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UNIQUE RANGE SETS FOR MEROMORPHIC FUNCTIONS CONSTRUCTED WITHOUT AN INJECTIVITY HYPOTHESIS

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Abstract. A set is called a unique range set (counting multiplicities) for a particular family of functions if the inverse image of the set counting multiplicities uniquely determines the function in the family. So far, almost all constructions of unique range sets for meromorphic functions are zero sets of polynomials which satisfy an injectivity condition introduced by Fujimoto. A polynomial $P(z)$ satisfies the *injectivity condition* if P is injective on the zeros of its derivative. In this paper, we will construct examples of unique range sets for meromorphic functions without assuming an injectivity condition.

1. Introduction

Let $\mathcal{M}^*(\mathbb{C})$ be the set of non-constant meromorphic functions defined on $\mathbb{C}$ and $\mathcal{F}$ be a non-empty subset of $\mathcal{M}^*(\mathbb{C})$. For $f \in \mathcal{F}$ and a set S in the range of f define

$$E(f, S) = \bigcup_{a \in S} \{(z, m) \in \mathbb{C} \times \mathbb{Z}^+ : f(z) = a \text{ with multiplicity } m\}.$$

A set S is called a *unique range set counting multiplicity* for $\mathcal{F}$, if the condition $E(f, S) = E(g, S)$ for $f, g \in \mathcal{F}$ implies that $f \equiv g$.

The first example of a unique range set was given by Gross and Yang [9], who considered the zero set of the equation $z + e^z = 0$. Note that this set has infinitely many zeros. Since then, there have been many efforts to study the problem of constructing unique range sets (for example [1, 5, 10, 12, 13], ...). There are two main problems related to the study of unique range sets. The first problem is

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determining the minimum cardinality of a unique range set for entire and also for meromorphic functions. The second problem is characterizing unique range sets. In fact, examples of unique range sets given by most authors are sets of the form $\{z \in \mathbb{C} \mid z^n + az^m + b = 0\}$ under suitable conditions on the constants a and b and on the positive integers n and m. So far, the smallest unique range set for meromorphic functions has 11 elements and was given by Frank and Reinders in [6]. They proved that the set

$$\left\{z \in \mathbb{C} \mid \frac{(n-1)(n-2)}{2}z^n + n(n-2)z^{n-1} + \frac{(n-1)n}{2}z^{n-2} + b = 0\right\},$$

where $n \geq 11$ and $b \neq 0, 1$, is a unique range set for meromorphic functions. Fujimoto [8] extended this result to zero sets of more general polynomials satisfying an *injectivity condition*. A polynomial $P(z)$ is said to satisfy the *injectivity condition* if P is injective on the zeros of its derivative, i.e., for any zeros $e_i \neq e_j$ of $P'(z)$, we have $P(e_i) \neq P(e_j)$ (in [8], Fujimoto referred to this as "condition H"). One can see that the zero set of the polynomial $z^n + az^{n-1} + b = 0$ is not a unique range set for meromorphic functions for any n; see [3]. Therefore, whether or not the zero set of a polynomial is a unique range set depends not only on the degree of the polynomial, but also on the form of the polynomial. Note that up to now, most constructions of unique range sets were zero sets of polynomials satisfying the injectivity condition made explicit in Fujimoto's work.

In this paper, we will construct unique range sets for meromorphic functions that are zero sets of polynomials that do not necessarily satisfy the injectivity condition. Moreover, for a large class of polynomials we give a necessary and sufficient condition for the zero set of the polynomial to be a unique range set for meromorphic functions.

We will let

$$P(z) = a_n z^n + \sum_{i=0}^{m} a_i z^i, \quad (1 \leq m < n, a_i \in \mathbb{C} \text{ and } a_m \neq 0)$$

be a polynomial of degree n in $\mathbb{C}[z]$ without multiple zeros. We will denote the distinct roots of the derivative $P'(z)$ by $\alpha_1, \alpha_2, ..., \alpha_l$ and use $m_1, m_2, ..., m_l$ to denote their respective multiplicities. The number l is called *the derivative index* of $P(z)$.

A subset S of $\mathbb{C}$ is said to be *affine rigid* if no non-trivial affine transformation of $\mathbb{C}$ preserves S.

Let

$$I = \{i \mid a_i \neq 0\}, \quad \lambda = \min\{i \mid i \in I\}, \quad \text{and} \quad J = \{i - \lambda \mid i \in I\}.$$

Our main results are as follows.

Theorem 1. *Let*

$$P(z) = a_n z^n + a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0, \quad (1 \le m < n, a_i \in \mathbb{C} \textit{ and } a_m \neq 0)$$

be a polynomial of degree n with only simple zeros, and let S be its zero set. If $n \ge \max\{m+4, 2l+7\}$, then the following statements are equivalent:

(i) *S is unique range set for meromorphic functions.*

(ii) *P is a strong uniqueness polynomial for meromorphic functions.*

(iii) *S is affine rigid.*

(iv) *The greatest common divisors of the indices respectively in I and J are both 1.*

Theorem 2. *Let $P(z) = a_n z^n + a_m z^m + a_{m-1} z^{m-1} + \cdots + a_p z^p + a_0$ with $n > m > p$, with $a_i \in \mathbb{C}$ and with $a_m a_p a_0 \neq 0$ be a polynomial of degree n with only simple zeros, and let S be its zero set. Assume that $n > 8 + 2m$ and $p \ge 4$. Then the following statements are equivalent:*

(i) *S is unique range set for meromorphic functions.*

(ii) *P is a strong uniqueness polynomial for meromorphic functions.*

(iii) *S is affine rigid.*

(iv) *The greatest common divisor of the indices in I is 1.*

2. Some Standard Notation in Nevanlinna's Theory

We recall some standard notation in Nevanlinna's theory.

Let f be a meromorphic function on $\mathbf{D}(R)$, the disk of radius $0 < R \le \infty$ centered at the origin. Denote the number of poles of f on the closed disc $\overline{\mathbf{D}}(r)$, $r < R$, by $n(f,r)$ and $n_1(f,r)$, counting multiplicity and without multiplicity, respectively. The *counting function* $N(f,r)$ and the *truncated counting function* $N_1(f,r)$ are defined, respectively by

$$N(f,r) = n(f,0)\log r + \int_0^r (n(f,t) - n(f,0))\frac{dt}{t}$$

and

$$N_1(f,r) = n_1(f,0)\log r + \int_0^r (n_1(f,t) - n_1(f,0))\frac{dt}{t}.$$

Here $n(f,0)$ is the order of any pole of f at $z=0$, and $n_1(f,0)$ is 1 if f has a pole at $z=0$ and is 0 otherwise. The Nevanlinna *characteristic function* $T(f,r)$ is defined by

$$T(f,r) = m(f,r) + N(f,r),$$

where $m(f,r)$ is the *proximity function* defined by

$$m(f,r) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi}.$$

We will use Nevanlinna's second main theorem as follows.

Truncated Second Main Theorem. Let $a_1, a_2, ..., a_q$ be a set of distinct complex numbers. Let f be a non-constant meromorphic function on $\mathbb{C}$. Then

$$(q-2)T(f,r) \leq \sum_{j=1}^{q} N_1(\frac{1}{f-a_j}, r) + O(\log T(f,r)),$$

for $r \to \infty$, except for r in a set of finite Lebesque measure.

3. Proof of Theorem 1

To prove Theorem 1, we need the following lemmas.

Lemma 1. *Let* $P(z) = z^n + \sum_{i=n-m}^{n} a_{n-i} z^{n-i}$ $(1 \leq m < n, a_i \in \mathbb{C}$ *and* $a_m \neq 0)$ *be a polynomial of degree* n. *Assume that* f *and* g *are non-constant meromorphic functions such that, for some constants* $c_0 \neq 0$ *and* c_1*,*

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1.$$

If $n \geq \max\{m+4, 7\}$, *then* $c_1 = 0$.

Proof. Assume that $c_1 \neq 0$. Consider the polynomial

$$Q(z) = P(z) + \frac{c_0}{c_1}.$$

Let $Q(z)$ have k distinct zeros $e_1, \ldots, e_k$ with multiplicities $n_1, \ldots, n_k$ respectively, i.e., $Q(z) = (z-e_1)^{n_1} \ldots (z-e_k)^{n_k}$. We have

$$\frac{P(g)}{c_1 P(f)} = P(g) + \frac{c_0}{c_1} = Q(g) = (g-e_1)^{n_1} \ldots (g-e_k)^{n_k}. \tag{1}$$

Since $c_1 \neq 0$, f and g have no common poles. Therefore, if $z_0 \in \mathbb{C}$ is a zero of $g - e_i$ for some $1 \leq i \leq k$, then z_0 is a pole of f. Moreover, from (1) we have $n_i \,\mathrm{ord}_{z_0}(g-e_i) = n \,\mathrm{ord}_{z_0}(\frac{1}{f})$, where the notation $\mathrm{ord}_z f$ denotes the multiplicity of

z such that $f(z) = 0$ for a meromorphic function f, with $\text{ord}_z f$ negative indicating a pole at z. This implies

$$\text{ord}_{z_0}(g - e_i) = \frac{n}{n_i}\,\text{ord}_{z_0}(\frac{1}{f}) \geq \frac{n}{n_i}. \tag{2}$$

Applying the Second Main Theorem for g and $e_1, \ldots, e_k$, we have

$$\begin{aligned}(k-2)T(g,r) &\leq \sum_{i=1}^{k} N_1(\frac{1}{g-e_i}, r) + O(\log T(g,r)) \\ &\leq \sum_{i=1}^{k} \frac{n_i}{n} T(g,r) + O(\log T(g,r)) \\ &\leq \frac{\sum_{i=1}^{k} n_i}{n} T(g,r) + O(\log T(g,r)) \\ &\leq T(g,r) + O(\log T(g,r)),\end{aligned}$$

which implies $k - 3 \leq 0$.

Case 1. $k = 1$.

We have $P(z) = (z - e_1)^n - \frac{c_o}{c_1}$ which contradicts the hypothesis that the coefficient in front of the term z^{n-1} in $P(z)$ is zero. Therefore $k \neq 1$.

Case 2. $k = 2$.

We have

$$\begin{aligned}P(z) &= (z-e_1)^{n_1}(z-e_2)^{n_2} - \frac{c_o}{c_1} \\ &= z^n - (n_1e_1 + n_2e_2)z^{n-1} + \Big(n_1n_2e_1e_2 + \frac{n_1(n_1-1)}{2}e_1^2 + \frac{n_2(n_2-1)}{2}e_2^2\Big)z^{n-2} \\ &\quad + \text{lower degree terms in } z.\end{aligned}$$

Therefore, by the hypothesis that $P(z)$ neither has a term of degree $n-1$ nor a term of degree $n-2$, we have

$$n_1e_1 + n_2e_2 = n_1n_2e_1e_2 + \frac{n_1(n_1-1)}{2}e_1^2 + \frac{n_2(n_2-1)}{2}e_2^2 = 0,$$

which contradicts $e_1, e_2 \neq 0$ and n_1, n_2 are positive. Hence $k \neq 2$.

Case 3. $k = 3$.

We will state this case as a separate lemma to be used again later.

Lemma 2. *Let $Q(z) = (z-e_1)^{n_1}(z-e_2)^{n_2}(z-e_3)^{n_3}$ be a polynomial of degree n where $n_1n_2n_3 \neq 0$ and e_1, e_2 and e_3 are distinct complex numbers. If there exist non-constant meromorphic functions f and g such that*

$$(g-e_1)^{n_1}(g-e_2)^{n_2}(g-e_3)^{n_3} = hf^n,$$

for some meromorphic function h which does not vanish on the zero sets of $g-e_i$, $i=1,2,3$, then only one of the following holds:

$$n_1 = n_2 = n_3 = \frac{n}{3}; \quad \textit{or} \quad n_1 = \frac{n}{2} \textit{ and } n_2 = n_3 = \frac{n}{4}; \quad \textit{or}$$
$$n_1 = \frac{n}{2} \textit{ and } n_2 = \frac{n}{3} \textit{ and } n_3 = \frac{n}{6}.$$

Proof of Lemma 2. Write $n = \alpha_i\beta_i$ and $n_i = \alpha_i\gamma_i$, where $(\beta_i, \gamma_i) = 1$, $\beta_i > \gamma_i \geq 1$, for $i = 1,2,3$.

Assume z_0 is a zero of $g - e_i$, for some $i = 1,2,3$. By the hypothesis $Q(g) = hf^n$ we have $\mathrm{ord}_{z_0}(g-e_i) = \frac{n}{n_i}\,\mathrm{ord}_{z_0}(f) = \frac{\beta_i}{\gamma_i}\,\mathrm{ord}_{z_0}(f) \geq \beta_i$.

We will consider the following cases.

(i) All of $\beta_1, \beta_2, \beta_3 \geq 3$ and one of them is at least 4; assume that $\beta_3 \geq 4$.

In this case, $\mathrm{ord}_{z_0}(g-e_3) \geq \beta_3 \geq 4$, and $\mathrm{ord}_{z_0}(g-e_i) \geq \beta_i \geq 3$ for $i = 1,2$. Hence, applying the Second Main Theorem to the function g and the complex numbers e_1, e_2, e_3, we have

$$T(g,r) \leq (\frac{1}{3} + \frac{1}{3} + \frac{1}{4})T(g,r) + O(\log T(g,r)),$$

which is impossible.

(ii) $\beta_1 = \beta_2 = \beta_3 = 3$.

Since $\beta_i > \gamma_i$ we have $\gamma_i = 1$ or 2. Assume that there exists an i, so without loss of generality $i = 1$, such that $\gamma_1 = 2$, then $n_1 = \frac{2n}{3}$, which contradicts the fact that $n_1 + n_2 + n_3 = n$ and $n_1n_2n_3 \neq 0$. Therefore $\gamma_i = 1$ for $i = 1,2,3$ and hence $n_1 = n_2 = n_3 = \frac{n}{3}$.

(iii) There exists at least one i, so without loss of generality $i = 1$, such that $\beta_1 = 2$.

By $2 = \beta_1 > \gamma_1$, we have $\gamma_1 = 1$ and hence $n_1 = \frac{n}{2}$. Since

$$n_1 + n_2 + n_3 = n \qquad \text{and} \qquad n_2n_3 \neq 0,$$

we have $\beta_2, \beta_3 \geq 3$.

If $\beta_2, \beta_3 \geq 5$ then the Second Main Theorem implies that

$$T(g,r) \leq (\frac{1}{2} + \frac{1}{5} + \frac{1}{5})T(g,r) + O(\log T(g,r)),$$

which is impossible.

If there exists i, so without loss of generality $i = 2$, such that $\beta_2 = 4$ then $\gamma_2 = 1$ or 3. If $\gamma_2 = 3$ then $n_2 = \frac{3n}{4}$ which is impossible since we would then have $n_1 + n_2 + n_3 > n_1 + n_2 = \frac{n}{2} + \frac{3n}{4} > n$. Therefore $\gamma_2 = 1$, which means $n_2 = \frac{n}{4}$ and hence $n_3 = \frac{n}{4}$.

If there exists i, so without loss of generality $i = 2$, such that $\beta_2 = 3$. Then, similarly we have $\gamma_2 = 1$, and hence $n_2 = \frac{n}{3}$, and hence $n_3 = \frac{n}{6}$. ∎

We now continue to prove Lemma 1.

When $k = 3$, by Lemma 2, we only have to consider the three following cases:

The first case is when $Q(z) = (z - e_1)^{\frac{n}{3}}(z - e_2)^{\frac{n}{3}}(z - e_3)^{\frac{n}{3}}$ which implies $P(z) = (z - e_1)^{n_1}(z - e_2)^{n_1}(z - e_3)^{n_1} - \frac{c_o}{c_1}$. In this case, $e_1 \neq 0$, otherwise $P(z)$ has a term of degree $n - 1$ or $n - 2$ in z. On the other hand, since $P(z)$ does not have a term of degree $n - 1$, we may assume $e_1 = 1$, $e_3 = -1 - e_2$. We have

$$\begin{aligned} P(z) &= (z-1)^{n_1}(z-e_2)^{n_1}(z+1+e_2)^{n_1} - \frac{c_o}{c_1} \\ &= \Big(z^3 - (e_2^2 + e_2 + 1)z + e_2(e_2+1)\Big)^{n_1} - \frac{c_o}{c_1}. \end{aligned}$$

In multiplying out the n_1-th power, the only way to get a term of degree $3n_1 - 2$ is to multiply the z^3 term $n_1 - 1$ times and the z term once. Since there are n_1 ways to do this, after multiplying out, the coefficient in front of z^{3n_1-2} is n_1 times the coefficient in front of z inside the product. Therefore, it is $-n_1(e_2^2 + e_2 + 1)$. Similarly, the only way to get a term of degree $3n_1 - 3$ is to multiply the z^3 term $n_1 - 1$ times and the constant term once. Again, there are n_1 ways to do this, and so the coefficient in front of z^{3n_1-3} after multiplying out is n_1 times the constant term inside the product, which is $n_1(e_2^2 + e_2)$. Thus, we have

$$\begin{aligned} P(z) = z^{3n_1} &- n_1(e_2^2 + e_2 + 1)z^{3n_1-2} + n_1(e_2^2 + e_2)z^{3n_1-3} + \\ &+ \text{ terms of lower degree.} \end{aligned}$$

By the hypothesis that $P(z)$ does not have terms of degree $n - 2$ nor $n - 3$, we have

$$e_2^2 + e_2 + 1 = e_2^2 + e_2 = 0,$$

which is impossible.

The second case is when $P(z) = Q(z) - \frac{c_o}{c_1} = (z - e_1)^{2v}(z - e_2)^v(z - e_3)^v - \frac{c_o}{c_1}$, with $v = \frac{n}{4}$. Since $P(z)$ does not have a term of degree $n - 1$ nor $n - 2$ in z we have $e_1 \neq 0$ and we may assume $e_1 = 1$, $e_3 = -2 - e_2$. Hence, we have

$$\begin{aligned} P(z) &= (z-1)^{2v}(z-e_2)^v(z+2+e_2)^v - \frac{c_o}{c_1} \\ &= \Big(z^4 - (e_2^2 + 2e_2 + 3)z^2 + (2e_2^2 + 4e_2 + 2)z - e_2^2 - 2e_2\Big)^v - \frac{c_o}{c_1}. \end{aligned}$$

In multiplying out the v-th power, the only way to get a term of degree $4v-2$ is to multiply the z^4 term $v-1$ times and the z^2 term once. Since there are v ways to do this, after multiplying out, the coefficient in front of z^{4v-2} is v times the coefficient in front of z^2 inside the product. Similarly, the only way to get a term of degree $4v-3$ is to multiply the z^4 term $v-1$ times and the z term once. Again, there are v ways to do this, and so the coefficient in front of z^{4v-3} after multiplying out is v times the coefficient in front of z inside the product. Thus, we have

$$P(z) = z^{4v} - v(e_2^2 + 2e_2 + 3)z^{4v-2} + v(2e_2^2 + 4e_2 + 2)z^{4v-3} + \\ + \text{terms of lower degree.}$$

By the hypothesis that $P(z)$ does not have terms of degree $n-2$ nor $n-3$, we have

$$e_2^2 + 2e_2 + 3 = 2e_2^2 + 4e_2 + 2 = 0,$$

which is impossible.

The last case is when $Q(z) = (z-e_1)^{\frac{n}{2}}(z-e_2)^{\frac{n}{3}}(z-e_3)^{\frac{n}{6}}$. In this case, we have $P(z) = Q(z) - \frac{c_o}{c_1} = (z-e_1)^{3v}(z-e_2)^{2v}(z-e_3)^{v} - \frac{c_o}{c_1}$, with $v = \frac{n}{6}$. Similar to the above case, we may assume without loss of generality that $e_1 = 1$, $e_3 = -3 - 2e_2$. Then, we have

$$\begin{aligned} P(z) &= \Big((z-1)^3(z-e_2)^2(z+3+2e_2)\Big)^v - \frac{c_o}{c_1} \\ &= \Big(z^6 - (3e_2^2 + 6e_2 + 6)z^4 + (2e_2^3 + 12e_2^2 + 18e_2 + 8)z^3 + \dots\Big)^v \\ &= z^{6v} - v(3e_2^2 + 6e_2 + 6)z^{6v-2} + v(2e_2^3 + 12e_2^2 + 18e_2 + 8)z^{6v-3} \\ &\quad + \text{terms of lower degree.} \end{aligned}$$

Therefore

$$3e_2^2 + 6e_2 + 6 = 2e_2^3 + 12e_2^2 + 18e_2 + 8 = 0,$$

which is impossible.

We conclude that $k \neq 3$.

Therefore $c_1 = 0$, and the proof of the Lemma 1 is complete. ∎

Lemma 3. (Theorem 6.8, [8]). *Let $S = \{a_1, a_2, \dots, a_n\}$ be a finite set in $\mathbb{C}$ and $P(z) = (z-a_1)(z-a_2)\dots(z-a_n)$. Assume that f and g are non-constant meromorphic functions such that $E(f,S) = E(g,S)$. If $n \geq 2l + 7$ then there exist constants $c_0 \in \mathbb{C}^*$ and c_1 such that*

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1.$$

A polynomial $P(z)$ with coefficients in $\mathbb{C}$ is called a *uniqueness polynomial* for $\mathcal{F}$ if the condition $P(f) = P(g)$ for $f, g \in \mathcal{F}$ implies that $f \equiv g$; P is called a *strong uniqueness polynomial* if the condition $P(f) = cP(g)$ for $f, g \in \mathcal{F}$ and some non-zero constant c implies that $c = 1$ and $f \equiv g$. The readers can find some related results in, for example, [2] and [7] for polynomials satisfying the injectivity condition and also in [3] for polynomials that need not satisfy the injectivity condition.

The following lemma is implied directly from [3, Theorem 1], [3, Proposition 3.1] and [3, Proposition 4.1].

Lemma 4. *Let*

$$P(z) = a_n z^n + \sum_{i=n-m}^{n} a_{n-i} z^{n-i} \quad (0 \le m < n, a_i \in \mathbb{C} \text{ and } a_n, a_m \ne 0)$$

be a polynomial of degree n. *Let* $I = \{i \mid a_i \ne 0\}, \lambda = \min\{i \mid i \in I\}$ *and* $J = \{i - \lambda \mid i \in I\}$. *If* $n \ge m + 4$, *then the following statements are equivalent:*

(i) P *is a strong uniqueness polynomial for meromorphic functions.*

(ii) S *is affine rigid.*

(iii) *The greatest common divisor of the indices in* I *is* 1 *and the greatest common divisor of the indices in* J *is also* 1.

Proof of Theorem 1. By Lemma 4, it is enough to prove that (i) is equivalent to (ii).

Assume that (i) holds, and assume that there are non-constant meromorphic functions f and g such that $P(f) = cP(g)$ for some non-zero constant c. Let $S = \{a_1, \ldots, a_n\}$ be the zero set of $P(z)$. For $i, 1 \le i \le n$, assume that $f(z_0) = a_i$ with multiplicity α then, by $P(f) = cP(g)$, there is $j, 1 \le j \le n$ such that $g(z_0) = a_j$ with multiplicity α. Therefore $E(f, S) = E(g, S)$. By the assumption S is a unique range set for meromorphic functions, we have $f = g$. Therefore, P is a strong uniqueness polynomial for meromorphic functions.

For the converse, assume that there exist non-constant meromorphic functions f and g such that $E(f, S) = E(g, S)$. Since $n \ge \max\{m+4, 2l+7\} \ge 2l+7$, Lemma 3 implies that there exist constants $c_0 \in \mathbb{C}^*$ and c_1 such that $\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1$. On the other hand, Lemma 1 and the assumption $n \ge \max\{m+4, 2l+7\} \ge \max\{m+4, 7\}$ imply c_1=0. Therefore $c_0 P(f) = P(g)$ which implies $f = g$ because of P is a strong uniqueness polynomial for meromorphic functions. ■

4. Proof of Theorem 2

The following lemmas will be needed in the proof of our second theorem.

Lemma 5. (see [14]). *Let $g_j(x_0, \ldots, x_s)$ be a homogeneous polynomial of degree δ_j for $0 \le j \le s$. Suppose there exists a holomorphic map $f : \mathbb{C} \longrightarrow \mathbb{P}^s$ so that its image lies in the curve described by*

$$\sum_{j=0}^{s} x_j^{n-\delta_j} g_j(x_0, \ldots, x_s) = 0, \quad \textit{and} \quad n > (s+1)(s-1) + \sum_{j=0}^{s} \delta_j.$$

Then the polynomials

$$x_1^{n-\delta_1} g_1(x_0, \ldots, x_s), \ldots, x_s^{n-\delta_s} g_s(x_0, \ldots, x_s)$$

are linearly dependent on the image of f.

Proof. [Proof of Theorem 2]. By Lemma 4, it is enough to prove that (i) is equivalent to (iv).

If the assertion (i) holds then similarly as in the proof of Theorem 1, the assertion (ii) holds. By Lemma 4, it means (iv) holds.

Now, we will prove that (iv) implies (i). Let $S = \{a_1, \ldots, a_n\}$ be the distinct zeros set of $P(z)$. Let $f = \dfrac{f_1}{f_2}$, $g = \dfrac{l_1}{l_2}$ be non-constant meromorphic functions such that $E_f(S) = E_g(S)$, where (f_1, f_2) and (l_1, l_2) are pairs of entire functions without common zeros. Then there exists an entire function h such that

$$(f_1 - a_1 f_2) \ldots (f_1 - a_n f_2) = e^h (l_1 - a_1 l_2) \ldots (l_1 - a_n l_2).$$

Put $g_1 = e^{\frac{l}{n}} l_1$, $g_2 = e^{\frac{l}{n}} l_2$, and define $\Phi = (f_1, f_2, g_1, g_2)$. Hence

$$f_1^n + f_2^{n-m} \sum_{i=n-m}^{n} a_{n-i} f_1^{n-i} f_2^i - g_1^n - g_2^{n-m} \sum_{i=n-m}^{n} a_{n-i} g_1^{n-i} g_2^i = 0. \tag{3}$$

Applying Lemma 5 in the case $s = 3$, $\delta_0 = \delta_2 = 0$, $\delta_1 = \delta_3 = m$ and $n > 8 + 2m$, we may assume without loss of generality that there are constants α_1, α_2, α_3, not all are zero, such that

$$\alpha_1 f_1^n + \alpha_2 f_2^{n-m} \sum_{i=n-m}^{n} a_{n-i} f_1^{n-i} f_2^i + \alpha_3 g_1^n = 0.$$

We consider the possible cases:

Case 1. $\alpha_1 \alpha_2 \alpha_3 \neq 0$.
Using again Lemma 5 (with $s = 2$, $\delta_0 = \delta_2 = 0$, $\delta_1 = m$), we obtain

$$\alpha_1' f_1^n + \alpha_2' f_2^{n-m} \sum_{i=n-m}^{n} a_{n-i} f_1^{n-i} f_2^i = 0,$$

where not all α_i' are zeros. This implies that f is constant.

Case 2. $\alpha_3 = 0$, which implies f must be constant.

Case 3. $\alpha_1 = 0$. Clearly, $\alpha_2\alpha_3 \neq 0$. Then

$$\alpha_2 f_2^{n-m} \sum_{i=n-m}^{n} a_{n-i} f_1^{n-i} f_2^i = -\alpha_3 g_1^n.$$

Dividing the above equation by f_2^n and recalling that $f = \frac{f_1}{f_2}$, we have

$$f^{n-m}\Big(a_m f^m + a_{m-1} f^{m-1} + \cdots + a_0\Big) = -\frac{\alpha_3}{\alpha_2}\big(\frac{g_1}{f_2}\big)^n.$$

Let $e_1, \ldots, e_q$ be the distinct zeros with multiplicity $n_1, \ldots, n_q$ respectively of

$$Q(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0.$$

Then

$$f^{n-m}(f - e_1)^{n_1} \ldots (f - e_q)^{n_q} = -\frac{\alpha_3}{\alpha_2}\big(\frac{g_1}{f_2}\big)^n.$$

By the hypothesis $a_0 \neq 0$, we have $e_i \neq 0$, and since $m > p \geq 4$, we have $q \geq 2$. On the other hand, by the same argument as in the beginning of the proof of Lemma 1, we have $q \leq 2$. Therefore $q = 2$. By Lemma 2, if $q = 2$ then one of the following cases must hold: $n - m = \frac{n}{2}$; $n - m = \frac{n}{3}$; $n - m = \frac{n}{4}$; or $n - m = \frac{n}{6}$. All of these cases contradict the hypothesis $n > 2m + 8$. Therefore f is constant.

Case 4. $\alpha_2 = 0$. It is clear that $\alpha_1\alpha_3 \neq 0$. Furthermore,

$$\alpha f_1 = g_1,$$

where $\alpha^n = \dfrac{\alpha_1}{\alpha_3}$. From (3) we have

$$(1 - \alpha) a_n f_1^n + f_2^{n-m} \sum_{i=n-m}^{n} a_{n-i} f_1^{n-i} f_2^i - g_2^{n-m} \sum_{i=n-m}^{n} a_{n-i} \alpha^{n-i} f_1^{n-i} g_2^i = 0.$$

If $\alpha \neq 1$, then using Lemma 5 for $\delta_0 = 0$, $\delta_1 = \delta_2 = m$, $s = 2$ we obtain that f_1^n and $f_2^{n-m} \sum_{i=0}^{m} a_i f_1^i f_2^{m-i}$ are linearly dependent, and hence f is constant. This is a contradiction. Therefore, $\alpha = 1$ and

$$a_n f_1^n + f_2^{n-m} \sum_{i=n-m}^{n} a_{n-i} f_1^{n-i} f_2^i = a_n g_1^n + g_2^{n-m} \sum_{i=n-m}^{n} a_{n-i} g_1^{n-i} g_2^i.$$

Consider the polynomial

$$H(z) = a_0 h_1^n + a_p h_1^{n-p} + \cdots + a_m h_1^{n-m} + a_n.$$

Divide both sides of the above equation by $f_1^n = g_1^n$ to get

$$H\left(\frac{1}{f}\right) = H\left(\frac{1}{g}\right). \tag{4}$$

On the other hand, since the hypothesis that the greatest common divisor of the indices in $I = \{i \mid a_i \neq 0\}$ is 1, we have that the greatest common divisor of the indices in $K := \{n - i \mid a_i \neq 0\}$ is also 1. From $a_n \neq 0$, we have

$$\min\{n - i \mid n - i \in K\} = 0.$$

Thus, we can apply Lemma 4 with $p \geq 4$ to get that $H(z)$ is a strong uniqueness polynomial for meromorphic functions. Therefore from (4), we have $f = g$. ■

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