TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 2, pp. 745-772, April 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

COMMON FIXED POINTS OF A FINITE FAMILY OF NONSELF GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

Shuechin Huang

Abstract. Suppose that C is a nonempty subset of a real Banach space X. In this article, we construct two types of iterative schemes with errors for a finite family $\{T_i\}_{i=1}^k$ of nonself generalized asymptotically quasi-nonexpansive mappings of C into X. Furthermore, not only a necessary and sufficient condition for $\{x_n\}$ generated by each of those iterations to converge to a common fixed point of $\{T_i\}_{i=1}^k$ is obtained, but also the weak and strong convergence theorems of $\{x_n\}$ in uniformly convex Banach spaces are established as well.

1. Introduction

Let X be a real Banach space and let C be a nonempty subset of X. Suppose that $T : C \to X$ is a mapping and denote the fixed point set of T by $F(T)$. We recall the following definition, when T is a self-mapping of C.

Definition 1.1. *A mapping $T : C \to C$ is said to be*

(i) *uniformly λ-Lipschitzian if there exists $\lambda > 0$ such that*

(1) $$\|T^n x - T^n y\| \leq \lambda \|x - y\|, \quad \forall x, y \in C,\ n \in \mathbf{N};$$

(ii) *nonexpansive if*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C,\ n \in \mathbf{N};$$

(iii) *quasi-nonexpansive if*

Received June 1, 2009, accepted November 5, 2009.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: Primary 47H10, 54H25.
Key words and phrases: Nonself generalized asymptotically quasi-nonexpansive mapping, Lipschitzian mapping, Retract, Retraction, Opial property, Kadec-Klee property, Demiclosedness, Semicompactness, Condition (A).
This research was partially supported by a grant from the National Science Council.

$$\|Tx - p\| \le \|x - p\|, \quad \forall x \in C,\ p \in F(T),\ n \in \mathbf{N};$$

(iv) *asymptotically nonexpansive if there exists a sequence* $\{r_n\}$ *in* $[0,1]$ *with* $\lim_{n\to\infty} r_n = 0$ *such that*

$$\|T^n x - T^n y\| \le (1 + r_n)\|x - y\|, \quad \forall x, y \in C,\ n \in \mathbf{N}; \tag{2}$$

(v) *asymptotically quasi-nonexpansive if there exists a sequence* $\{r_n\}$ *in* $[0,1]$ *with* $\lim_{n\to\infty} r_n = 0$ *such that*

$$\|T^n x - p\| \le (1 + r_n)\|x - p\|, \quad \forall x \in C,\ p \in F(T),\ n \in \mathbf{N}; \tag{3}$$

(vi) *generalized asymptotically quasi-nonexpansive if there exist two sequences* $\{r_n\}$ *and* $\{s_n\}$ *in* $[0,1]$ *with* $\lim_{n\to\infty} r_n = 0$ *and* $\lim_{n\to\infty} s_n = 0$ *such that*

$$\|T^n x - p\| \le (1 + r_n)\|x - p\| + s_n\|x - T^n x\|,\ \forall x \in C,\ p \in F(T),\ n \in \mathbf{N}. \tag{4}$$

Suppose that C is a convex subset of a real Banach space X. There are three classical iterations used to approximate a fixed point of a nonexpansive mapping $T : C \to C$. That is,

(i) *Halpern* iteration [11]: Choose $u, x_1 \in C$ and define
$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \in \mathbf{N},$$
where $\{\alpha_n\} \subset [0,1]$.

(ii) *Mann* iteration [17]: Choose $x_1 \in C$ and define
$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbf{N},$$
where $\{\alpha_n\} \subset [0,1]$.

(iii) *Ishikawa* iteration [12]: Choose $x_1 \in C$ and define
$$y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$
$$x_{n+1} = \beta_n x_n + (1 - \beta_n)Ty_n, \quad n \in \mathbf{N},$$
where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$.

If T is yet a mapping from C into X, the preceding iterations may not be well defined. A motivation of this paper was to construct an iterative scheme with errors for a finite family of nonself mappings and generate a sequence to approximate a common fixed point of them.

In 2003, Chidume, Ofoedu and Zegeye [4] introduced the notion of nonself asymptotically nonexpansive mappings as an important generalization of asymptotically nonexpansive self-mappings.

Definition 1.2. *A subset A of a topological space X is said to be a retract of X if there exists a continuous mapping $R : X \to A$ (called a retraction) such that $R(a) = a$, for all $a \in A$. If, in addition, R is nonexpansive, then A is said to be a nonexpansive retract of X.*

If $R : X \to A$ is a retraction, then $R^2 = R$. A retract of a Hausdorff space must be a closed subset. Every closed convex subset of a uniformly convex Banach space is a retract.

Definition 1.3. *Let C be a nonempty subset of a real Banach space X and let $R : X \to C$ be a nonexpansive retraction of X onto C. A nonself mapping $T : C \to X$ is said to be*

(i) *uniformly λ-Lipschitzian if there exists $\lambda > 0$ such that*

$$\|T(RT)^{n-1}x - T(RT)^{n-1}y\| \leq \lambda\|x - y\|, \quad \forall x, y \in C,\ n \in \mathbf{N}; \tag{5}$$

(ii) *asymptotically nonexpansive if there exists a sequence $\{r_n\}$ in $[0, 1]$ with $\lim_{n\to\infty} r_n = 0$ such that*

$$\|T(RT)^{n-1}x - T(RT)^{n-1}y\| \leq (1 + r_n)\|x - y\|, \quad \forall x, y \in C,\ n \in \mathbf{N}; \tag{6}$$

(iii) *asymptotically quasi-nonexpansive if there exists a sequence $\{r_n\}$ in $[0, 1]$ with $\lim_{n\to\infty} r_n = 0$ such that*

$$\|T(RT)^{n-1}x - p\| \leq (1 + r_n)\|x - p\|, \quad \forall x \in C,\ p \in F(T),\ n \in \mathbf{N}; \tag{7}$$

(iv) *generalized asymptotically quasi-nonexpansive if there exist two sequences $\{r_n\}$ and $\{s_n\}$ in $[0, 1]$ with $\lim_{n\to\infty} r_n = 0$ and $\lim_{n\to\infty} s_n = 0$ such that*

$$\|T(RT)^{n-1}x - p\| \leq (1 + r_n)\|x - p\| + s_n\|x - T(RT)^{n-1}x\|, \tag{8}$$

for all $x \in C$, $p \in F(T)$, $n \in \mathbf{N}$.

If T is a self-mapping of C in Definition 1.3, then (5)-(8) in which R becomes the identity mapping are exactly the cases (1)-(4) respectively. Suppose that $T : C \to X$ is generalized asymptotically quasi-nonexpansive with respective to $\{r_n\}$ and $\{s_n\}$. If $s_1 < 1$, then using (8) with $n = 1$, we see that $F(T)$ is closed. Also, we have the following implications from this definition:

$$
\begin{array}{ccccc}
 & & \begin{array}{c}\text{uniformly}\\ \text{Lipschitzian}\end{array} & \Longrightarrow & \begin{array}{c}\text{nonself uniformly}\\ \text{Lipschitzian}\end{array} \\
 & & \Uparrow & & \\
\text{nonexpansive} & \Longrightarrow & \begin{array}{c}\text{asymptotically}\\ \text{nonexpansive}\end{array} & \Longrightarrow & \begin{array}{c}\text{nonself asymptotically}\\ \text{nonexpansive}\end{array} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{quasi-nonexpansive} & \Longrightarrow & \begin{array}{c}\text{asymptotically}\\ \text{quasi-nonexpansive}\end{array} & \Longrightarrow & \begin{array}{c}\text{nonself asymptotically}\\ \text{quasi-nonexpansive}\end{array} \\
 & & \Downarrow & & \Downarrow \\
 & & \begin{array}{c}\text{generalized}\\ \text{asymptotically}\\ \text{quasi-nonexpansive}\end{array} & \Longrightarrow & \begin{array}{c}\textit{nonself generalized}\\ \textit{asymptotically}\\ \textit{quasi-nonexpansive}\end{array}
\end{array}
$$

Let X be a real Banach space and let C be a nonempty subset of X and let $\{T_i : C \to X\}_{i=1}^{k}$ be a family of (not necessarily distinct) nonself generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$. Suppose that, for $i = 1, \ldots, k$, $\{u_{in}\}_{n=1}^{\infty} \subset X$ and $\{\alpha_{in}\}_{n=1}^{\infty}, \{\beta_{in}\}_{n=1}^{\infty} \subset [0,1]$ such that $\alpha_{in} + \beta_{in} \leq 1$. Let $R : X \to C$ be a nonexpansive retraction. This work is devoted to study the following two types of iterative schemes with errors for $\{T_i\}_{i=1}^{k}$:

I. Choose x_1 arbitrarily in C. Define an iterative sequence by

$$
\begin{aligned}
y_{1n} &= R((1-\alpha_{1n}-\beta_{1n})x_n + \alpha_{1n}T_1(RT_1)^{n-1}x_n + \beta_{1n}u_{1n}),\\
y_{in} &= R((1-\alpha_{in}-\beta_{in})x_n + \alpha_{in}T_i(RT_i)^{n-1}y_{(i-1)n} + \beta_{in}u_{in}),\ \ 2 \leq i \leq k-1,\\
x_{n+1} = y_{kn} &= R((1-\alpha_{kn}-\beta_{kn})x_n + \alpha_{kn}T_k(RT_k)^{n-1}y_{(k-1)n} + \beta_{kn}u_{kn}),\ \ n \in \mathbf{N}.
\end{aligned} \tag{9}
$$

II. Choose x_1 arbitrarily in C. Define an iterative sequence by

$$
\begin{aligned}
y_{1n} &= R((1 - \alpha_{1n} - \beta_{1n})x_n + \alpha_{1n}T_1(RT_1)^{n-1}x_n + \beta_{1n}u_{1n}),\\
y_{in} &= R((1-\alpha_{in}-\beta_{in})y_{(i-2)n} + \alpha_{in}T_i(RT_i)^{n-1}y_{(i-1)n} + \beta_{in}u_{in}),\ \ 2 \leq i \leq k-1,\\
x_{n+1} = y_{kn} &= R((1-\alpha_{kn}-\beta_{kn})y_{(k-2)n} + \alpha_{kn}T_k(RT_k)^{n-1}y_{(k-1)n} + \beta_{kn}u_{kn}),\ \ n \in \mathbf{N},
\end{aligned} \tag{10}
$$

where $y_{0n} = x_n$.

The algorithm (9) is the generalized process of the classical finite-step iterations, while the algorithm (10) is a new iterative scheme. In fact, the problem of the convergence of $\{x_n\}$ generated by (9) to a fixed point of (generalized) asymptotically quasi-nonexpansive self-mappings has been tremendously studied. In particular, if $k = 1$, the iteration (9) is precisely the Mann iteration. When $k = 2$ and T_1 and T_2 are (possibly the same) self-mappings of C, the iterative scheme (9) is reduced to the Ishikawa iteration; see, e.g., [8, 14, 15, 28]. If $k = 3$ and $T_1 = T_2 = T_3$, the iterative scheme (9) is the three-step iteration, i.e., the modified Mann and Ishikawa iteration introduced by Xu and Noor [29]; see, e.g., [19, 27]. It is worthy of note that the three-step iteration was extended to the k-step iteration, where $k \geq 4$, and many nice results have been established; see, e.g., [2, 3, 13, 20]. On the other hand, the implicit iterative scheme is another new iteration introduced by Xu and Ori [30] for a finite family of nonexpansive mappings; also see, e.g., [5, 23, 24]. In [6], Deng and Liu gave a new iterative process, the modified Ishikawa iteration with finite steps, and established the strong convergence theorems for two nonself generalized asymptotically quasi-nonexpansive mappings.

In this paper, a necessary and sufficient condition for $\{x_n\}$ generated by each of iterations (9) and (10) to converge to a common fixed point of $\{T_i\}_{i=1}^k$ is obtained. Not only the strong convergence theorems of $\{x_n\}$ in uniformly convex Banach spaces, but also the weak convergence theorems in uniformly convex Banach spaces which satisfies the Opial property, or whose dual space has the Kadec-Klee property are established as well. In fact, a dual space of a reflexive Banach space with a Fréchet differentiable norm or the Opial property also satisfies the Kadec-Klee property [7]. There exist uniformly convex Banach spaces which have neither a Fréchet differentiable norm nor the Opial property, but their dual spaces do have the Kadec-Klee property; see [9]. Therefore the Opial property is independent of uniform convexity. To a certain extent, a part of this work based on (9) can be viewed as an extension of the results in the literature; see, e.g., [13, 19, 20, 28].

2. Preliminaries

Suppose that X is a real Banach space and C is a subset of X. Then X is said to be *uniformly convex* [1] if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for $x, y \in X$ with $\|x\| \leq 1$ and $\|y\| \leq 1$,

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\epsilon), \quad \text{whenever } \|x-y\| \geq \epsilon.$$

A mapping $T : C \to X$ is *semicompact* (or *hemicompact*) [26] if for any sequence $\{x_n\}$ in C with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence of $\{x_n\}$ which converges strongly to a point of C.

A family $\{T_i : C \to X\}_{i=1}^k$ of nonself mappings with $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$ is said to satisfy *Condition* (A) *with respect to* E [5, 6], where $E \subset C$, if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\max_{1\leq i\leq k}\{\|x - T_i x\|\} \geq f(d(x,F)), \quad \forall x \in E,$$

where $d(x,F) = \inf\{\|x-y\| : y \in F\}$. When $k = 1$ and T_1 is a self-mapping of C, Condition (A) reduces to the one discussed in [16, 22, 25].

A Banach space X is said to satisfy the *Opial property* [10] if whenever a sequence $\{x_n\}$ in X converges weakly to x, then $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$, for $y \neq x$. We say that X has the *Kadec-Klee property* if, for any sequence $\{x_n\}$ in E which converges weakly to $x \in X$ and $\|x_n\| \to \|x\|$, we have $\{x_n\}$ converges strongly to x. Every locally uniformly convex normed space, for instance, L_p spaces, $1 < p < \infty$, has this property. We also remark that L_p spaces, $1 < p < \infty$ and $p \neq 2$, do not satisfy the Opial property, but their dual spaces have the Kadec-Klee property.

The lemmas stated in this section will be required in the sequel.

Lemma 2.1. [15]. *Let $\{a_n\}$, $\{\epsilon_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the following condition:*

$$a_{n+1} \leq (1 + \epsilon_n)a_n + \delta_n.$$

If $\sum_{n=1}^{\infty} \epsilon_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to 0, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. (Schu's lemma [21]). *Let X be a uniformly convex Banach space. Suppose that $0 < a \leq t_n \leq b < 1$ for all $n \in \mathbf{N}$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\limsup_{n\to\infty} \|x_n\| \leq c$, $\limsup_{n\to\infty} \|y_n\| \leq c$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = c$, for some number $c \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Recall that a mapping $T : C \to X$ is *demiclosed* at $y \in X$ if, for any sequence $\{x_n\}$ in C, the conditions $x_n \to x \in C$ weakly and $T(x_n) \to y$ strongly together imply $T(x) = y$.

Lemma 2.3. (Demiclosed principle for nonself mappings [4]). *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and $T : C \to X$ an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

The following result enables us to establish the weak convergence theorems of iterative schemes in a uniformly convex Banach space whose dual space has the Kadec-Klee property. We will denote $\omega_w\{x_n\}$ the set of the weak subsequential limits of $\{x_n\}$, i.e., the set of all limits of all weakly convergent subsequences.

Lemma 2.4. [9]. *Let X be a uniformly convex Banach space such that its dual X^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in X. If $\lim_{n\to\infty} \|tx_n + (1 - t)p - q\|$ exists, for all $t \in [0, 1]$ and $p, q \in \omega_w\{x_n\}$, then $p = q$.*

Lemma 2.5. [9, 18]. *Let X be a uniformly convex Banach space, C a bounded closed convex subset of X and $T : C \to C$ an L-Lipschitz mapping. Then there exists a strictly increasing continuous convex function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that*

$$\|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\| \leq L\phi^{-1}\left(\|x-y\| - \frac{1}{L}\|Tx - Ty\|\right),$$

for all $x, y \in C$ and $\lambda \in (0, 1)$.

3. A Necessary and Sufficient Condition

Theorem 3.1. *Let X be a real Banach space, C a closed convex subset of X, $\{u_{in}\}_{n=1}^{\infty}$, $i = 1, \ldots, k$, bounded sequences in X, $R : X \to C$ a nonexpansive retraction, and $\{T_i : C \to X\}_{i=1}^{k}$ a family of nonself generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ is closed. Suppose that*

(*i*) $\sum_{n=1}^{\infty} r_{in} < \infty$ *and* $\sum_{n=1}^{\infty} s_{in} < \infty$, *for* $i = 1, \ldots, k$;

(*ii*) $\sum_{n=1}^{\infty} \beta_{in} < \infty$, *for* $i = 1, \ldots, k$.

Then the sequence $\{x_n\}$ defined by (9) converges strongly to a common fixed point of $\{T_i\}_{i=1}^{k}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. Since the necessity is clear, we only prove the sufficiency. It may be assumed that $s_{in} < 1$, for $1 \leq i \leq k$ and $n \in \mathbf{N}$. Let $r_n = \max\{r_{in} : i = 1, \ldots, k\}$, $s_n = \max\{s_{in} : i = 1, \ldots, k\}$ so that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ which is equivalent to the condition that

$$\sum_{n=1}^{\infty} \frac{r_n + 2s_n}{1 - s_n} < \infty.$$

Let $\lambda_n = (r_n + 2s_n)/(1 - s_n)$ and $\tau_n = 1 + \lambda_n = (1 + r_n + s_n)/(1 - s_n)$ for $n \in \mathbf{N}$. Then $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\lim_{n\to\infty} \tau_n = 1$. Fix any $p \in F$ and set $M_p = \sup\{\|u_{in} - p\| : 1 \leq i \leq k, n \in \mathbf{N}\} < \infty$ (depending on p). For any $x \in C$, $1 \leq i \leq k$ and $n \in \mathbf{N}$, we have

$$\begin{aligned}
\|x - T_i(RT_i)^{n-1}x\| &\leq \|x - p\| + \|T_i(RT_i)^{n-1}x - p\| \\
&\leq (2 + r_{in})\|x - p\| + s_{in}\|x - T_i(RT_i)^{n-1}x\| \\
&\leq (2 + r_n)\|x - p\| + s_n\|x - T_i(RT_i)^{n-1}x\|
\end{aligned}$$

which implies that

$$\|x - T_i(RT_i)^{n-1}x\| \leq \frac{2 + r_n}{1 - s_n}\|x - p\|.$$

It follows that for all $x \in C$, $1 \leq i \leq k$ and $n \in \mathbf{N}$,

$$
\begin{aligned}
\|T_i(RT_i)^{n-1}x - p\| &\leq (1 + r_{in})\|x - p\| + s_{in}\|x - T_i(RT_i)^{n-1}x\| \\
&\leq \left[1 + r_n + \frac{s_n(2 + r_n)}{1 - s_n}\right]\|x - p\| \\
&= \tau_n\|x - p\|.
\end{aligned} \tag{11}
$$

This asserts that

$$
\begin{aligned}
\|y_{1n} - p\| &= \|R((1 - \alpha_{1n} - \beta_{1n})x_n + \alpha_{1n}T_1(RT_1)^{n-1}x_n + \beta_{1n}u_{1n}) - R(p)\| \\
&\leq \|(1 - \alpha_{1n} - \beta_{1n})x_n + \alpha_{1n}T_1(RT_1)^{n-1}x_n + \beta_{1n}u_{1n} - p\| \\
&\leq (1 - \alpha_{1n} - \beta_{1n})\|x_n - p\| + \alpha_{1n}\|T_1(RT_1)^{n-1}x_n - p\| \\
&\quad + \beta_{1n}\|u_{1n} - p\|, \\
&\leq (1 - \alpha_{1n} - \beta_{1n} + \alpha_{1n}\tau_n)\|x_n - p\| + \beta_{1n}\|u_{1n} - p\| \\
&\leq \tau_n\|x_n - p\| + \beta_{1n}\tau_n\|u_{1n} - p\|.
\end{aligned} \tag{12}
$$

Therefore (11) and (12) imply that

$$
\begin{aligned}
\|y_{2n} - p\| &\leq (1 - \alpha_{2n} - \beta_{2n})\|x_n - p\| + \alpha_{2n}\|T_2(RT_2)^{n-1}y_{1n} - p\| \\
&\quad + \beta_{2n}\|u_{2n} - p\| \\
&\leq (1 - \alpha_{2n} - \beta_{2n})\|x_n - p\| + \alpha_{2n}\tau_n\|y_{1n} - p\| + \beta_{2n}\|u_{2n} - p\| \\
&\leq (1 - \alpha_{2n} - \beta_{2n} + \alpha_{2n}\tau_n^2)\|x_n - p\| + \alpha_{2n}\beta_{1n}\tau_n^2\|u_{1n} - p\| \\
&\quad + \beta_{2n}\|u_{2n} - p\| \\
&\leq \tau_n^2\|x_n - p\| + \tau_n^2[\beta_{1n}\|u_{1n} - p\| + \beta_{2n}\|u_{2n} - p\|].
\end{aligned}
$$

Now repeat this step inductively to get

$$
\begin{aligned}
\|y_{in} - p\| &\leq (1 - \alpha_{in} - \beta_{in})\|x_n - p\| + \alpha_{in}\|T_i(RT_i)^{n-1}y_{(i-1)n} - p\| \\
&\quad + \beta_{in}\|u_{in} - p\| \\
&\leq (1 - \alpha_{in} - \beta_{in})\|x_n - p\| + \alpha_{in}\tau_n\|y_{(i-1)n} - p\| + \beta_{in}\|u_{in} - p\| \\
&\leq (1 - \alpha_{in} - \beta_{in} + \alpha_{in}\tau_n^i)\|x_n - p\| + \tau_n^i[\beta_{1n}\|u_{1n} - p\| + \cdots \\
&\quad + \beta_{(i-1)n}\|u_{(i-1)n} - p\|] + \beta_{in}\|u_{in} - p\| \\
&\leq \tau_n^i\|x_n - p\| + \tau_n^i \textstyle\sum_{j=1}^{i} \beta_{jn}\|u_{jn} - p\|,
\end{aligned} \tag{13}
$$

for $i = 1, \ldots, k$, where $y_{0n} = x_n$. In particular,

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|y_{kn} - p\| \\
&\le \tau_n^k \|x_n - p\| + \tau_n^k \sum_{j=1}^{k} \beta_{jn} \|u_{in} - p\| \\
&\le (1+\lambda_n)^k \|x_n - p\| + \tau_n^k M_p \sum_{j=1}^{k} \beta_{in} \\
&= \left[1 + \lambda_n \sum_{j=0}^{k-1} (1+\lambda_n)^j\right] \|x_n - p\| + \tau_n^k M_p \sum_{j=1}^{k} \beta_{in} \\
&\le (1+\epsilon_n)\|x_n - p\| + M_p \delta_n,
\end{aligned} \tag{14}
$$

where $\sigma = \sup\{\tau_n : n \in \mathbf{N}\}$, $\epsilon_n = \lambda_n \sum_{j=0}^{k-1}(1+\lambda_n)^j$ and $\delta_n = \sigma^k \sum_{j=1}^{k} \beta_{jn}$. Note that by hypotheses,

$$
\sum_{n=1}^{\infty} \epsilon_n < \infty, \quad \sum_{n=1}^{\infty} \delta_n < \infty. \tag{15}
$$

Now we claim that $\{x_n\}$ is a Cauchy sequence in X. To see this, applying the inequality $1 + t \le e^t$ for all $t \ge 0$, we derive from (14) that

$$
\begin{aligned}
\|x_{n+1} - p\| &\le (1+\lambda_n)^k \|x_n - p\| + M_p \delta_n \\
&\le e^{k\lambda_n} \|x_n - p\| + M_p \delta_n,
\end{aligned}
$$

which shows that for $m, n \ge 1$,

$$
\begin{aligned}
\|x_{n+m} - p\| &\le e^{k\lambda_{n+m-1}} \|x_{n+m-1} - p\| + M_p \delta_{n+m-1} \\
&\le e^{k(\lambda_{n+m-1}+\lambda_{n+m-2})} \|x_{n+m-2} - p\| + e^{k\lambda_{n+m-1}} M_p \delta_{n+m-2} \\
&\quad + M_p \delta_{n+m-1} \\
&\le e^{k(\lambda_{n+m-1}+\lambda_{n+m-2})} \|x_{n+m-2} - p\| \\
&\quad + e^{k(\lambda_{n+m-1}+\lambda_{n+m-2})} M_p (\delta_{n+m-2} + \delta_{n+m-1}) \\
&\le \cdots \\
&\le e^{k\sum_{i=1}^{\infty}\lambda_i} \|x_n - p\| + e^{k\sum_{i=1}^{\infty}\lambda_i} M_p \sum_{i=n}^{n+m-1} \delta_i.
\end{aligned} \tag{16}
$$

Set $e^{k\sum_{i=1}^{\infty}\lambda_i} = L$. Given $\epsilon > 0$, it follows from $\liminf_{n\to\infty} d(x_n, F) = 0$ and (15) that there exist a positive integer n_0 and a point $q \in F$ such that

$$\|x_{n_0} - q\| < \epsilon, \qquad \sum_{i=n_0}^{n_0+m-1} \delta_i < \epsilon. \tag{17}$$

Therefore, according to (16) and (17), for all $m \geq 1$,

$$\begin{aligned}\|x_{n_0+m} - x_{n_0}\| &\leq \|x_{n_0+m} - q\| + \|x_{n_0} - q\| \\ &\leq L\|x_{n_0} - q\| + LM_q \sum_{i=n_0}^{n_0+m-1} \delta_i + \epsilon \\ &< (L + LM_q + 1)\epsilon\,;\end{aligned}$$

hence $\{x_n\}$ is a Cauchy sequence in X. The completeness of X assures that $\{x_n\}$ converges strongly to a point, say x^*. Also, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} d(x_{n_j}, F) = 0$, since $\liminf_{n\to\infty} d(x_n, F) = 0$. Therefore the continuity of the mapping $z \mapsto d(z, F)$ and the closedness of F imply that $d(x^*, F) = 0$ and so $x^* \in F$, as required. ∎

As shown in the preceding proof, the property needed to assure that $x^* \in F$ is exactly the following one. Given any sequence $\{a_n\}$ of real numbers there is a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $\lim_{j\to\infty} a_{n_j} = \liminf_{n\to\infty} a_n$. In general, if $\{a_{m_j}\}$ is a convergent subsequence of $\{a_n\}$, then $\liminf_{n\to\infty} a_n \leq \lim_{j\to\infty} a_{m_j}$. This immediately yields the following result.

Corollary 3.2. *Let X be a real Banach space, C a closed convex subset of X, $\{u_{in}\}_{n=1}^{\infty}$, $i = 1, \ldots, k$, bounded sequences in X, $R : X \to C$ a nonexpansive retraction, and $\{T_i : C \to X\}_{i=1}^{k}$ a family of nonself generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ is closed. Suppose that*

(i) $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$, for $i = 1, \ldots, k$;

(ii) $\sum_{n=1}^{\infty} \beta_{in} < \infty$, for $i = 1, \ldots, k$.

Then the sequence $\{x_n\}$ defined by (9) converges strongly to a point $p \in F$ if and only if there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging strongly to p.

We now consider the iteration generated by (10) and obtain a necessary and sufficient condition for the sequence $\{x_n\}$ to converge strongly to a common fixed point of finitely many nonself generalized asymptotically quasi-nonexpansive mappings.

Theorem 3.3. *Let X be a real Banach space, C a closed convex subset of X, $\{u_{in}\}_{n=1}^{\infty}$, $i = 1, \ldots, k$, bounded sequences in X, $R : X \to C$ a nonexpansive retraction, and $\{T_i : C \to X\}_{i=1}^{k}$ a family of nonself generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ is closed. Suppose that*

(*i*) $\sum_{n=1}^{\infty} r_{in} < \infty$ *and* $\sum_{n=1}^{\infty} s_{in} < \infty$*, for* $i = 1, \ldots, k$*;*
(*ii*) $\sum_{n=1}^{\infty} \beta_{in} < \infty$*, for* $i = 1, \ldots, k$.

Then the sequence $\{x_n\}$ *defined by (10) converges strongly to a common fixed point of* $\{T_i\}_{i=1}^{k}$ *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. We use the same arguments and notation r_n, s_n, λ_n and τ_n as in the proof of Theorem 3.1. Fix any $p \in F$ and set $M_p = \sup\{\|u_{in} - p\| : 1 \le i \le k, n \in \mathbf{N}\}$. It follows from (11) and (12) that

$$\begin{aligned}\|y_{2n} - p\| &\le (1 - \alpha_{2n} - \beta_{2n})\|y_{01} - p\| + \alpha_{2n}\|T_2(RT_2)^{n-1}y_{1n} - p\| \\ &\quad + \beta_{2n}\|u_{2n} - p\| \\ &\le (1 - \alpha_{2n} - \beta_{2n})\|x_n - p\| + \alpha_{2n}\tau_n\|y_{1n} - p\| + \beta_{2n}\|u_{2n} - p\| \\ &\le \tau_n^2\|x_n - p\| + \tau_n^2[\beta_{1n}\|u_{1n} - p\| + \beta_{2n}\|u_{2n} - p\|].\end{aligned}$$

Therefore we derive inductively to get

$$\begin{aligned}\|y_{in} - p\| &\le (1 - \alpha_{in} - \beta_{in})\|y_{(i-2)n} - p\| + \alpha_{in}\|T_i(RT_i)^{n-1}y_{(i-1)n} - p\| \\ &\quad + \beta_{in}\|u_{in} - p\| \\ &\le (1 - \alpha_{in} - \beta_{in})\|y_{(i-2)n} - p\| + \alpha_{in}\tau_n\|y_{(i-1)n} - p\| + \beta_{in}\|u_{in} - p\| \\ &\le (1 - \alpha_{in} - \beta_{in})\tau_n^{i-2}\left(\|x_n - p\| + \sum_{j=1}^{i-2}\beta_{jn}\|u_{jn} - p\|\right) \\ &\quad + \alpha_{in}\tau_n^{i}\left(\|x_n - p\| + \sum_{j=1}^{i-1}\beta_{jn}\|u_{jn} - p\|\right) + \beta_{in}\|u_{in} - p\| \\ &\le \tau_n^{i}\|x_n - p\| + \tau_n^{i}\sum_{j=1}^{i}\beta_{jn}\|u_{jn} - p\|,\end{aligned} \tag{18}$$

for $i = 1, \ldots, k$. In particular,

$$\begin{aligned}\|x_{n+1} - p\| &\le \tau_n^{k}\|x_n - p\| + \tau_n^{k}\sum_{j=1}^{k}\beta_{jn}\|u_{jn} - p\| \\ &\le (1 + \epsilon_n)\|x_n - p\| + M_p\delta_n,\end{aligned} \tag{19}$$

where $\sigma = \sup\{\tau_n : n \in \mathbf{N}\}$, $\epsilon_n = \lambda_n \sum_{j=0}^{k-1}(1 + \lambda_n)^j$ and $\delta_n = \sigma^k \sum_{j=1}^{k}\beta_{jn}$. Observe that

$$\sum_{n=1}^{\infty}\epsilon_n < \infty, \quad \sum_{n=1}^{\infty}\delta_n < \infty. \tag{20}$$

Therefore the rest of the proof is the same as that of Theorem 3.1. ∎

As mentioned before Corollary 3.2, the following result is an immediate consequence of Theorem 3.3.

Corollary 3.4. *Let X be a real Banach space, C a closed convex subset of X, $\{u_{in}\}_{n=1}^{\infty}$, $i = 1, \ldots, k$, bounded sequences in X, $R : X \to C$ a nonexpansive retraction, and $\{T_i : C \to X\}_{i=1}^{k}$ a family of nonself generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ is closed. Suppose that*

(i) $\sum_{n=1}^{\infty} r_{in} < \infty$ *and* $\sum_{n=1}^{\infty} s_{in} < \infty$, *for* $i = 1, \ldots, k$;

(ii) $\sum_{n=1}^{\infty} \beta_{in} < \infty$, *for* $i = 1, \ldots, k$.

Then the sequence $\{x_n\}$ defined by (10) converges strongly to a point $p \in F$ if and only if there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging to p.

4. Applications to Uniformly Convex Banach Spaces

In this section, we will apply the previous results in Section 3 to present the strong and weak convergence theorems in uniformly convex Banach spaces.

Theorem 4.1. *Let X be a uniformly convex Banach space, C a closed convex subset of X, $\{u_{in}\}_{n=1}^{\infty}$, $i = 1, \ldots, k$, bounded sequences in X, $R : X \to C$ a nonexpansive retraction, and $\{T_i : C \to X\}_{i=1}^{k}$ a family of nonself uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (9). Suppose that*

(i) $\sum_{n=1}^{\infty} r_{in} < \infty$ *and* $\sum_{n=1}^{\infty} s_{in} < \infty$, *for* $i = 1, \ldots, k$;

(ii) $0 < \liminf_{n\to\infty} \alpha_{in} \leq \limsup_{n\to\infty} \alpha_{in} < 1$ *and* $\sum_{n=1}^{\infty} \beta_{in} < \infty$, *for* $i = 1, \ldots, k$;

(iii) *the family $\{T_i\}_{i=1}^{k}$ satisfies Condition (A) with respect to $\{x_n\}$, or $T_{i_0}(RT_{i_0})^{m-1}$ is semicompact, for some $1 \leq i_0 \leq k$ and for some $m \geq 1$.*

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{k}$.

Proof. Note that F is closed because the fixed point set of a Lipschitzian mapping is closed. Let $p \in F$. By Lemma 2.1, (14) and (15) in the proof of Theorem 3.1 show that $\lim_{n\to\infty} \|x_n - p\|$ exists and hence $\{x_n\}$ is bounded. Set $\lim_{n\to\infty} \|x_n - p\| = c$, or equivalently,

$$\lim_{n\to\infty} \|R((1 - \alpha_{kn} - \beta_{kn})x_n + \alpha_{kn} T_k (RT_k)^{n-1} y_{(k-1)n} + \beta_{kn} u_{kn}) - R(p)\| = c,$$

which means that

$$c \le \liminf_{n\to\infty} \|\alpha_{kn}[T_k(RT_k)^{n-1}y_{(k-1)n} - p + \beta_{kn}(u_{kn} - x_n)] \\ +(1-\alpha_{kn})[x_n - p + \beta_{kn}(u_{kn} - x_n))]\|, \tag{21}$$

because R is nonexpansive. The inequality (13) and assumption (ii) imply that

$$\limsup_{n\to\infty} \|y_{in} - p\| \le c, \quad i = 1, \ldots, k. \tag{22}$$

By (11),

$$\|T_k(RT_k)^{n-1}y_{(k-1)n} - p + \beta_{kn}(u_{kn} - x_n)\| \le \tau_n\|y_{(k-1)n} - p\| + \beta_{kn}\|u_{kn} - x_n\|$$

which shows that

$$\limsup_{n\to\infty} \|T_k(RT_k)^{n-1}y_{(k-1)n} - p + \beta_{kn}(u_{kn} - x_n)\| \le c. \tag{23}$$

Also, since $\|x_n - p + \beta_{kn}(u_{kn} - x_n)]\| \le \|x_n - p\| + \beta_{kn}\|u_{kn} - x_n\|$, we have

$$\limsup_{n\to\infty} \|x_n - p + \beta_{kn}(u_{kn} - x_n)\| \le c. \tag{24}$$

Combining (23) with (24) yields that

$$c \ge \limsup_{n\to\infty} \|\alpha_{kn}[T_k(RT_k)^{n-1}y_{(k-1)n} - p + \beta_{kn}(u_{kn} - x_n)] \\ +(1-\alpha_{kn})[x_n - p + \beta_{kn}(u_{kn} - x_n))]\|. \tag{25}$$

Therefore (21) and (25) assert that

$$\lim_{n\to\infty} \|\alpha_{kn}[T_k(RT_k)^{n-1}y_{(k-1)n} - p + \beta_{kn}(u_{kn} - x_n)] \\ +(1-\alpha_{kn})[x_n - p + \beta_{kn}(u_{kn} - x_n))]\| = c. \tag{26}$$

According to Lemma 2.2 together with (23), (24) and (26), we obtain that

$$\lim_{n\to\infty} \|x_n - T_k(RT_k)^{n-1}y_{(k-1)n}\| = 0.$$

Since

$$\begin{aligned} \|x_n - p\| &\le \|x_n - T_k(RT_k)^{n-1}y_{(k-1)n}\| + \|T_k(RT_k)^{n-1}y_{(k-1)n} - p\| \\ &\le \|x_n - T_k(RT_k)^{n-1}y_{(k-1)n}\| + \tau_n\|y_{(k-1)n} - p\|, \end{aligned}$$

we have

$$c \le \liminf_{n\to\infty} \|y_{(k-1)n} - p\|. \tag{27}$$

By (22) and (27), $\lim_{n\to\infty}\|y_{(k-1)n}-p\|=c$. That is,

$$\lim_{n\to\infty}\|R((1-\alpha_{(k-1)n}-\beta_{(k-1)n})x_n+\alpha_{(k-1)n}T_{k-1}(RT_{k-1})^{n-1}y_{(k-2)n}$$
$$+\beta_{(k-1)n}(u_{(k-1)n})-R(p)\|=c.$$

As shown in (21)-(26), since

$$\limsup_{n\to\infty}\|T_{k-1}(RT_{k-1})^{n-1}y_{(k-2)n}-p+\beta_{(k-1)n}(u_{(k-1)n}-x_n)\|\le c$$

and

$$\limsup_{n\to\infty}\|x_n-p+\beta_{(k-1)n}(u_{(k-1)n}-x_n)\|\le c,$$

we see that

$$\lim_{n\to\infty}\|\alpha_{(k-1)n}[T_{k-1}(RT_{k-1})^{n-1}y_{(k-2)n}-p+\beta_{(k-1)n}(u_{(k-1)n}-x_n)]$$
$$+(1-\alpha_{(k-1)n})[x_n-p+\beta_{(k-1)n}(u_{(k-1)n}-x_n)]\|=c;$$

hence

$$\lim_{n\to\infty}\|x_n-T_{k-1}(RT_{k-1})^{n-1}y_{(k-2)n}\|=0.$$

Continuously proceed this process k times to conclude that

$$\lim_{n\to\infty}\|x_n-T_i(RT_i)^{n-1}y_{(i-1)n}\|=0,\quad i=1,\ldots,k,$$

where $y_{0n}=x_n$, from which it follows that

$$\begin{aligned}&\|x_n-x_{n+1}\|\\ &=\|R(x_n)-x_{n+1}\|\\ &\le\alpha_{kn}\|T_k(RT_k)^{n-1}y_{(k-1)n}-x_n\|+\beta_{kn}\|u_{kn}-x_n\|\to 0 \text{ as } n\to\infty,\end{aligned}\tag{28}$$

and

$$\begin{aligned}\|x_n-T_i(RT_i)^{n-1}x_n\| &\le \|x_n-T_i(RT_i)^{n-1}y_{(i-1)n}\|\\ &\quad+\|T_i(RT_i)^{n-1}y_{(i-1)n}-T_i(RT_i)^{n-1}x_n\|\\ &\le \|x_n-T_i(RT_i)^{n-1}y_{(i-1)n}\|+L\|y_{(i-1)n}-x_n\|\\ &\le \|x_n-T_i(RT_i)^{n-1}y_{(i-1)n}\|\\ &\quad+L\alpha_{(i-1)n}\|T_{i-1}(RT_{i-1})^{n-1}y_{(i-2)n}-x_n\|\\ &\quad+L\beta_{(i-1)n}\|u_{(i-1)n}-x_n\|\to 0\quad\text{as } n\to\infty,\end{aligned}\tag{29}$$

for $i = 1, \ldots, k$. Since

$$\begin{aligned}\|x_n - T_i x_n\| \le\ & \|x_n - x_{n+1}\| + \|x_{n+1} - T_i(RT_i)^n x_{n+1}\| \\ & + \|T_i(RT_i)^n x_{n+1} - T_i(RT_i)^n x_n\| + \|T_i(RT_i)^n x_n - (T_iR)x_n\| \\ \le\ & (1+L)\|x_n - x_{n+1}\| + \|x_{n+1} - T_i(RT_i)^n x_{n+1}\| \\ & + L\|T_i(RT_i)^{n-1}x_n - x_n\|,\end{aligned}$$

we use (28) and (29) to get

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad i = 1, \ldots, k. \tag{30}$$

To verify the strong convergence of $\{x_n\}$ to a point of F, we need to discuss two cases. First, suppose that $\{T_i\}_{i=1}^k$ satisfies Condition (A) with respect to $\{x_n\}$. Let f be the corresponding nondecreasing function for $\{T_i\}_{i=1}^k$ which satisfy Condition (A) with respect to $\{x_n\}$. According to (30),

$$f(d(x_n, F)) \le \max_{i\le i\le k} \|x_n - T_i x_n\| \to 0 \quad \text{as } n \to \infty,$$

and hence $\liminf_{n\to\infty} d(x_n, F) = 0$. Theorem 3.1 assures that $\{x_n\}$ converges strongly to a point of F.

Second, if $T_{i_0}^m$ is semicompact, for some $1 \le i_0 \le k$ and for some $m \ge 1$, it follows from (30) that

$$\begin{aligned}\|x_n - T_{i_0}(RT_{i_0})^{m-1}x_n\| \le\ & \|x_n - T_{i_0}x_n\| + \|T_{i_0}x_n - T_{i_0}(RT_{i_0})x_n\| + \cdots \\ & + \|T_{i_0}(RT_{i_0})^{m-2}x_n - T_{i_0}(RT_{i_0})^{m-1}x_n\| \\ \le\ & \|x_n - T_{i_0}x_n\| + (m-1)L\|Rx_n - RT_{i_0}x_n\| \\ \le\ & [1 + (m-1)L]\|x_n - T_{i_0}x_n\| \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

and thus there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to $x^* \in C$. Using (30), we have $T_i x^* = x^*$, $i = 1, \ldots, k$, and so $x^* \in F$. Therefore $\liminf_{n\to\infty} d(x_n, F) = 0$. By Theorem 3.1, $\{x_n\}$ converges strongly to x^*. ■

The following two results are the weak convergence theorems in a uniformly convex Banach space such that either it satisfies the Opial property or its dual space has the Kadec-Klee property.

Theorem 4.2. *Let X be a uniformly convex Banach space satisfying the Opial property, C a closed convex subset of X, $\{u_{in}\}_{n=1}^\infty$, $i = 1, \ldots, k$, bounded sequences in X, $R : X \to C$ a nonexpansive retraction, and $\{T_i : C \to X\}_{i=1}^k$ a* *family of nonself uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Suppose that*

(i) $\sum_{n=1}^{\infty} r_{in} < \infty$ *and* $\sum_{n=1}^{\infty} s_{in} < \infty$, *for* $i = 1, \ldots, k$;

(ii) $0 < \liminf_{n\to\infty} \alpha_{in} \leq \limsup_{n\to\infty} \alpha_{in} < 1$ *and* $\sum_{n=1}^{\infty} \beta_{in} < \infty$, *for* $i = 1, \ldots, k$;

(iii) $I - T_i$, $i = 1, \ldots, k$, *is demiclosed at 0, for* $i = 1, \ldots, k$.

Then the sequence $\{x_n\}$ defined by (9) converges weakly to a common fixed point of $\{T_i\}_{i=1}^{k}$.

Proof. Since a uniformly convex Banach space is reflexive, the bounded sequence $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ which converges weakly to a point x^* of C. The demiclosedness of each $I - T_i$ and (30) imply that $T_i x^* = x^*$, $i = 1, \ldots, k$, i.e., $x^* \in F$. To prove that $\{x_n\}$ converges weakly to x^*, let $\{x_{m_j}\}$ be any subsequence of $\{x_n\}$ which converges weakly to a point $\bar{x}$ so that $\bar{x} \in F$. Assume that $x^* \neq \bar{x}$. Then it follows from the Opial property that

$$\begin{aligned}\lim_{j\to\infty} \|x_{m_j} - x^*\| &= \lim_{n\to\infty} \|x_n - x^*\| = \lim_{j\to\infty} \|x_{n_j} - x^*\| \\ &< \lim_{j\to\infty} \|x_{n_j} - \bar{x}\| = \lim_{n\to\infty} \|x_n - \bar{x}\| = \lim_{j\to\infty} \|x_{m_j} - \bar{x}\|,\end{aligned}$$

which is a contradiction. Consequently, any subsequence of $\{x_n\}$ has a weakly convergent subsequence with limit x^*, and hence $\{x_n\}$ converges weakly to x^*. ∎

We consider an iteration defined by (9) for nonself asymptotically nonexpansive mappings, where $\beta_{in} = 0$, for all $1 \leq i \leq k$ and $n \in \mathbf{N}$. That is, choose x_1 arbitrarily in C. Define an iterative sequence as follows:

$$\begin{aligned} y_{1n} &= R((1-\alpha_{1n})x_n + \alpha_{1n} T_1 (RT_1)^{n-1} x_n), \\ y_{in} &= R((1-\alpha_{in})x_n + \alpha_{in} T_i (RT_i)^{n-1} y_{(i-1)n}), \quad 2 \leq i \leq k-1, \\ x_{n+1} = y_{kn} &= R((1-\alpha_{kn})x_n + \alpha_{kn} T_k (RT_k)^{n-1} y_{(k-1)n}), \quad n \in \mathbf{N}. \end{aligned} \tag{31}$$

Theorem 4.3. *Let X be a uniformly convex Banach space whose dual X^* has the Kadec-Klee property, C a closed convex subset of X, $R : X \to C$ a nonexpansive retraction, and $\{T_i : C \to X\}_{i=1}^{k}$ a family of nonself asymptotically nonexpansive mappings with respect to $\{r_{in}\}$ such that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Suppose that*

(i) $\sum_{n=1}^{\infty} r_{in} < \infty$, *for* $i = 1, \ldots, k$;

(*ii*) $0 < \liminf_{n\to\infty} \alpha_{in} \leq \limsup_{n\to\infty} \alpha_{in} < 1$, *for* $i = 1, \ldots, k$;

Then the sequence $\{x_n\}$ *defined by (31) converges weakly to a common fixed point of* $\{T_i\}_{i=1}^k$.

Proof. The idea of proof was referred to that of Lemma 3.8 in [2]. As discussed in Theorem 3.1, the sequence $\{x_n\}$ is bounded and hence there exists a closed ball B with $F \cap B \neq \emptyset$ such that $\{x_n\} \subset K = B \cap C$. Then K is a closed bounded convex subset of C. Let $r_n = \max\{r_{in} : i = 1, \ldots, k\}$ so that $\sum_{n=1}^{\infty} r_n < \infty$. Let $p, q \in F \cap K$. For each $n \in \mathbf{N}$, define a function

$$a_n(t) = \|tx_n + (1-t)p - q\|, \quad \text{where } t \in [0,1].$$

Then $\lim_{n\to\infty} a_n(0) = \lim_{n\to\infty} \|p - q\|$ and $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - q\|$ exist. To prove that $\lim_{n\to\infty} a_n(t)$ exists, for $t \in (0,1)$, define a mapping $S_n : K \to K$ by

$$S_n v = R((1-\alpha_{kn})v + \alpha_{kn} T_k (RT_k)^{n-1} v_{(k-1)n}), \quad v \in K,$$

where

$$v_{1n} = R((1-\alpha_{1n})v + \alpha_{1n} T_1 (RT_1)^{n-1} v),$$

$$v_{in} = R((1-\alpha_{in})v + \alpha_{in} T_i (RT_i)^{n-1} v_{(i-1)n}, \quad 2 \leq i \leq k-1.$$

Observe that if $v = x_n$ and $v_{in} = y_{in}$, for $1 \leq i \leq k-1$ and $n \in \mathbf{N}$, then $S_n x_n = x_{n+1}$. Also, $F \cap K \subset F(S_n)$, $n \in \mathbf{N}$. For any $v, w \in K$, we have

$$\begin{aligned} \|v_{1n} - w_{1n}\| &\leq (1-\alpha_{1n})\|v - w\| + \alpha_{1n}(1+r_n)\|v - w\| \\ &\leq (1+r_n)\|v - w\| \end{aligned}$$

and then

$$\begin{aligned} \|v_{2n} - w_{2n}\| &\leq (1-\alpha_{2n})\|v - w\| + \alpha_{2n}(1+r_n)\|v_{1n} - w_{1n}\| \\ &\leq (1+r_n)^2\|v - w\|. \end{aligned}$$

We repeat this process inductively to obtain that

$$\begin{aligned} \|v_{in} - w_{in}\| &\leq (1-\alpha_{in})\|v - w\| + \alpha_{in}(1+r_n)\|v_{(i-1)n} - w_{(i-1)n}\| \\ &\leq (1-\alpha_{in})\|v - w\| + \alpha_{in}(1+r_n)^i\|v - w\| \\ &\leq (1+r_n)^i\|v - w\|, \end{aligned}$$

for $i = 1, \ldots, k-1$. Therefore

$$\begin{aligned}\|S_n v - S_n w\| &\le (1-\alpha_{kn})\|v-w\| + \alpha_{kn}(1+r_n)\|v_{(k-1)n} - w_{(k-1)n}\| \\ &\le [(1-\alpha_{kn}) + \alpha_{kn}(1+r_n)^k]\|v-w\| \\ &\le (1+r_n)^k\|v-w\| \\ &= \left[1 + r_n \sum_{j=0}^{k-1}(1+r_n)^j\right]\|v-w\| \\ &= (1+\epsilon_n)\|v-w\|,\end{aligned} \tag{32}$$

where $\epsilon_n = r_n \sum_{j=0}^{k-1}(1+r_n)^j$. Note that by hypothesis,

$$\sum_{n=1}^{\infty} \epsilon_n < \infty. \tag{33}$$

For $m \in \mathbf{N}$, define a mapping $Q_{nm} : K \to K$ by

$$Q_{nm} = S_{n+m}S_{n+m-1}\cdots S_n$$

and

$$b_{nm} = \|Q_{nm}(tx_n + (1-t)p) - [tQ_{nm}x_n + (1-t)p]\|.$$

By (32), for $v, w \in K$,

$$\begin{aligned}\|Q_{nm}v - Q_{nm}w\| &\le (1+\epsilon_{n+m})(1+\epsilon_{n+m-1})\cdots(1+\epsilon_n)\|v-w\| \\ &= \delta_{nm}\|v-w\|,\end{aligned}$$

where

$$\delta_{nm} = (1+\epsilon_{n+m})(1+\epsilon_{n+m-1})\cdots(1+\epsilon_n), \quad \text{for } n, m \in \mathbf{N},$$

so that $\lim_{n,m\to\infty} \delta_{nm} = 1$ by (33). According to Lemma 2.5, there is a strictly increasing, continuous and convex function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\begin{aligned}b_{nm} &\le \delta_{nm}\phi^{-1}\left(\|x_n - p\| - \frac{1}{\delta_{nm}}\|Q_{nm}x_n - Q_{nm}p\|\right) \\ &= \delta_{nm}\phi^{-1}\left(\|x_n - p\| - \frac{1}{\delta_{nm}}\|x_{n+m+1} - p\|\right),\end{aligned}$$

since $Q_{nm}x_n = x_{n+m+1}$ and $F \cap K \subset F(Q_{nm})$. Therefore $\lim_{n,m\to\infty} b_{nm} = 0$. Now from

$$\begin{aligned} a_{n+m+1}(t) &= \|tQ_{nm}x_n + (1-t)p - q\| \\ &\le \|[tQ_{nm}x_n + (1-t)p] - Q_{nm}(tx_n + (1-t)p)\| \\ &\quad + \|Q_{nm}(tx_n + (1-t)p) - Q_{nm}q\| \\ &= b_{nm} + \delta_{nm}a_n(t), \end{aligned}$$

we take the limit superior as $m \to \infty$ and then the limit inferior as $n \to \infty$ to obtain

$$\limsup_{n\to\infty} a_n(t) \le \liminf_{n\to\infty} a_n(t).$$

Consequently, $\lim_{n\to\infty} a_n(t)$ exists.

The reflexivity of X implies that the bounded sequence $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ which converges weakly to some point $x^* \in K$ because K is closed and convex. Lemma 2.3 asserts that $T_i x^* = x^*$, for $1 \le i \le k$, and so $x^* \in F \cap K$. We can see at once that $\omega_w\{x_n\} \subset F \cap K$. To prove that $\{x_n\}$ converges weakly to x^*, let $\{x_{m_j}\}$ be any subsequence of $\{x_n\}$ which converges weakly to a point $\bar{x}$ so that $\bar{x} \in F \cap K$. Since $x^*, \bar{x} \in \omega_w\{x_n\}$ and $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists, for all $t \in [0,1]$ and $p, q \in F \cap K$, it follows from Lemma 2.4 that $x^* = \bar{x}$. We conclude that any subsequence of $\{x_n\}$ has a weakly convergent subsequence with limit x^*, and hence $\{x_n\}$ converges weakly to x^*, as assured. ■

The following three theorems are analogs of the preceding results for the iteration defined by (10).

Theorem 4.4. *Let X be a uniformly convex Banach space, C a closed convex subset of X, $\{u_{in}\}_{n=1}^{\infty}$, $i = 1, \ldots, k$, bounded sequences in X, $R : X \to C$ a nonexpansive retraction, and $\{T_i : C \to X\}_{i=1}^k$ a family of nonself uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (10). Suppose that*

- (i) $\sum_{n=1}^{\infty} r_{in} < \infty$ *and* $\sum_{n=1}^{\infty} s_{in} < \infty$, *for* $i = 1, \ldots, k$;
- (ii) $0 < \liminf_{n\to\infty} \alpha_{in} \le \limsup_{n\to\infty} \alpha_{in} < 1$ *and* $\sum_{n=1}^{\infty} \beta_{in} < \infty$, *for* $i = 1, \ldots, k$;
- (iii) *the family* $\{T_i\}_{i=1}^k$ *satisfies Condition* (A) *with respect to* $\{x_n\}$, *or* $T_{i_0}(RT_{i_0})^{m-1}$ *is semicompact, for some* $1 \le i_0 \le k$ *and for some* $m \ge 1$.

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^k$.

Proof. We may assume by hypothesis that $\rho \le \alpha_{in} \le 1 - \rho$, for some $\rho > 0$, where $i = 1, \ldots, k$, and $n \in \mathbf{N}$. Let $p \in F$. It follows from (19) and (20) in the proof of Theorem 3.3 that $\lim_{n\to\infty}\|x_n - p\|$ exists and hence $\{x_n\}$ is bounded. Set $\lim_{n\to\infty}\|x_n - p\| = c$ which can be written as

$$\lim_{n\to\infty}\|R((1-\alpha_{kn}-\beta_{kn})y_{(k-2)n}+\alpha_{kn}T_k(RT_k)^{n-1}y_{(k-1)n}+\beta_{kn}u_{kn})-R(p)\| = c.$$

So

$$\begin{aligned} c \le \liminf_{n\to\infty} &\|\alpha_{kn}[T_k(RT_k)^{n-1}y_{(k-1)n} - p + \beta_{kn}(u_{kn} - y_{(k-2)n})] \\ &+(1-\alpha_{kn})[y_{(k-2)n} - p + \beta_{kn}(u_{kn} - y_{(k-2)n})]\|. \end{aligned} \tag{34}$$

From the inequality (18) and assumption (ii) we obtain that

$$\limsup_{n\to\infty}\|y_{in} - p\| \le c, \quad i = 1, \ldots, k\,; \tag{35}$$

so the sequence $\{y_{in}\}$ is bounded, for $i = 1, \ldots, k$. By (11),

$$\begin{aligned} &\|T_k(RT_k)^{n-1}y_{(k-1)n} - p + \beta_{kn}(u_{kn} - y_{(k-2)n)}\| \\ \le&\tau_n\|y_{(k-1)n} - p\| + \beta_{kn}\|u_{kn} - y_{(k-2)n}\| \end{aligned}$$

which shows that

$$\limsup_{n\to\infty}\|T_k(RT_k)^{n-1}y_{(k-1)n} - p + \beta_{kn}(u_{kn} - y_{(k-2)n)})\| \le c. \tag{36}$$

Since $\|y_{(k-2)n)} - p + \beta_{kn}(u_{kn} - y_{(k-2)n})]\| \le \|y_{(k-2)n} - p\| + \beta_{kn}\|u_{kn} - y_{(k-2)n}\|$, we have

$$\limsup_{n\to\infty}\|y_{(k-2)n} - p + \beta_{kn}(u_{kn} - y_{(k-2)n})\| \le c. \tag{37}$$

We obtain from (34), (36) and (37) that

$$\begin{aligned} \lim_{n\to\infty} &\|\alpha_{kn}[T_k(RT_k)^{n-1}y_{(k-1)n} - p + \beta_{kn}(u_{kn} - y_{(k-2)n})] \\ &\qquad + (1-\alpha_{kn})[y_{(k-2)n} - p + \beta_{kn}(u_{kn} - y_{(k-2)n})]\| = c. \end{aligned}$$

According to Lemma 2.2,

$$\lim_{n\to\infty}\|y_{(k-2)n} - T_k(RT_k)^{n-1}y_{(k-1)n}\| = 0.$$

On the other hand, from (18) we have

$$\begin{aligned}\|x_{n+1}-p\| &\le (1-\alpha_{kn}-\beta_{kn})\|y_{(k-2)n}-p\|+\alpha_{kn}\tau_n\|y_{(k-1)n}-p\| \\ &\quad+\beta_{kn}\|u_{kn}-p\| \\ &\le (1-\alpha_{kn})\tau_n^k\|x_n-p\|+\tau_n^k[\beta_{1n}\|u_{1n}-p\|+\beta_{2n}\|u_{2n}-p\| \\ &\quad+\cdots+\beta_{(k-2)n}\|u_{(k-2)n}-p\|]+\alpha_{kn}\tau_n^k\|y_{(k-1)n}-p\| \\ &\quad+\beta_{kn}\tau_n^k\|u_{kn}-p\| \\ &\le (1-\alpha_{kn})\tau_n^k\|x_n-p\|+\tau_n^k\sum_{j=1}^{k}\beta_{jn}\|u_{jn}-p\|+\alpha_{kn}\tau_n^k\|y_{(k-1)n}-p\|;\end{aligned}$$

hence

$$\begin{aligned}\|x_n-p\| &\le \frac{1}{\alpha_{kn}\tau_n^k}\left(\tau_n^k\|x_n-p\|-\|x_{n+1}-p\|+\tau_n^k\sum_{j=1}^{k}\beta_{jn}\|u_{jn}-p\|\right) \\ &\quad+\|y_{(k-1)n}-p\| \\ &\le \frac{1}{\rho\tau_n^k}\left(\tau_n^k\|x_n-p\|-\|x_{n+1}-p\|+\tau_n^k\sum_{j=1}^{k}\beta_{jn}\|u_{jn}-p\|\right) \\ &\quad+\|y_{(k-1)n}-p\|,\end{aligned}$$

because (19) implies that the expression in the square bracket is nonnegative. This asserts that $c \le \liminf_{n\to\infty}\|y_{(k-1)n}-p\|$ and so by (35),

$$\lim_{n\to\infty}\|y_{(k-1)n}-p\|=c.$$

Therefore

$$\begin{aligned}\lim_{n\to\infty}\|R((1-\alpha_{(k-1)n}-\beta_{(k-1)n})y_{(k-3)n}+\alpha_{(k-1)n}T_{k-1}(RT_{k-1})^{n-1}y_{(k-2)n} \\ +\beta_{(k-1)n}u_{(k-1)n})-R(p)\|=c.\end{aligned}$$

Again, since

$$\limsup_{n\to\infty}\|T_{k-1}(RT_k)^{n-1}y_{(k-2)n}-p+\beta_{(k-1)n}(u_{(k-1)n}-y_{(k-3)n})\|\le c$$

and

$$\limsup_{n\to\infty}\|y_{(k-3)n}-p+\beta_{(k-1)n}(u_{(k-1)n}-y_{(k-3)n})\|\le c,$$

we see that

$$\lim_{n\to\infty} \|\alpha_{(k-1)n}[T_{k-1}(RT_k)^{n-1}y_{(k-2)n} - p + \beta_{(k-1)n}(u_{(k-1)n} - y_{(k-3)n})] + (1-\alpha_{(k-1)n})[y_{(k-3)n} - p + \beta_{(k-1)n}(u_{(k-1)n} - y_{(k-3)n})]\| = c,$$

and so

$$\lim_{n\to\infty} \|y_{(k-3)n} - T_{k-1}(RT_k)^{n-1}y_{(k-2)n}\| = 0.$$

Furthermore,

$$\begin{aligned}\|y_{(k-1)n} - p\| &\le (1-\alpha_{(k-1)n} - \beta_{(k-1)n})\|y_{(k-3)n} - p\| + \alpha_{(k-1)n}\tau_n\|y_{(k-2)n} - p\| \\ &\quad + \beta_{(k-1)n}\|u_{(k-1)n} - p\| \\ &\le (1-\alpha_{(k-1)n})\tau_n^{k-1}\|x_n - p\| + \tau_n^{k-1}\sum_{j=1}^{k-1}\beta_{jn}\|u_{jn} - p\| \\ &\quad + \alpha_{(k-1)n}\tau_n^{k-1}\|y_{(k-2)n} - p\|\end{aligned}$$

and hence

$$\begin{aligned}\|x_n - p\| &\le \frac{1}{\alpha_{(k-1)n}\tau_n^{k-1}}\left(\tau_n^{k-1}\|x_n - p\| - \|y_{(k-1)n} - p\| + \tau_n^{k-1}\sum_{j=1}^{k-1}\beta_{jn}\|u_{jn} - p\|\right) \\ &\quad + \|y_{(k-2)n} - p\| \\ &\le \frac{1}{\rho\tau_n^{k-1}}\left(\tau_n^{k-1}\|x_n - p\| - \|y_{(k-1)n} - p\| + \tau_n^{k-1}\sum_{j=1}^{k-1}\beta_{jn}\|u_{jn} - p\|\right) \\ &\quad + \|y_{(k-2)n} - p\|.\end{aligned}$$

Thus

$$\lim_{n\to\infty} \|y_{(k-2)n} - p\| = c.$$

Repeat this process inductively k times to conclude that

$$\lim_{n\to\infty} \|x_n - T_1(RT_1)^{n-1}x_n\| = 0$$

and

$$\lim_{n\to\infty} \|y_{(i-2)n} - T_i(RT_i)^{n-1}y_{(i-1)n}\| = 0, \quad i = 2, \ldots, k, \tag{38}$$

from which it follows that

$$\lim_{n\to\infty} \|x_n - y_{1n}\| = 0 \tag{39}$$

and for $i = 2, \ldots, k$,

$$\begin{aligned} \|y_{in} - y_{(i-2)n}\| &\le \alpha_{in}\|T_i(RT_i)^{n-1}y_{(i-1)n} - y_{(i-2)n}\| \\ &\quad +\beta_{in}\|u_{in} - y_{(i-2)n}\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{40}$$

If i is an even integer, then

$$\|x_n - y_{in}\| \le \|y_{0n} - y_{2n}\| + \|y_{2n} - y_{4n}\| + \cdots + \|y_{(i-2)n} - y_{in}\|;$$

if i is an odd integer, then

$$\|x_n - y_{in}\| \le \|y_{0n} - y_{1n}\| + \|y_{1n} - y_{3n}\| + \cdots + \|y_{(i-2)n} - y_{in}\|.$$

Consequently, by (39) and (40),

$$\lim_{n\to\infty} \|x_n - y_{in}\| = 0, \quad i = 1, \ldots, k. \tag{41}$$

In particular, since $x_{n+1} = y_{kn}$,

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{42}$$

Also note that from (38) and (41),

$$\begin{aligned} \|x_n - T_i(RT_i)^{n-1}x_n\| &\le \|x_n - y_{(i-2)n}\| + \|y_{(i-2)n} - T_i(RT_i)^{n-1}y_{(i-1)n}\| \\ &\quad + \|T_i(RT_i)^{n-1}y_{(i-1)n} - T_i(RT_i)^{n-1}x_n\| \\ &\le \|x_n - y_{(i-2)n}\| + \|y_{(i-2)n} - T_i(RT_i)^{n-1}y_{(i-1)n}\| \\ &\quad + L\|y_{(i-1)n} - x_n\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{43}$$

From

$$\begin{aligned} \|x_n - T_ix_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_i(RT_i)^n x_{n+1}\| \\ &\quad + \|T_i(RT_i)^n x_{n+1} - T_i(RT_i)^n x_n\| + \|(T_iR)T_i(RT_i)^{n-1}x_n - (T_iR)x_n\| \\ &\le (1+L)\|x_n - x_{n+1}\| + \|x_{n+1} - T_i(RT_i)^n x_{n+1}\| \\ &\quad + L\|T_i(RT_i)^{n-1}x_n - x_n\|, \end{aligned}$$

(42) and (43) imply that

$$\lim_{n\to\infty} \|x_n - T_ix_n\| = 0, \quad i = 1, \ldots, k.$$

Using the same argument as in the last part of the proof of Theorem 4.1, we can assure that $\{x_n\}$ converges strongly to a point of F. ■

Theorem 4.5. *Let X be a uniformly convex Banach space satisfying the Opial property, C a closed convex subset of X, $\{u_{in}\}_{n=1}^{\infty}$, $i = 1, \ldots, k$, bounded sequences in X, $R : X \to C$ a nonexpansive retraction, and $\{T_i : C \to X\}_{i=1}^{k}$ a family of nonself uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Suppose that*

(i) $\sum_{n=1}^{\infty} r_{in} < \infty$ *and* $\sum_{n=1}^{\infty} s_{in} < \infty$, *for* $i = 1, \ldots, k$;

(ii) $0 < \liminf_{n\to\infty} \alpha_{in} \leq \limsup_{n\to\infty} \alpha_{in} < 1$ *and* $\sum_{n=1}^{\infty} \beta_{in} < \infty$, *for* $i = 1, \ldots, k$;

(iii) $I - T_i$, $i = 1, \ldots, k$, *is demiclosed at 0, for* $i = 1, \ldots, k$.

Then the sequence $\{x_n\}$ defined by (10) converges weakly to a common fixed point of $\{T_i\}_{i=1}^{k}$.

Proof. The proof is the same as that of Theorem 4.2 and so is omitted. ∎

Setting $\beta_{in} = 0$ for the iteration defined by (10), for all $1 \leq i \leq k$ and $n \in \mathbf{N}$, we obtain a sequence $\{x_n\}$ as follows: choose x_1 arbitrarily in C,

$$
\begin{aligned}
y_{1n} &= R((1-\alpha_{1n})x_n + \alpha_{1n}T_1(RT_1)^{n-1}x_n), \\
y_{in} &= R((1-\alpha_{in})y_{(i-2)n} + \alpha_{in}T_i(RT_i)^{n-1}y_{(i-1)n}), \quad 2 \leq i \leq k-1, \\
x_{n+1} = y_{kn} &= R((1-\alpha_{kn})y_{(k-2)n} + \alpha_{kn}T_k(RT_k)^{n-1}y_{(k-1)n}), \quad n \in \mathbf{N},
\end{aligned} \tag{44}
$$

where $y_{0n} = x_n$.

Theorem 4.6. *Let X be a uniformly convex Banach space whose dual X^* has the Kadec-Klee property, C a closed convex subset of X, $R : X \to C$ a nonexpansive retraction, and $\{T_i : C \to X\}_{i=1}^{k}$ a family of nonself asymptotically nonexpansive mappings with respect to $\{r_{in}\}$ such that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Suppose that*

(i) $\sum_{n=1}^{\infty} r_{in} < \infty$, *for* $i = 1, \ldots, k$;

(ii) $0 < \liminf_{n\to\infty} \alpha_{in} \leq \limsup_{n\to\infty} \alpha_{in} < 1$, *for* $i = 1, \ldots, k$;

Then the sequence $\{x_n\}$ defined by (44) converges weakly to a common fixed point of $\{T_i\}_{i=1}^{k}$.

Proof. It follows from (19) and (20) in Theorem 3.3 that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$ and so the sequence $\{x_n\}$ is bounded. Hence there exists a closed ball B with $F \cap B \neq \emptyset$ such that $\{x_n\} \subset K = B \cap C$. So K is a closed bounded convex subset of C. Let $r_n = \max\{r_{in} : i = 1, \ldots, k\}$ so that $\sum_{n=1}^{\infty} r_n < \infty$. Let $p, q \in F \cap K$. For each $n \in \mathbf{N}$, define a function

$$a_n(t) = \|tx_n + (1-t)p - q\|, \quad \text{where } t \in [0,1].$$

Then $\lim_{n\to\infty} a_n(0) = \lim_{n\to\infty} \|p - q\|$ and $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - q\|$ exist. To prove that $\lim_{n\to\infty} a_n(t)$ exists, for $t \in (0,1)$, define a mapping $S_n : K \to K$ by

$$S_n v = R((1-\alpha_{kn})v_{(k-2)n} + \alpha_{kn} T_k (RT_k)^{n-1} v_{(k-1)n}), \quad v \in K,$$

where

$$\begin{aligned} v_{0n} &= v, \\ v_{1n} &= R((1-\alpha_{1n})v + \alpha_{1n} T_1 (RT_1)^{n-1} v), \\ v_{in} &= R((1-\alpha_{in})v_{(i-2)n} + \alpha_{in} T_i (RT_i)^{n-1} v_{(i-1)n}), \quad 2 \le i \le k-1. \end{aligned}$$

If $v_{in} = y_{in}$, for $0 \le i \le k-1$ and $n \in \mathbf{N}$, then $S_n x_n = y_{kn} = x_{n+1}$. Moreover, $F \cap K \subset F(S_n)$, $n \in \mathbf{N}$. For any $v, w \in K$, we have

$$\begin{aligned} \|v_{1n} - w_{1n}\| &\le (1-\alpha_{1n})\|v - w\| + \alpha_{1n}(1+r_n)\|v - w\| \\ &\le (1+r_n)\|v - w\| \end{aligned}$$

and so

$$\begin{aligned} \|v_{2n} - w_{2n}\| &\le (1-\alpha_{2n})\|v_{0n} - w_{0n}\| + \alpha_{2n}(1+r_n)\|v_{1n} - w_{1n}\| \\ &\le (1+r_n)^2\|v - w\|. \end{aligned}$$

Inductively continuing this process, we obtain

$$\begin{aligned} \|v_{in} - w_{in}\| &\le (1-\alpha_{in})\|v_{(i-2)n} - w_{(i-2)n}\| + \alpha_{in}(1+r_n)\|v_{(i-1)n} - w_{(i-1)n}\| \\ &\le (1-\alpha_{in})(1+r_n)^{i-2}\|v - w\| + \alpha_{in}(1+r_n)^i\|v - w\| \\ &\le (1+r_n)^i\|v - w\|, \end{aligned}$$

for $i = 1, \ldots, k-1$. In particular,

$$\begin{aligned} &\|S_n v - S_n w\| \\ &\le (1-\alpha_{kn})\|v_{(k-2)n} - w_{(k-2)n}\| + \alpha_{kn}(1+r_n)\|v_{(k-1)n} - w_{(k-1)n}\| \\ &\le (1+r_n)^k\|v - w\| \\ &= (1+\epsilon_n)\|v - w\|, \end{aligned} \tag{45}$$

where $\epsilon_n = r_n \sum_{j=0}^{k-1}(1+r_n)^j$. By hypotheses,

$$\sum_{n=1}^{\infty} \epsilon_n < \infty. \tag{46}$$

For $m \in \mathbf{N}$, define a mapping $Q_{nm} : K \to K$ by

$$Q_{nm} = S_{n+m}S_{n+m-1}\cdots S_n$$

and

$$b_{nm} = \|Q_{nm}(tx_n + (1-t)p) - [tQ_{nm}x_n + (1-t)p]\|.$$

By (45), for $v, w \in K$,

$$\begin{aligned}\|Q_{nm}v - Q_{nm}w\| &\leq (1+\epsilon_{n+m})(1+\epsilon_{n+m-1})\cdots(1+\epsilon_n)\|v-w\| \\ &= \delta_{nm}\|v-w\|,\end{aligned}$$

where

$$\delta_{nm} = (1+\epsilon_{n+m})(1+\epsilon_{n+m-1})\cdots(1+\epsilon_n), \quad \text{for } n, m \in \mathbf{N},$$

so that by (46), $\lim_{n,m\to\infty} \delta_{nm} = 1$.

It remains to show that $\lim_{n\to\infty} a_n(t)$ exists, for $t \in [0,1]$, and hence $\{x_n\}$ converges weakly to a point of $F \cap K$. The argument of this part is the same as that in Theorem 4.3 and so is omitted. ∎

References

1. A. G. Aksoy and M. A. Khamsi, *Nonstandard Mathods in Fixed Point Theory*, Springer-Verlag, New York, 1990.
2. C. E. Chidume and B. Ali, Weak and strong convergence theorems for finite families of asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.*, **330** (2007), 377-387.
3. C. E. Chidume and B. Ali, Approximation of common fixed points for finite families of nonself asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.*, **326** (2007), 960-973.
4. C. E. Chidume, E. U. Ofoedu and H. Zegeye, Strong and weak convergence theorems for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.*, **280** (2003), 364-374.
5. C. E. Chidume and N. Shahzad, Strong convergence of an implicit iteration process for finite family of nonexpansive mappings, *Nonlinear Anal.*, **62(6)** (2005), 1149-1156.

6. L. Deng and Q. Liu, Iterative scheme for nonself generalized asymptotically quasi-nonexpansive mappings, *Applied Mathematics and Computation*, **205** (2008), 317-324.

7. J. Diestel, *Geometry of Banach Spaces-Selected Topics*, Lecture Notes in Mathematics, vol. 485, Springer, New York, 1975.

8. H. Fukhar-ud-din and S. H. Khan, Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications, *J. Math. Anal. Appl.*, **328** (2007), 821-829.

9. J. Garcia Falset, W. Kaczor, T. Kuczumow and S. Reich, Weak convergence theorems for asymptotically nonexpansive mappings and semigroups, *Nonlinear Anal.*, **43** (2001), 377-401.

10. K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, England, 1990.

11. B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.*, **73(6)** (1967), 957-961.

12. S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44** (1974), 147-150.

13. A. R. Khan, A. A. Domlo and H. Fukhar-ud-din, Common fixed points Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.*, **341** (2008), 1-11.

14. H. Y. Lan, Common fixedpoint iterative processes with errors for generalized asymptotically quasi-nonexpansive mappings, *Comput. Math. Appl.*, **52** (2006), 1403-1412.

15. Q. H. Liu, Iterative sequences for asymptotically quasi-nonexpansive mapping with error member, *J. Math Anal. Appl.*, **259** (2001), 18-24.

16. M. Maiti and B. Saha, Approximating fixed points of nonexpansive and generalized nonexpansive mappings, *Int. J. Math. Math. Sci.*, **16(1)** (1993), 81-86.

17. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506-510.

18. H. Oka, A nonlinear ergodic theorem for asymptotically nonexpansive mappings in Banach space, *Proc. Japan Acad. Ser. A*, **65** (1998), 284-287.

19. S. Plubtieng and R. Wangkeeree, Fixed point iteration for asymptotically quasi-nonexpansive mappings in Banach spaces, *International Journal of Mathematics and Mathematical Sciences*, **11** (2005), 1685-1692.

20. J. Quan, S. S. Chang and X. J. Long, Approximation common fixed point of asymptotically quasi-nonexpansive-type mappings by the finite steps iterative sequences, *Fixed Point Theory and Applications*, **2006** (2006), 1-8.

21. J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Austral. Math. Soc.*, **43** (1991), 153-159.

22. H. F. Senter and W. G. Dotson, Approximating fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.*, **44(2)** (1974), 375-380.

23. N. Shahzad and H. Zegeye, Strong convergence of an implicit iteration process for a finite family of generalized asymptotically quasi-nonexpansive maps, *Applied Mathematics and Computation*, **189** (2007), 1058-1065.

24. Z. H. Sun, Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings, *J. Math. Anal. Appl.*, **286** (2003), 351-358.

25. K. K. Tan and H. K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.*, **178** (1993), 301-308.

26. K. K. Tan and X. Z. Yuan, Random fixed point theorems and approximation in cone, *J. Math. Anal. Appl.*, **185** (1994), 378-390.

27. Y. X. Tian and C. D. Yang, Convergence theorems of three-step iterative scheme for a finite family of uniformly quasi-lipschitzian mappings in convex metric spaces, *Fixed Point Theory and Applications*, **2009** (2009), 1-12.

28. L. Wang, Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, *J. Math. Anal. Appl.*, **323** (2006), 550-557.

29. B. Xu and M. A. Noor, Fixed-point iterations for asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.*, **267(2)** (2002), 444-453.

30. H. K. Xu and R. G. Ori, An implicit iteration process for nonexpansive mappings, Numer. *Funct. Anal. Optim.*, **22** (2001), 767-773.

Shuechin Huang
Department of Applied Mathematics
National Dong Hwa University
Hualien 97401, Taiwan
E-mail: shuang@mail.ndhu.edu.tw