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SEQUENTIAL PURITY AND INJECTIVITY OF ACTS OVER SOME CLASSES OF SEMIGROUPS

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Abstract. The notion of sequential purity for acts over the monoid $\mathbb{N}^\infty$, called projection algebras, was introduced and studied by Mahmoudi and Ebrahimi. This paper is devoted to the study of this notion and its relation to injectivity of S-acts for a semigroup S. We prove that in general injectivity implies absolute sequential purity and they are equivalent for acts over some classes of semigroups.

1. Introduction

One of the very useful notions in many branches of mathematics as well as in computer science is the action of a semigroup on a set. Purity of acts has been studied by Gould, Normak, and others (see [4], [8], [6]). Sequential purity was introduced and studied by Mahmoudi and Ebrahimi in [7]. In this paper we study the injectivity of acts using sequential purity.

Recall that for a semigroup S, a (right) S-*act* is a set A together with a function $\alpha : A \times S \to A$, called the *action* of S (or the S-action) on A, such that for $x \in A$ and $s, t \in S$ (denoting $\alpha(x, s)$ by xs), $x(st) = (xs)t$. A *subact* of an act A is simply a subset B of A which is closed under the action. A *homomorphism* $f : A \to B$ between S-acts A, B is a function such that for each $x \in X$, $s \in S$, $f(xs) = f(x)s$. We denote the category of all S-acts and homomorphisms between them by **Act**$-S$. An element a of an S-act A is called a *fixed* or a *zero* element if $as = a$ for all $s \in S$. The category **Act**$-S$ is clearly a variety. So monomorphisms are exactly one-one homomorphisms. Thus we can consider the monomorphisms in this category as inclusions.

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Definition 1.1. An S-act B containing (an isomorphic copy of) an S-act A as a subact is called an *extension* of A. The S-act A is said to be a *retract* of B if there exists a homomorphism $f : B \to A$ such that $f\restriction_A = id_A$, in which case f is said to be a *retraction*.

A is called an *absolute retract* if it is a retract of each of its extensions. An S-act A is said to be *injective* if for every monomorphism $h : B \to C$ and each homomorphism $f : B \to A$ there exists a homomorphism $g : C \to A$ such that $gh = f$. An S-act A is said to be S*-injective* if for every homomorphism $f : S \to A$ there exists an element $a \in A$ such that $f = \lambda_a$, where $\lambda_a(s) = as$ for all $s \in S$ (this notion was first defined and applied in [5] and [7]). A minimal injective extension of an S-act is called its *injective hull*.

Remark 1.2. Notice that, for any semigroup S, in the category **Act**$-S$, pushouts transfer monomorphisms. To see this, let $f : A \to B$ and $g : A \to C$ be homomorphisms in this category. We know that the pushout of f and g is $(B \sqcup C)/\theta$ together with $k = \pi u_B : B \to (B \sqcup C)/\theta,\ h = \pi u_C : C \to (B \sqcup C)/\theta$, where θ is the smallest congruence generated by $X = \{(u_B f(a), u_C g(a)) : a \in A\}$ and $\pi : (B \sqcup C) \to (B \sqcup C)/\theta$ is the natural homomorphism and u_B, u_C are injection homomorphisms from B, C to $(B \sqcup C)$, respectively (see also [2]). Let f be a monomorphism, to show that h is also a monomorphism, let for $c, c' \in C$, $h(c) = h(c')$. Then $(c, c') \in \theta$, hence we have $c = c'$ or there exist $a_1, a_2, ...a_n \in A, s_1, s_2, ..., s_n \in S$ such that $c = g(a_1 s_1), f(a_1 s_1) = f(a_2 s_2), g(a_2 s_2) = g(a_3 s_3), f(a_3 s_3) = f(a_4 s_4), ..., g(a_n s_n) = c'$. If $c = c'$ then the proof is complete. Otherwise, since f is a monomorphism we can write $a_1 s_1 = a_2 s_2, a_3 s_3 = a_4 s_4, ..., a_{n-1} s_{n-1} = a_n s_n$ and so $g(a_1 s_1) = g(a_2 s_2), g(a_3 s_3) = g(a_4 s_4), ..., g(a_{n-1} s_{n-1}) = g(a_n s_n)$. Thus $c = g(a_1 s_1) = g(a_2 s_2) = g(a_3 s_3) = g(a_4 s_4) = ... = g(a_n s_n) = c'$.

Using the above lemma and the results of [1] we get the following theorem:

Theorem 1.3. *The category* **Act**$-S$ *has enough injectives, and* A *is injective if and only if* A *is an absolute retract.*

2. S-Injectivity Versus Sequential Purity

In this section we show that absolute sequential purity is in fact equivalent to S-injectivity.

Definition 2.1. Let A be a subact of an S-act B. Then A is said to be *sequentially pure,* or s*-pure,* in B if every "sequential" system $xs = a_s,\ s \in S$, of equations over A is solvable in A whenever it is solvable in B. A is called an *absolute* s*-pure* if it is s-pure in every extension of A. A homomorphism $f : A \to B$ is called s*-pure* if $f(A)$ is s-pure in B.

Theorem 2.2. *For an S-acts A, the following are equivalent:*

(i) *A is s-pure in its injective hull.*

(ii) *A is an absolute s-pure.*

(iii) *A is S-injective.*

Proof. (i)⇔(ii): To prove the non clear direction, let B be an extension of A and the sequential system $xs = a_s,\ s \in S$, of equations over A has a solution b in B. Let $E(A)$ be the injective hull of A. Then there exists a homomorphism $f : B \to E(A)$ extending the inclusion map of A into $E(A)$. We thus have $f(b)s = f(bs) = f(a_s) = a_s$. That is, $xs = a_s$ has a solution $f(b)$ in $E(A)$ and so has a solution in A, by (i).

(ii)⇒(iii): Let $f : S \to A$ be a homomorphism. Consider the extension $B = A \cup \{b\}$ of A with the action $b.t = f(t)t \in S$. Since A is an absolute s-pure and the sequential system $xs = f(s),\ s \in S$, has a solution b in B, it must have a solution in A, that is, there exists $a \in A$ such that for all $s \in S$ we have $as = f(s)$, which means that $f = \lambda_a$.

(iii)⇒(ii): Let B be an extension of A and the sequential system $xs = a_s,\ s \in S$, has a solution b in B. Then we can easily see that $f : S \to A$ defined by $f(s) = a_s$, $s \in S$, is a homomorphism. Now, since A is S-injective, $f = \lambda_a$ for some a in A, which is easily seen to be a solution of the given sequential system. ■

As a corollary of the last two parts of the above proof we have:

Corollary 2.3. *The sequential system $xs = a_s,\ s \in S$, of equations over an S-act A has a solution in some extension of A if and only if the corresponding map $f : S \to A$, $f(s) = a_s,\ s \in S$, is a homomorphism.*

3. Injectivity Versus Sequential Purity

In this section, we first show that if an act A is injective (equivalently, absolute retract) then it is an absolute s-pure. Trying to answer the question about the converse of this result and hence get a result similar to Theorem 2.2 we give a counter example and show that the converse is true for acts over some special classes of semigroups. Hence, for these classes of semigroups, the notion of injectivity can be studied using sequential purity (compare with [4], [8], or [7]).

Lemma 3.1. *If an S-act A is a retract of an S-act B then A is s-pure in B.*

Proof. Let $f : B \to A$ be a homomorphism such that $f\restriction_A = id$, and the sequential system $xs = a_s$, $s \in S$, have a solution b in B. Then $f(b)$ belongs to A and for all $s \in S$ we have $f(b)s = f(bs) = f(a_s) = a_s$, that is A is s-pure in B. ■

The converse of the above lemma is not necessarily true:

Example 3.2. Let $S = \{s, t\}$ be a left zero semigroup (that is, $xy = x$, for all x, y) and $A = \{a, b\}$ be an S-act where a and b are zero elements, and $B = \{a, b, c, d, e\}$ be an extension of A where c is a zero element and $ds = a, dt = c, es = b, et = c$. The S-act A is s-pure in B, since the only sequential systems over A are (i) $xs = a, xt = a$, (ii) $xs = a, xt = b$, (iii) $xs = b, xt = a$, and (iv) $xs = b, xt = b$, and none of them has a solution in B. Hence A is trivially s-pure in B. But there is no retraction $f : B \to A$, because if $f(c) = a$ then the definition of the action of S on the elements of B yields that $f(d) = a = f(e)$ and so $f(b) = a$ which is impossible since $f\restriction_A = id$. Similarly the case $f(c) = b$ is not possible.

Theorem 3.3. *Every injective S-act is an absolute s-pure.*

Proof. This is in fact a corollary of the above lemma. ∎

The converse of the above theorem is also not true:

Example 3.4. Consider the semigroup $S = (\mathbb{N}, +)$ and the S-act $A = \{a, b\}$ with the action given by $a(2n) = a, a(2n-1) = b$ and $b(2n) = b, b(2n-1) = a$ for $n \in \mathbb{N}$. Now we see that there are exactly two homomorphisms f and g from S to A, given by the sequences $\{a, b, a, b, ...\}$ and $\{b, a, b, a, ...\}$, respectively. This is because for a homomorphism $h : \mathbb{N} \to A$ with $h(n) = a$ (or b) we have $h(n+1) = h(n).1 = b$ (or a). Now we have $f = \lambda_b$ and $g = \lambda_a$. Therefore A is S-injective and hence, by Theorem 2.2, is an absolute s-pure. But A having no zero element is not injective.

The following example shows that there is an absolute s-pure act with a zero element which is not injective.

Example 3.5. Consider the semigroup $S = (\{2, 3, 4, ...\}, .)$ and S-act $A = \{0, 1, 2, 3, ...\}$ with product as its action. Then every homomorphism $f : S \to A$, $f(s) = a_s,\ s \in S$ is of the form λ_m, for some $m \geq 0$. This is because, for all $t, s \in S$, $a_t.s = a_{ts} = a_{st} = a_s.t$. So, a_2 is even, for if $a_2 = 2k + 1$ for some $k \geq 0$ then since $a_2.3 = a_3.2$ we have $3(2k + 1) = 2a_3$ which is impossible. Let $a_2 = 2m$, $m \geq 0$. Then the equation $a_2.s = a_s.2$ gives $a_s = ms$, for all $s \in S$, as required. Therefore, A is S-injective, and hence is an absolute s-pure. But, A is not injective, for if we consider the extension B of A consisting of all the elements of A and all the multiples of 1.5, then there does not exist a homomorphism $f : B \to A$ with $f\restriction_A = id$. This is because, if f is such a homomorphism and $f(1.5) = n$ then we must have $3 = f(3) = 2f(1.5) = 2n$ which is impossible.

We now present some classes of semigroups such that the converse of the above theorem is true for the acts over them.

Theorem 3.6. *Let S be a cyclic semigroup. Then every S-act A with a zero element is an absolute s-pure if and only if it is injective.*

Proof. First recall that every infinite cyclic semigroup is isomorphic to $(\mathbb{N}, +)$ (see [6]). Now, we prove the result for acts over $S = (\mathbb{N}, +)$. Let A be an absolute s-pure S-act with a zero element a_0. To prove that A is an absolute retract, let B be an extension of A. For $b \in B - A$, take S_1^b and S_2^b to be the subsets of S with $bS_1^b \subseteq A$, $bS_2^b \subseteq B - A$, and $S_1^b \cup S_2^b = S$. The retraction map $g : B \to A$ is defined on A as identity map, and for $b \in B - A$ with $bS \subseteq B - A$ is given by $g(b) = a_0$. To define g for the other elements b of $B - A$, first notice that S_2^b is finite whenever S_1^b is nonempty. This is because, taking $m \in S_1^b$, we get that for every $l,\ l \geq m,\ bl = b(m + (l - m)) \in A$. In fact, if n is the least element of $\mathbb{N}$ with $bn \in A$ then $S_1^b = \{n, n+1, n+2, ...\}$ and $S_2^b = \{1, 2, ..., n-1\}$. Therefore, the subset of $B - A$ consisting of the elements b with $bS \nsubseteq B - A$, is the union of disjoint subsets $B_n, n \geq 0$ of $B - A$, where $B_n = \{b \in B - A : \ card(S_2^b) = n\}$. So we define maps $g_n : B_n \to A$, for $n \geq 0$, and then the retraction map g on the elements of $B - A$ with $bS \nsubseteq B - A$ is given by $\bigcup_{n, n \geq 0} g_n$. The family $g_n,\ n \geq 0$ is defined by induction on n as: First step: $g_0(b)$, since $b \in B_0$ is a solution of the sequential system $xl = bl$, $l \geq 1$, and A is an absolute s-pure, the sequential system has a solution in A, namely $g_0(b)$. Hence we have $g_0(b)l = bl$ for all $l \geq 1$. Induction step: Let $g_0, ..., g_{k-1}$ be defined, then define $g_k(b)$ for $b \in B_k$ as a solution of the sequential system

$$(*) \qquad \begin{cases} xk = g_0(bk) \\ \quad \vdots \\ x1 = g_{k-1}(b1) \\ xs = bs \ (s \geq k+1) \end{cases}$$

Notice that using Corollary 2.3, the system $(*)$ over A has a solution in some extension of A, and hence has a solution in A, since the map $f : S \to A$ corresponding to the system is a homomorphism and A is an absolute s-pure. Now it is enough to prove that the defined map g is an act map. Let $b \in B$ and $n \in \mathbb{N}$. If $b \in A$ then $g(bn) = bn = g(b)n$, if $b \in B - A$ with $bS \subseteq B - A$ then $g(bn) = a_0 = a_0 n = g(b)n$. So, let $b \in B_k$, for some $k \geq 0$. If $n \leq k$ then $bn \in B_{k-n}$ so $g(bn) = g_{k-n}(bn)$, also $g(b)n = g_k(b)n$ where since $g_k(b)$ is a solution of the system $(*)$ we have $g_k(b)n = g_{k-n}(bn)$ as required. Finally, if $n > k$ then $bn \in A$ so $g(bn) = bn$, and again since $g_k(b)$ is a solution of $(*)$ we get $g(b)n = g_k(b)n = bn$.

To prove the result for the case where S is finite, let $S = \langle s_0 \rangle = \{s_0, s_0^2, ..., s_0^m, s_0^{m+1}, ..., s_0^{m+r-1}\}$ where m is the index of s_0 and r is the period of s_0 see [6]. Continuation of the proof is similar to the above discussion, just replace all bl by bs_0^l, for $1 \leq l \leq m + r - 1$. ■

Theorem 3.7. *Let S be a zero semigroup with s_0 as the zero of S. Then every S-act A with a zero element is an absolute s-pure if and only if it is injective.*

Proof. Let A be an S-act with a zero element a_0 which is an absolute s-pure. First note that for every $a \in A$, as_0 is a zero element, since for every $s \in S$ we can write $(as_0)s = a(s_0 s) = as_0$. To prove that A is an absolute retract, let B be an extension of A. Note that for every $b \in B - A$ we have $bs_0 \in B - A$ if and only if $bS \subseteq B - A$, since if for an $s \in S$, $bs \in A$ then $bs_0 = b(ss_o) = (bs)s_0 \in A$. Define $f : B \to A$ such that $f \restriction_A = id$ and for $b \in B - A$,

$$f(b) = \begin{cases} a_b & \text{if } bS \nsubseteq B - A, \exists t \in S,\ bt \neq bs_0 \\ bs_0 & \text{if } bS \nsubseteq B - A, \forall t \in S,\ bt = bs_0 \\ a_0 & \text{if } bS \subseteq B - A \end{cases}$$

where a_b is an element of A that obtains as follows: Let $bS \nsubseteq B - A$ then there exist nonempty subsets S_1 and S_2 of S such that $S_1 \cup S_2 = S$ and $bS_1 \subseteq A$, $bS_2 \subseteq B - A$. Now consider the sequential system $xs = a_s,\ s \in S$, with $a_s = bs$, for $s \in S_1$, and $a_s = bs_0$ for $s \in S_2$. Take a_b to be a solution of this system in A, which exists by Corollary 2.3, since the map $f : S \to A$, $f(s) = a_s,\ s \in S$, is a homomorphism and A is an absolute s-pure.

Now to see that f is a homomorphism, let $b \in B - A$ and $s \in S$. If $bS \subseteq B - A$ then $bsS \subseteq B - A$ hence $f(bs) = a_0 = a_0 s = f(b)s$. If $bS \nsubseteq B - A$ and there exists $t \in S$ with $bt \neq bs_0$ then two cases may happen:

Case 1. $s \in S_1$, then we have $a_b s = bs$ and hence $f(bs) = f(a_b s) = a_b s = f(b)s$.

Case 2. $s \in S_2$, that is $bs \in B - A$, then for all $t \in S$, $bst = bs_0$ hence we can write $f(bs) = bs_0 = bs_0 s = f(b)s$.

Finally, if $bS \nsubseteq B - A$ and for all $t \in S,\ bt = bs_0$ then we have $bsS = \{bs_0\} \nsubseteq B - A$, and for all $t \in S$ have $bst = bs_0$ hence $f(bs) = bs_0 = bs_0 s = f(b)s$. ■

Theorem 3.8. *Let S be left zero semigroup. Then every S-act A with a zero element is an absolute s-pure if and only if it is injective.*

Proof. Let A be an S-act with a zero element a_0 which is an absolute s-pure. To prove that A is an absolute retract, let B be an extension of A. Define $g : B \to A$ by

$$g(b) = \begin{cases} a_0 & \text{if } bS \subseteq B - A \\ a_b & \text{if } bS \nsubseteq B - A \end{cases}$$

where a_b is a solution of the sequential system

$$\begin{cases} xs = bs & \text{for } s \in S \text{ with } bs \in A \\ xs = a_0 & \text{for } s \in S \text{ with } bs \notin A \end{cases}$$

in A which exists applying Corollary 2.3, since the map $f : S \to A$ corresponding to the system is a homomorphism and A is an absolute s-pure.

To see that g is homomorphism, let $b \in B - A$ and $t \in S$. If $bt \in A$ then $g(bt) = bt = a_b t = g(b)t$, and if $bt \notin A$ then $btS = \{bt\} \subseteq B - A$ so $g(bt) = a_0 = a_b t = g(b)t$. ■

For the above theorem, one can also apply results of [3].

Theorem 3.9. *Let S be in one of the above three classes of semigroups. Then the following are equivalent for any S-act A:*

(*i*) *A is injective.*

(*ii*) *A is an absolute retract.*

(*iii*) *A is S-injective and has a zero element.*

(*iv*) *A is an absolute s-pure and has a zero element.*

Proof. Equivalence of (i) and (ii) follows by Theorem 1.3, (iii) and (iv) are equivalent by Theorem 2.2. Finally, (i) and (iv) are equivalent by the previous four Theorems. ■

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