= \nabla^2 Z(X,Y) - \nabla_X(\frac{div\, Z}{n} id)(Y) \\
&= \nabla^2 Z(X,Y) - \nabla_X \frac{div\, Z}{n} id(Y) + \frac{div\, Z}{n} id(\nabla_X Y) \\
&= \nabla^2 Z(X,Y) - \nabla_X \frac{div\, Z}{n} Y + \frac{div\, Z}{n} \nabla_X Y \\
&= \nabla^2 Z(X,Y) - X(\frac{div\, Z}{n})Y - \frac{div\, Z}{n} \nabla_X Y + \frac{div\, Z}{n} \nabla_X Y \\
&= \nabla^2 Z(X,Y) - \frac{1}{n} X(div\, Z)Y \\
&= \nabla^2 Z(X,Y) - \frac{1}{n} g(\nabla div\, Z, X)Y
\end{aligned}$$

$$
\begin{aligned}
&= -2\lambda g(Z,X)Y - \lambda g(Y,Z)X - \lambda g(X,Y)Z - \frac{1}{n} g(\nabla div\, Z, X)Y \\
&= -2\lambda \frac{1}{-2(n+1)\lambda} g(\nabla div\, Z, X)Y - \lambda \frac{1}{-2(n+1)\lambda} g(Y, \nabla div\, Z)X \\
&\quad -\lambda \frac{1}{-2(n+1)\lambda} g(X,Y)\nabla div\, Z - \frac{1}{n} g(\nabla div\, Z, X)Y \\
&= \frac{1}{n+1} g(\nabla div\, Z, X)Y + \frac{1}{2(n+1)} g(Y, \nabla div\, Z)X \\
&\quad + \frac{1}{2(n+1)} g(X,Y)\nabla div\, Z - \frac{1}{n} g(\nabla div\, Z, X)Y \\
&= (\frac{1}{(n+1)} - \frac{1}{n}) g(X, \nabla div\, Z)Y + \frac{1}{2(n+1)} g(Y, \nabla div\, Z)X \\
&\quad + \frac{1}{2(n+1)} g(X,Y)\nabla div\, Z \\
&= \frac{-1}{n(n+1)} g(X, \nabla div\, Z,)Y + \frac{1}{2(n+1)} g(Y, \nabla div\, Z)X \\
&\quad + \frac{1}{2(n+1)} g(X,Y)\nabla div\, Z
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(\nabla^2 \nabla div\, Z)(X,Y) \\
&= -2(n+1)\lambda(\nabla^2 Z)(X,Y) \\
&= -2(n+1)\lambda \nabla(\frac{div\, Z}{n} id + \sigma)(X,Y) \\
&= -2(n+1)\lambda[\nabla \frac{div\, Z}{n} id) + \nabla\sigma](X,Y) \\
&= -2(n+1)\lambda[(\frac{1}{n}) g(\nabla div\, Z, X)Y + \nabla\sigma(X,Y)] \\
&= -2\frac{n+1}{n}\lambda g(\nabla div\, Z, X)Y - 2(n+1)\lambda[\frac{-1}{n(n+1)} g(X, \nabla div\, Z)Y \\
&\quad + \frac{1}{2(n+1)} g(\nabla div\, Z, Y)X + \frac{1}{2(n+1)} g(X,Y)\nabla div\, Z] \\
&= -2(\frac{n+1}{n} - \frac{1}{n})\lambda g(\nabla div\, Z, X)Y - \lambda g(X, \nabla div\, Z)Y \\
&\quad -\lambda g(X,Y)\nabla div\, Z \\
&= -\lambda[2g(X, \nabla div\, Z)Y + g(\nabla div\, Z, Y)X + g(X,Y)\nabla div\, Z]. \qquad \blacksquare
\end{aligned}
$$

Corollary 3.12. *Let (M, g) be an n-dimensional Riemannian manifold. If Z is a non-zero vector field on (M, g) satisfying the equation*

$$(\nabla^2 Z)(X,Y) + \lambda[2g(Z,X)Y + g(Y,Z)X + g(X,Y)Z] = 0,$$

for all $X, Y \in \Gamma(TM)$, then

$$\Delta\nabla div\, Z = -(n+3)\lambda\nabla div\, Z.$$

Proof. If we take the trace of the equation

$$(\nabla^2\nabla div\, Z)(X,Y) = -\lambda[2g(X,\nabla div\, Z)Y + g(\nabla div\, Z, Y)X + g(X,Y)\nabla div\, Z],$$

with respect to g on (M, g) we obtain another differential equation

$$\begin{aligned}
\Delta\nabla div\, Z &= tr(\nabla^2\nabla div\, Z) \\
&= \sum_{i=1}^{n}(\nabla^2 Z)(X_i, X_i) \\
&= \sum_{i=1}^{n} -\lambda[2g(X_i,\nabla div\, Z)X_i + g(\nabla div\, Z, X_i)X_i + g(X_i,X_i)\nabla div\, Z \\
&= -\lambda\sum_{i=1}^{n}[3g(\nabla div\, Z, X_i)X_i + g(X_i,X_i)\nabla div\, Z] \\
&= -\lambda(3\nabla div\, Z + n\nabla div\, Z) \\
&= -\lambda(n+3)\nabla div\, Z.
\end{aligned}$$

■

Lemma 3.13. *Let (M, g) be an n-dimensional Riemannian manifold. If Z is a non-zero vector field on (M, g) satisfying the equation*

$$(\nabla^2 Z)(X,Y) + \lambda[2g(Z,X)Y + g(Z,Y)X + g(X,Y)Z] = 0,$$

for all $X, Y \in \Gamma(TM)$, then

$$\Box Z = -4\lambda Z.$$

Proof. It follows from Lemma 3.1 and Lemma 3.4 that,

$$\begin{aligned}
\Box Z &= \Delta Z + Ric(Z) \\
&= -(n+3)\lambda Z + (n-1)\lambda Z \\
&= -4\lambda Z.
\end{aligned}$$

■

Remark 3.14. Let (M, g) be a compact $n(\geq 2)$-dimensional Riemannian manifold. Recall that the tension operator $\Box$ on $\Gamma(TM)$ is also a linear, self-adjoint, elliptic operator with respect to the inner product $<,>$ on the vector space $\Gamma(TM)$ of vector fields on M defined by $< X, Y >= \int_M g(X, Y)$.

Corollary 3.15. *Let (M, g) be an n-dimensional Riemannian manifold. If Z is a non-zero vector field on (M, g) satisfying the equation*

$$(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Z, Y)X + g(X, Y)Z] = 0,$$

for all $X, Y \in \Gamma(TM)$, then it also satisfies the equation

$$\Box Z - \frac{2}{n+1}\nabla div\, Z = 0.$$

Proof. By Lemma 3.4 and Lemma 3.13,

$$\begin{aligned} \Box Z - \frac{2}{n+1}\nabla div\, Z &= -4\lambda Z - \frac{2}{n+1}(-2)\lambda(n+1)Z \\ &= -4\lambda Z + 4\lambda Z \\ &= 0. \end{aligned}$$

■

Corollary 3.16. *Let (M, g) be an n-dimensional compact Riemannian manifold. If Z is a non-zero vector field on (M, g) satisfying the equation*

$$(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Z, Y)X + g(X, Y)Z] = 0,$$

for all $X, Y \in \Gamma(TM)$, then Z is a projective vector field.

Proof. This can easily be obtained from Corollary 3.15 (see page 45 of [11]). ■

Lemma 3.17. *Let (M, g) be an Einstein n-dimensional Riemannian manifold with scalar curvature τ. If Z is a non-zero vector field satisfying the equation*

$$(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Z, Y)X + g(X, Y)Z] = 0, \lambda > 0,$$

for all $X, Y \in \Gamma(TM)$, then

$$\lambda = \frac{\tau}{n(n-1)}.$$

Proof. If (M, g) is an Einstein n-dimensional Riemannian manifold with scalar curvature τ and Z be a vector field on (M, g) then

$$div\, \Delta Z = \frac{\tau}{n} div\, Z + \Delta div\, Z,$$

by *Lemma 3.8* of [4]. On the other hand, $\Delta Z = -(n+3)\lambda Z$ by Lemma 3.1. Hence

$$\begin{aligned} div\,\Delta Z &= div\,[-(n+3)\lambda Z] \\ &= -(n+3)\lambda div\,Z \\ &= \frac{\tau}{n} div\,Z + \Delta div\,Z, \end{aligned}$$

which implies

$$\begin{aligned} \Delta div\,Z &= -(n+3)\lambda div\,Z - \frac{\tau}{n} div\,Z \\ &= -[(n+3)\lambda + \frac{\tau}{n}] div\,Z. \end{aligned}$$

Comparing this with

$$\Delta div\,Z = -2(n+1)\lambda div\,Z,$$

by Corollary 3.9 yields

$$\begin{aligned} -[(n+3)\lambda + \frac{\tau}{n}] = -2(n+1)\lambda \Rightarrow \frac{\tau}{n} &= [2(n+1) - (n+3)]\lambda \\ \Rightarrow \lambda &= \frac{\tau}{n(n-1)}. \end{aligned}$$ ■

Theorem 3.18. *Let (M, g) be a connected, simply connected, complete, $n(\geq 2$-dimensional Riemannian manifold. Then, a necessary and a sufficient condition for (M, g) to be isometric with the Euclidian sphere of radius $\frac{1}{\sqrt{\lambda}}, \lambda > 0$, is the existence of a nonzero vector field Z on M satisfying the equation*

$$(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Z, Y)X + g(X, Y)Z] = 0, \lambda > 0,$$

for all $X, Y \in \Gamma(TM)$.

Proof. It follows from Theorem A of [10] together with *Lemma* 3.13 for $f = div\,Z$. ■

Remark 3.19. Note that, the differential equation $(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Z, Y)X + g(X, Y)Z] = 0, \lambda > 0$, can also be considered as an analytic characterization (or representative) of Euclidian spheres in the class of connected, simply connected, complete Riemannian manifolds by Theorem 3.18.

Theorem 3.20. *Let (M, g) be an, $n(\geq 2)$-dimensional Riemannian manifold. If there exist a nonzero vector field Z on (M, g) satisfying the equation*

$$(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Z, Y)X + g(X, Y)Z] = 0, \lambda > 0,$$

for all $X, Y \in \Gamma(TM)$ and if (M, g) contains the whole trajectory of Z with its limit points, then (M, g) is of constant curvature at each point of the trajectory.

Proof. It follows from Theorem B of [10] together with Lemma 3.13 for $f = div\, Z$. ∎

Remark 3.21. The assumption $\lambda > 0$ implies that $\tau > 0$ in Lemma 3.17 and hence below.

Theorem 3.22. *Let (M, g) be a complete, $n(\geq 2)$-dimensional Einstein space of (positive) constant scalar curvature τ. If there exist a nonzero vector field Z on (M, g) satisfying the equation*

$$(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Z, Y)X + g(X, Y)Z] = 0, \lambda > 0,$$

for all $X, Y \in \Gamma(TM)$, then (M, g) is of constant curvature λ.

Proof. It follows from [7] together with Corollary 3.8 or Corollary 3.16 and Lemma 3.17 (see Theorem 9.1 in [10] also). ∎

Theorem 3.23. *Let (M, g) be a complete, $n(\geq 2)$-dimensional Riemannian manifold of (positive) constant scalar curvature τ. If there exist a nonzero vector field Z on (M, g) satisfying the equation*

$$(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Z, Y)X + g(X, Y)Z] = 0, \lambda > 0,$$

for all $X, Y \in \Gamma(TM)$, then (M, g) is of constant curvature $\lambda = \frac{\tau}{n(n-1)}$.

Proof. It follows from Theorem 9.2 of [10] together with *Corollary* 3.8 or Corollary 3.16 and *Lemma* 3.17. ∎

Remark 3.24. Let (M, g) be a compact $n(\geq 2)$-dimensional Riemannian manifold. Recall that the tension operator $\Box$ is also a linear, self-adjoint, elliptic operator with respect to the inner product on $\Gamma(TM)$ defined by

$$< X, Y >= \int_M g(X, Y),$$

where X, Y are vector fields on (M, g). Hence furthermore, if (M, g) is Einstein with $\tau > 0$ then eigenvalues of $\Box$ bounded from above by $\tau(\frac{n-2}{n(n-1)})$ by Theorem 3.9 of [4]. That is, if Z is a nonzero vector field satisfying the eigenvalue equation $\Box Z = \mu Z$, then $\mu \leq \tau(\frac{n-2}{n(n-1)})$.

Also see [3] for a survey on characterizing specific Riemannian manifolds by differential equations.

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