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NEW ITERATIVE ALGORITHM FOR SOLVING A SYSTEM OF GENERALIZED MIXED IMPLICIT EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract. A new system of generalized mixed implicity equilibrium problems is introduced and studied in real Banach spaces. The notion of the Yosida approximation introduced by Moudafi in Hilbert spaces is first generalized to reflexive Banach spaces. By using the notion of the Yosida approximation, a system of generalized equation problems is considered and its equivalence with the system of generalized mixed implicity equilibrium problems is also proved. By using the system of generalized equation problems, a new iterative algorithm for solving the system of generalized mixed implicity equilibrium problems is suggested and analyzed. The strong convergence of the iterative sequences generated by the algorithm is proved under suitable conditions. These results are new and unify and generalize some recent results in this field.

1. INTRODUCTION

It is well known that equilibrium problem includes variational inequality, optimization problem, problems of Nash equilibria, saddle point problems, fixed point problems and complementarity problems as special cases, for example, see [1-5] and the references therein. In the theory of variational inequalities, variational inclusions and equilibrium problems, the development of an efficient and implementable iterative algorithm is interesting and important. Various kinds of iterative algorithms to solve the equilibrium problems have been developed by many authors. For details, we can refer to [3, 6-25, 35-51] and the references therein. By using the viscosity

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approximation method and its variants, many authors studied the iterative algorithms for finding a common element of the solution set of (mixed) equilibrium problems and the fixed point set of nonexpansive mappings in Hilbert spaces, for example, see [12-21]. By using the auxiliary principle technique, Ding [9], Ding et al. [10], and Xia and Ding [11] studied the approximation solvability of some mixed equilibrium problems in Hilbert spaces. Recently, By using Yosida approximation and Wiener-Hopf equation technique, Moudafi [3] and Huang et al. [22] studied the sensitivity analysis of solutions for generalized mixed implicit equilibrium problems in Hilbert spaces. Kazmi and Khan [23] studied the approximation solvability of generalized mixed equilibrium problems in Hilbert spaces. As pointed out by Moudafi [3], "But up to now no sensitivity analysis and only few iterative methods to solve such problems have been done. It is worth mentioning that the new algorithm developed here can be applied to solve the system of generalized mixed equilibrium problems".

Inspired and motivated by the recent works [3, 13, 14, 22, 23], in this paper, we introduce and study a new system of generalized mixed implicit equilibrium problems involving non-monotone set-valued mappings with non-compact values in real reflexive Banach spaces, which includes the system of generalized implicit variational inequalities, the system of generalized implicit variational inclusions as special cases. We first generalize the notion of the Yosida approximation introduced by Moudafi [3] in Hilbert spaces to reflexive Banach spaces. By using the notion of the Yosida approximation, we consider a system of generalized equations problems and show its equivalence with the system of generalized mixed implicit equilibrium problems. Using the system of generalized equations problems, we construct a new iterative algorithm for solving the system of generalized mixed implicit equilibrium problems. Furthermore, we prove the existence of solutions and the convergence of the iterative sequences generated by the algorithm. These results generalize and improve the corresponding results in [3, 13, 14, 22, 23].

2. Preliminaries

Let B be a real Banach space with norm $\|\cdot\|$, B^* be its dual space and $R = (-\infty, +\infty)$. Let $\langle x, \varphi\rangle$ denote the duality pairing between B and B^*, where $x \in B$ and $\varphi \in B^*$. Let K be a nonempty, closed and convex subset of B and let $CB(B)$ be the family of all nonempty, closed and bounded subsets of B.

Definition 2.1. Let K be a closed convex subset of a Hausdorff topological vector space E. A real valued bifunction $F : K \times K \to (-\infty, \infty)$ is said to be

(i) monotone if
$$F(x, y) + F(y, x) \leq 0, \ \forall\ x,\ y \in K;$$

(ii) strictly monotone if

$$F(x,y)+F(y,x)<0,\ \forall\ x,\ y\in K \text{ with } x\neq y;$$

(iii) α-strongly monotone if there exists a $\alpha>0$ such that

$$F(x,y)+F(y,x)\leq -\alpha\|x-y\|,\ \forall\ x,\ y\in K;$$

(iv) upper-hemicontinuous if

$$\lim\ \sup_{t\to 0}F(tz+(1-t)x,y)\leq F(x,y),\ \forall\ x,y,z\in K,\ t\in[0,1].$$

Remark 2.1. Clearly the strong monotonicity of F implies the strict monotonicity of F.

Definition 2.2. A mapping $\eta: B\times B\to B^*$ is said to be

(i) monotone if

$$\langle x-y,\eta(x,y)\rangle\geq 0,\ \forall\ x,y\in B;$$

(ii) strictly monotone if

$$\langle x-y,\eta(x,y)\rangle>0,\ \forall\ x,\ y\in K \text{ with } x\neq y;$$

(iii) δ-strongly monotone if there exists a $\delta>0$ such that

$$\langle x-y,\eta(x,y)\rangle\geq\delta\|x-y\|^2,\ \forall\ x,y\in B;$$

(iv) affine in second argument if

$$\eta(y,\beta x+(1-\beta)z)=\beta\eta(y,x)+(1-\beta)\eta(y,z),\ \forall\ \beta\in[0,1],\ x,y,z\in K.$$

(v) τ-Lipschitz continuous if there exists a constant $\tau>0$ such that

$$\|\eta(x,y)\|\leq\tau\|x-y\|,\ \forall\ x,y\in B.$$

Remark 2.2. If $B=H$ is a Hilbert space, then the concepts (i),(iii) and (v) reduces to the corresponding concepts in [22, 23].

Definition 2.3. The bifunction $\varphi: B\times B\to(-\infty,+\infty]$ is said to be skew-symmetric if

$$\varphi(u,u)-\varphi(u,v)-\varphi(v,u)+\varphi(v,v)\geq 0,\ \forall\ u,v\in B.$$

The skew-symmetric bifunctions have the properties which can be considered an analogs of monotonicity of gradient and nonnegativity of a second derivative for the convex function. For the properties and applications of the skew-symmetric bifunction, the reader may consult Antipin [24].

The following result is a special case of Theorem 3.9.3 of Chang [25].

Lemma 2.1. *Let K be a closed convex subset of a Hausdorff topological vector space E, $F: K\times K\to R$ be a bifunction. If the following conditions hold:*

(i) $F(x,x) \geq 0, \ \forall \ x \in K$,

(ii) F *is monotone and for each* $y \in K$, $x \mapsto F(x,y)$ *is upper hemicontinuous;*

(iii) *for each* $x \in K$, $y \mapsto F(x,y)$ *is convex and lower-semicontinuous;*

(iv) *there exist a compact subset* D *of* E *and* $y_0 \in K \bigcap D$ *such that* $F(x,y_0) < 0$ *for each* $x \in K \setminus D$.

Then the solution set of the following equilibrium problem (EP): find $\hat{x} \in K$ *such that*

$$F(\hat{x},y) \geq 0, \ \forall \ y \in K,$$

is nonempty, convex and compact.

Lemma 2.2. Let K be a closed convex subset of a reflexive Banach space B. Let $F : K \times K \to R$ and $\varphi : B \times B \to R$ be two bifunctions, $\eta : B \times B \to B^*$ be a mapping and $\rho > 0$ be a positive number. Suppose the following conditions are satisfied:

(i) F satisfies the conditions (i)-(iii) in Lemma 2.1;

(ii) η be monotone with $\eta(x,y) + \eta(y,x) = 0, \ \forall \ x,y \in B$;

(iii) η is affine in second argument and continuous from weak topology in B to weak* topology in B^* in first argument,

(iv) φ is skew symmetric and weakly continuous, and φ is convex in first argument.

(v) for each $x \in B$ there exist a compact subset D_x of B and $y_0 \in K \bigcap D_x$ such that $F(z,y_0) + \varphi(y_0,z) - \varphi(z,z)) + \langle z - x, \eta(y_0,z)\rangle < 0$ for each $x \in K \setminus D_x$.

Then for each $x \in B$, there exists a point $z_x \in K$ such that

$$\rho(F(z_x,y) + \varphi(y,z_x) - \varphi(z_x,z_x)) + \langle z_x - x, \eta(y,z_x)\rangle \geq 0, \ \forall \ y \in K. \tag{2.1}$$

Proof. For each fixed $x \in B$, define $\psi : K \times K \to R$ by

$$\psi(z,y) = \rho(F(z,y) + \varphi(y,z) - \varphi(z,z)) + \langle z - x, \eta(y,z)\rangle, \ \forall \ z,y \in K.$$

Since $\eta(x,y) + \eta(y,x) = 0, \ \forall \ x,y \in B$, we have $\eta(z,z) = 0$ for all $z \in B$ and hence

$$\psi(z,z) = \rho F(z,z) + \langle z - x, \eta(z,z)\rangle \geq 0, \ \forall \ z \in K.$$

The condition (i) of Lemma 2.1 is satisfied. Since η is monotone, φ is skew symmetric and $\eta(z,y)+\eta(y,z)=0$, then for each $(z,y)\in K\times K$, we have

$$\begin{aligned}\psi(z,y)+\psi(y,z) =\ & \rho(F(z,y)+F(y,z))+\rho(\varphi(y,z)-\varphi(z,z)+\varphi(z,y)-\varphi(y,y))\\ & +\langle z-x,\eta(y,z)\rangle+\langle y-x,\eta(z,y)\rangle\\ \le\ & \rho(F(z,y)+F(y,z))-\langle z-y,\eta(z,y)\rangle\le 0,\end{aligned}$$

i.e., ψ is monotone. Since η is affine in second argument and φ is weakly continuous, by the condition (ii) in Lemma 2.1, we have that for each $u,y,z\in K$,

$$\begin{aligned}&\limsup_{t\to 0}\psi(tu+(1-t)z,y)\\ \le\ & \limsup_{t\to 0}\rho F(((tu+(1-t)z),y)\\ & +\limsup_{t\to 0}\rho(\varphi(tu+(1-t)z,y)-\varphi(tu+(1-t)z,tu+(1-t)z))\\ & +\limsup_{t\to 0}\langle t(u-x)+(1-t)(z-x),\eta(y,tu+(1-t)z)\rangle\\ \le\ & \rho(F(z,y)+\varphi(z,y)-\varphi(z,z))\\ & +\limsup_{t\to 0}[\langle t(u-x)+(1-t)(z-x),t\eta(y,u)+(1-t)\eta(y,z)\rangle]\\ \le\ & \rho(F(z,y)+\varphi(z,y)-\varphi(z,z))\\ & +\limsup_{t\to 0}[t^2\langle u-x,\eta(y,u)\rangle+t(1-t)\langle z-x,\eta(y,u)\rangle\\ & +t(1-t)\langle u-x,\eta(y,z)\rangle+(1-t)^2\langle z-x,\eta(y,z)\rangle]\\ \le\ & \rho(F(z,y)+\varphi(z,y)-\varphi(z,z))+\langle z-x,\eta(y,z)\rangle=\psi(z,y).\end{aligned}$$

Therefore ψ is upper hemicontinuous in first argument, the condition (ii) of Lemma 2.1 is satisfied. Since for each $z\in K$, $y\mapsto F(z,y)$ is convex and lower semi-continuous, φ is weakly continuous and convex in first argument, and η is affine in second argument and continuous from weak topology in B to weak* topology in B^* in first argument, It is easy to see that for each $z\in K$, $y\mapsto\psi(z,y)$ is convex and lower semicontinuous, the condition (iii) of Lemma 2.1 is satisfied. Clearly the condition (v) implies that ψ also satisfies the condition (iv) of Lemma 2.1. By Lemma 2.1, for each $x\in B$, there exists a point $z_x\in K$ such that

$$\psi(z_x,y)\ge 0,\ \forall\ y\in K.$$

By the definition of ψ, we obtain that for each $x\in B$, there exists a point $z_x\in K$ such that

$$\rho(F(z_x,y)+\varphi(y,z_x)-\varphi(z_x,z_x))+\langle z_x-x,\eta(y,z_x)\rangle\ge 0,\ \forall\ y\in K.$$

Remark 2.3. If F or η is strictly monotone, then the solution of the MEP (2.1) in Lemma 2.2 is unique, i.e., for each $x \in B$, there exists unique a point $z_x \in K$ such that the inequality (2.1) holds.

Remark 2.4. By Lemma 2.2 and Remark 2.3, we obtain that for each $x \in B$, there exists a unique $z_x = J_\rho^{F,\varphi}(x) \in K$ such that

$$\begin{aligned} &\rho(F(J_\rho^{F,\varphi}(x), y) + \varphi(y, J_\rho^{F,\varphi}(x)) - \varphi(J_\rho^{F,\varphi}(x), J_\rho^{F,\varphi}(x))) \\ &\quad + \langle J_\rho^{F,\varphi}(x) - x, \eta(y, J_\rho^{F,\varphi}(x))\rangle \geq 0,\ \forall\, y \in K, \end{aligned} \tag{2.2}$$

and hence $J_\rho^{F,\varphi} : X \to K$ is a well-defined single-valued mapping.

Theorem 2.1. Let K be a closed convex subset of a reflexive Banach space B and $\rho > 0$ be a positive number. Let $F : K \times K \to R$, $\varphi : B \times B \to R$ and $\eta : B \times B \to B^*$ satisfy the following conditions:

(i) F is α-strongly monotone and satisfy the conditions (i)-(iii) of Lemma 2.1;

(ii) η is δ-strongly monotone and τ-Lipschitz continuous with $\eta(x, y) + \eta(y, x) = 0,\ \forall\, x, y \in B$;

(iii) η is affine in second argument and continuous from weak topology in B to weak* topology in B^* in first argument.

(iv) φ is skew symmetric and weakly continuous, and φ is convex in first argument.

(v) for each $x \in B$ there exist a compact subset D_x of B and $y_0 \in K \bigcap D_x$ such that $F(z, y_0) + \varphi(y_0, z) - \varphi(z, z)) + \langle z - x, \eta(y_0, z)\rangle < 0$ for each $x \in K \setminus D_x$.

Then the mapping $J_\rho^{F,\varphi}$ is $\frac{\tau}{\delta+\rho\alpha}$-Lipschitz continuous.

Proof. By Lemma 2.2, we have

$$\begin{aligned} &\rho(F(J_\rho^{F,\varphi}(x), J_\rho^{F,\varphi}(y)) + \varphi(J_\rho^{F,\varphi}(y), J_\rho^{F,\varphi}(x)) - \varphi(J_\rho^{F,\varphi}(x), J_\rho^{F,\varphi}(x))) \\ &\quad + \langle J_\rho^{F,\varphi}(x) - x, \eta(J_\rho^{F,\varphi}(y), J_\rho^{F,\varphi}(x))\rangle \geq 0,\ \forall\, x, y \in B, \\ &\rho(F(J_\rho^{F,\varphi}(y), J_\rho^{F,\varphi}(x)) + \varphi(J_\rho^{F,\varphi}(x), J_\rho^{F,\varphi}(y)) - \varphi(J_\rho^{F,\varphi}(y), J_\rho^{F,\varphi}(y))) \\ &\quad + \langle J_\rho^{F,\varphi}(y) - y, \eta(J_\rho^{F,\varphi}(x), J_\rho^{F,\varphi}(y))\rangle \geq 0,\ \forall\, x, y \in B. \end{aligned}$$

Note that φ is skew symmetric, by adding the above two inequalities, we have

$$\begin{aligned} &\rho(F(J_\rho^{F,\varphi}(x), J_\rho^{F,\varphi}(y)) + F(J_\rho^{F,\varphi}(y), J_\rho^{F,\varphi}(x))) \\ &\quad + \langle J_\rho^{F,\varphi}(x) - x, \eta(J_\rho^{F,\varphi}(y), J_\rho^{F,\varphi}(x))\rangle \end{aligned}$$

$$+\langle J_\rho^{F,\varphi}(y) - y, \eta(J_\rho^{F,\varphi}(x), J_\rho^{F,\varphi}(y))\rangle$$
$$\geq \rho[\varphi(J_\rho^{F,\varphi}(x), J_\rho^{F,\varphi}(x)) - \varphi(J_\rho^{F,\varphi}(y), J_\rho^{F,\varphi}(x))$$
$$-\varphi(J_\rho^{F,\varphi}(x), J_\rho^{F,\varphi}(y)) + \varphi(J_\rho^{F,\varphi}(y), J_\rho^{F,\varphi}(y))] \geq 0.$$

By using α-strongly monotonicity of F and $\eta(x,y)+\eta(y,x)=0,\ \forall\ x,y \in B$, we have

$$-\rho\alpha\|J_\rho^{F,\varphi}(x) - J_\rho^{F,\varphi}(y)\|^2 + \langle x-y, \eta(J_\rho^{F,\varphi}(x), J_\rho^{F,\varphi}(y))\rangle$$
$$\geq \langle J_\rho^{F,\varphi}(x) - J_\rho^{F,\varphi}(y), \eta(J_\rho^{F,\varphi}(x), J_\rho^{F,\varphi}(y))\rangle.$$

Since η is α-strongly monotone and τ-Lipschitz continuous, it follows from the above inequality that

$$-\rho\alpha\|J_\rho^{F,\varphi}(x)-J_\rho^{F,\varphi}(y)\|^2+\tau\|x-y\|\|J_\rho^{F,\varphi}(x)-J_\rho^{F,\varphi}(y)\| \geq \delta\|J_\rho^{F,\varphi}(x)-J_\rho^{F,\varphi}(y)\|^2.$$

Hence, we have

$$\|J_\rho^{F,\varphi}(x) - J_\rho^{F,\varphi}(y)\| \leq \frac{\tau}{\delta+\rho\alpha}\|x-y\|\ \forall x,y \in B.$$

Remark 2.5. Theorem 2.1 generalizes Theorem 3.1 of Kazmi and Khan [23] in following way: (1) from Hilbert spaces to reflexive Banach spaces; (2) the MEP (2.1) is more general than the model of MEP (3.2) in Definition 3.1 of Kazmi and Khan [23].

3. System of Generalized Mixed Implicit Equilibrium Problems

For each $i \in \{1,2\}$, let B_i be a real reflexive Banach space with the dual space B_i^* and $\langle\cdot,\cdot\rangle_i$ be the dual pair between B_i and B_i^*. We denote the norm of B_i and B_i^* by $\|\cdot\|_i$. Let K_i be a nonempty closed convex subset of B_i, and $CB(B_i)$ denote the family of all nonempty bounded closed subsets of B_i. For each $i \in \{1,2\}$, Let $\widetilde{H}_i(\cdot,\cdot)$ be the Hausdorff metric on $CB(B_i)$ defined by

$$\widetilde{H}_i(A,B) = \max\{\sup_{a\in A} d_i(a,B); \sup_{b\in B} d_i(A,b)\}, \forall\ A,B \in CB(B_i),$$

where $d_i(a,B) = \inf_{b\in B}\|a-b\|_i$ and $d_i(A,b) = \inf_{a\in A}\|a-b\|_i$.

For each $i \in \{1,2\}$, let $\varphi_i : B_i \times B_i \to R$ and $F_i : K_i \times K_i \to R$ be two bifunctions, $g_i : K_i \to K_i$, $\eta_i : B_i \times B_i \to B_i^*$ and $N_i,\ G_i : B_1 \times B_2 \to B_i$ be single-valued mappings, and $T_i : B_1 \to CB(B_1)$ and $S_i : B_2 \to CB(B_2)$ be set-valued mappings. We consider the following system of generalized mixed implicit equilibrium problems (SGMIEP): find $(x_1, x_2) \in K_1 \times K_2$, $u_1 \in T_1(x_1)$, $u_2 \in T_2(x_2)$, $v_1 \in S_1(x_2)$ and $v_2 \in S_2(x_2)$ such that

$$
(3.1) \qquad \begin{cases} F_1(g_1(x_1), y_1) + \varphi_1(y_1, g_1(x_1)) - \varphi_1(g_1(x_1), g_1(x_1)) \\ \quad +\langle G_1(x_1, x_2) + N_1(u_1, v_1), \eta_1(y_1, g_1(x_1))\rangle \geq 0, \ \forall \ y_1 \in K_1, \\ F_2(g_2(x_2), y_2) + \varphi_2(y_2, g_2(x_2)) - \varphi_1(g_2(x_2), g_2(x_2)) \\ \quad +\langle G_2(x_1, x_2) + N_2(u_2, v_2), \eta_2(y_2, g_2(x_2))\rangle \geq 0, \ \forall \ y_2 \in K_2. \end{cases}
$$

Special cases:

(I) If for $i = 1, 2$, let $G_i \equiv 0$, then the SGMIEP (3.1) reduces to the following system of generalized mixed implicit equilibrium problems problems: find $(x_1, x_2) \in K_1 \times K_2$, $u_1 \in T_1(x_1)$, $u_2 \in T_2(x_2)$, $v_1 \in S_1(x_2)$ and $v_2 \in S_2(x_2)$ such that

$$
(3.2) \qquad \begin{cases} F_1(g_1(x_1), y_1) + \varphi_1(y_1, g_1(x_1)) - \varphi_1(g_1(x_1), g_1(x_1)) \\ \quad +\langle N_1(u_1, v_1), \eta_1(y_1, g_1(x_1))\rangle \geq 0, \ \forall \ y_1 \in K_1, \\ F_2(g_2(x_2), y_2) + \varphi_2(y_2, g_2(x_2)) - \varphi_1(g_2(x_2), g_2(x_2)) \\ \quad +\langle N_2(u_2, v_2), \eta_2(y_2, g_2(x_2))\rangle \geq 0, \ \forall \ y_2 \in K_2. \end{cases}
$$

The SGMIEP (3.2) is new and includes many known models of the system of generalized mixed equilibrium problems and the system of generalized mixed variational-like inequality problems as special cases.

(II) If for $i = 1, 2$, let $F_i \equiv 0$ and $G_i \equiv 0$, then the SGMIEP (3.1) reduces to the system of generalized mixed variational-like inequality problems (SGMVLIP): Find $(x_1, x_2) \in K_1 \times K_2$, $u_1 \in T_1(x_1)$, $u_2 \in T_2(x_2)$, $v_1 \in S_1(x_2)$ and $v_2 \in S_2(x_2)$ such that

$$
(3.3) \qquad \begin{cases} \langle N_1(u_1, v_1), \eta_1(y_1, g_1(x_1))\rangle + \varphi_1(y_1, g_1(x_1)) \\ \quad -\varphi_1(g_1(x_1), g_1(x_1)) \geq 0, \ \forall \ y_1 \in K_1, \\ \langle N_2(u_2, v_2), \eta_2(y_2, g_2(x_2))\rangle + \varphi_2(y_2, g_2(x_2)) \\ \quad -\varphi_1(g_2(x_2), g_2(x_2)) \geq 0, \ \forall \ y_2 \in K_2. \end{cases}
$$

The SGMVLIP (3.3) is also new and different from that in the known literature. If for $i = 1, 2$, $B_i = H_i$ is Hilbert space, then the SGMVLIP (3.3) includes many known models of the system of generalized mixed variational-like inequality problems in known literature as special cases.

(III) If for $i = 1, 2$, let $B_i = B$, $K_i = K$ $G_i = 0$, $F_i = F$, $\varphi_i = \varphi$, $N_i = N$, $T_i = T$, $S_i = S$, $K_i = K$, $\eta_i = \eta$ and $g_i = g$ then the SGMIEP (3.1) reduce to the generalized mixed implicit equilibrium problem (GMIEP): Find $x \in K$, $u \in T(x)$ and $v \in S(x)$ such that

(3.4) $$F(g(x), y)+\varphi(y, g(x))-\varphi(g(x), g(x))+\langle N(u, v), \eta(y, g(x))\rangle \geq 0, \ \forall \ y\in K.$$

The GMIEP (3.4) with $B = H$ being a Hilbert space and $\varphi \equiv 0$ is called the generalized mixed equilibrium problem (GMEP) by Kazmi and Khan [23].

(IV) If $F \equiv 0$, then the GMIEP (3.4) reduces to the generalized mixed variational-like problems (GMVLIP): Find $x \in K$, $u \in T(x)$ and $v \in S(x)$ such that

(3.5) $$\langle N(u, v), \eta(y, g(x))\rangle + \varphi(y, g(x)) - \varphi(g(x), g(x)) \geq 0, \ \forall \ y \in K,$$

Some similar problems have introduced and studied by many author in Hilbert spaces and Banach spaces, see [25-27].

(V) If for $N \equiv 0$, then the GMIEP (3.4) reduces to the following general mixed equilibrium problem (GMEP): Find $x \in K$ such that

(3.6) $$F(g(x), y) + \varphi(y, g(x)) - \varphi(g(x), g(x)) \geq 0, \ \forall \ y \in K.$$

The GMEP (3.6) is also new.

(VI) If let $\varphi(x, y) = f(x)$ for all $(x, y) \in K \times K$, then the GMEP (3.7) reduces the following mixed equilibrium problem MEP: Find $x \in K$ such that

(3.7) $$F(g(x), y) + f(y) - f(g(x)) \geq 0, \ \forall \ y \in K.$$

The MEP (3.7) have been introduced and studied in Hilbert spaces or Banach spaces by many authors in [3, 13, 14, 22, 23].

Now, related to SGMIEP (3.1), we consider the following system of generalized equation problems (SGEP): Find $(x_1, x_2) \in K_1 \times K_2$, $u_1 \in T_1(x_1)$, $v_1 \in S_1(x_2)$, $u_2 \in T_2(x_1)$ and $v_2 \in S_2(x_2)$ such that

(3.8) $$\begin{cases} g_1(x_1) = J_{\rho_1}^{F_1,\varphi_1}(g_1(x_1) - \rho_1(G_1(x_1, x_2) + N_1(u_1, v_1))), \\ g_2(x_2) = J_{\rho_2}^{F_2,\varphi_2}(g_2(x_2) - \rho_2(G_2(x_1, x_2) + N_2(u_2, v_2))), \end{cases}$$

where, for each $i = 1, 2$, $J_{\rho_i}^{F_i,\varphi_i} : B_i \to K_i$ is the single-valued mapping defined in Remark 2.4.

Related to the GMEP (3.6), we consider the following generalized equation problem (GEP): Find $x \in K$ such that

$$\left\{ \begin{array}{l} g(x) = J_{\rho}^{F,\varphi}(g(x)), \end{array} \right. \tag{3.9}$$

where, $J_{\rho}^{F,\varphi} : B \to K$ is the single-valued mapping defined in Remark 2.4.

Lemma 3.1. $(x_1, x_2, u_1, v_1, u_2, v_2)$ with $(x_1, x_2) \in K_1 \times K_2$, $u_1 \in T_1(x_1)$, $v_1 \in S_1(x_2)$, $u_2 \in T_2(x_1)$ and $v_2 \in S_2(x_2)$ is a solution of the SGEP (3.8) if and only if $(x_1, x_2, u_1, v_1, u_2, v_2)$ with $(x_1, x_2) \in K_1 \times K_2$, $u_1 \in T_1(x_1)$, $v_1 \in S_1(x_2)$, $u_2 \in T_2(x_1)$ and $v_2 \in S_2(x_2)$ is a solution of the SGMIEP (3.1).

Proof. If $(x_1, x_2, u_1, v_1, u_2, v_2)$ with $(x_1, x_2) \in K_1 \times K_2$, $u_1 \in T_1(x_1)$, $v_1 \in S_1(x_2)$, $u_2 \in T_2(x_1)$ and $v_2 \in S_2(x_2)$ is a solution of the SGEP (3.8), then we have

$$\left\{ \begin{array}{l} g_1(x_1) = J_{\rho_1}^{F_1,\varphi_1}(g_1(x_1) - \rho_1(G_1(x_1, x_2) + N_1(u_1, v_1))), \\ g_2(x_2) = J_{\rho_2}^{F_2,\varphi_2}(g_2(x_2) - \rho_2(G_2(x_1, x_2) + N_2(u_2, v_2))). \end{array} \right.$$

Let $z_1 = g_1(x_1) - \rho_1(G_1(x_1, x_2) + N_1(u_1, v_1))$ and $z_2 = g_2(x_2) - \rho_2(G_2(x_1, x_2) + N_2(u_2, v_2))$, then $g_1(x_1) = J_{\rho_1}^{F_1,\varphi_1}(z_1)$ and $g_2(x_2) = J_{\rho_2}^{F_2,\varphi_2}(z_2)$. By the definition of $J_{\rho_1}^{F_1,\varphi_1}$ and $J_{\rho_2}^{F_2,\varphi_2}$ in Remark 2.4, we obtain

$$\left\{ \begin{array}{l} \rho_1(F_1(J_{\rho_1}^{F_1,\varphi_1}(z_1), y_1) \\ \quad +\varphi_1(y_1, J_{\rho_1}^{F_1,\varphi_1}(z_1)) - \varphi_1(J_{\rho_1}^{F_1,\varphi_1}(z_1), J_{\rho_1}^{F_1,\varphi_1}(z_1))) \\ \quad +\langle J_{\rho_1}^{F_1,\varphi_1}(z_1) - z_1, \eta(y_1, J_{\rho_1}^{F_1,\varphi_1}(z_1))\rangle \geq 0, \ \forall\, y_1 \in K_1, \\ \rho_2(F_2(J_{\rho_2}^{F_2}(z_2), y_2) \quad + \varphi_2(y_2, J_{\rho_2}^{F_2,\varphi_2}(z_2)) \\ \quad -\varphi_2(J_{\rho_2}^{F_2,\varphi_2}(z_2), J_{\rho_2}^{F_2,\varphi_2}(z_2))) \\ \quad +\langle J_{\rho_2}^{F_2}(z_2) - z_2, \eta(y_2, J_{\rho_2}^{F_2}(z_2))\rangle \geq 0, \ \forall\, y_2 \in K_2, . \end{array} \right. \tag{3.10}$$

It follows from $z_1 = g_1(x_1) - \rho_1(G_1(x_1, x_2) + N_1(u_1, v_1))$, $z_2 = g_2(x_2) - \rho_2(G_2(x_1, x_2) + N_2(u_2, v_2))$, $g_1(x_1) = J_{\rho_1}^{F_1,\varphi_1}(z_1)$ and $g_2(x_2) = J_{\rho_2}^{F_2,\varphi_2}(z_2)$ that

$$\left\{ \begin{array}{l} F_1(g_1(x_1), y_1) + \varphi_1(y_1, g_1(x_1)) - \varphi_1(g_1(x_1), g_1(x_1)) \\ \quad +\langle G_1(x_1, x_2) + N_1(u_1, v_1), \eta(y_1, g_1(x_1))\rangle \geq 0, \ \forall\, y_1 \in K_1, \\ F_2(g_2(x_2), y_2) + \varphi_2(y_2, g_2(x_2)) - \varphi_2(g_2(x_2), g_2(x_2)) \\ \quad +\langle G_2(x_1, x_2) + N_2(u_2, v_2), \eta(y_2, g_2(x_2))\rangle \geq 0, \ \forall\, y_2 \in K_2, . \end{array} \right. \tag{3.11}$$

Hence $(x_1, x_2, u_1, v_1, u_2, v_2)$ with $(x_1, x_2) \in K_1 \times K_2$, $u_1 \in T_1(x_1)$, $v_1 \in S_1(x_2)$, $u_2 \in T_2(x_1)$ and $v_2 \in S_2(x_2)$ is a solution of the SGMIEP (3.1).

Conversely, if $(x_1, x_2, u_1, v_1, u_2, v_2)$ with $(x_1, x_2) \in K_1 \times K_2$, $u_1 \in T_1(x_1)$, $v_1 \in S_1(x_2)$, $u_2 \in T_2(x_1)$ and $v_2 \in S_2(x_2)$ is a solution of the SGMIEP (3.1), then the system of inequalities (3.11) holds. For $\rho_1, \rho_2 > 0$, it follows from (3.11) that

$$\begin{cases} \rho_1(F_1(g_1(x_1), y_1) + \varphi_1(y_1, g_1(x_1)) - \varphi_1(g_1(x_1), g_1(x_1))) \\ \quad +\langle g_1(x_1) - (g_1(x_1) - \rho_1(G_1(x_1, x_2) \\ \quad +N_1(u_1, v_1))), \eta(y_1, g_1(x_1))\rangle \geq 0, \ \forall\ y_1 \in K_1, \\ \rho_2(F_2(g_2(x_2), y_2) + \varphi_2(y_2, g_2(x_2)) - \varphi_2(g_2(x_2), g_2(x_2))) \\ \quad +\langle g_2(x_2) - (g_2(x_2) - \rho_2(G_2(x_1, x_2) \\ \quad +N_2(u_2, v_2))), \eta(y_2, g_2(x_2))\rangle \geq 0, \ \forall\ y_2 \in K_2, . \end{cases} \tag{3.12}$$

Let $z_1 = g_1(x_1) - \rho_1(G_1(x_1, x_2) + N_1(u_1, v_1)) \in B_1$ and $z_2 = g_2(x_2) - \rho_2(G_2(x_1, x_2) + N_2(u_2, v_2)) \in B_2$, then we have

$$\begin{cases} \rho_1(F_1((g_1(x_1), y_1) + \varphi_1(y_1, g_1(x_1)) - \varphi_1(g_1(x_1), g_1(x_1))) \\ \quad +\langle g_1(x_1) - z_1, \eta(y_1, g_1(x_1))\rangle \geq 0, \ \forall\ y_1 \in K_1, \\ \rho_2(F_2(g_2(x_2), y_2) + \varphi_2(y_2, g_2(x_2)) - \varphi_2(g_2(x_2), g_2(x_2))) \\ \quad +\langle g_2(x_2) - z_2, \eta(y_2, g_2(x_2)))\rangle \geq 0, \ \forall\ y_2 \in K_2, \end{cases} \tag{3.13}$$

But, by Lemma 2.2 and Remark 2.4, we have

$$\begin{cases} \rho_1(F_1(J_{\rho_1}^{F_1,\varphi_1}(z_1), y_1) \\ \quad +\varphi_1(y_1, J_{\rho_1}^{F_1,\varphi_1}(z_1)) - \varphi_1(J_{\rho_1}^{F_1,\varphi_1}(z_1), J_{\rho_1}^{F_1,\varphi_1}(z_1))) \\ \quad +\langle \eta(y_1, J_{\rho_1}^{F_1,\varphi_1}(z_1)), J_{\rho_1}^{F_1,\varphi_1}(z_1) - z_1\rangle \geq 0, \ \forall\ y_1 \in K_1, \\ \rho_2(F_2(J_{\rho_2}^{F_2,\varphi_2}(z_2), y_2) \\ \quad +\varphi_2(y_2, J_{\rho_2}^{F_2,\varphi_2}(z_2)) - \varphi_2(J_{\rho_2}^{F_2,\varphi_2}(z_2), J_{\rho_2}^{F_2,\varphi_2}(z_2))) \\ \quad +\langle \eta(y_2, J_{\rho_2}^{F_2,\varphi_2}(z_2)), J_{\rho_2}^{F_2,\varphi_2}(z_2) - z_2\rangle \geq 0, \ \forall\ y_2 \in K_2 \end{cases} \tag{3.14}$$

By Remark 2.4, the solution of the SGMIEP (3.14) is unique. Hence we must have $g_1(x_1) = J_{\rho_1}^{F_1,\varphi_1}(z_1)$ and $g_2(x_2) = J_{\rho_2}^{F_2,\varphi_2}(z_2)$. It follows that

$$\begin{cases} g_1(x_1) = J_{\rho_1}^{F_1,\varphi_1}(g_1(x_1) - \rho_1(G_1(x_1, x_2) + N_1(u_1, v_1))), \\ g_2(x_2) = J_{\rho_2}^{F_2,\varphi_2}(g_2(x_2) - \rho_2(G_2(x_1, x_2) + N_2(u_2, v_2))), \end{cases}$$

i.e., $(x_1, x_2, u_1, v_1, u_2, v_2)$ with $(x_1, x_2) \in K_1 \times K_2$, $u_1 \in T_1(x_1)$, $v_1 \in S_1(x_2)$, $u_2 \in T_2(x_1)$ and $v_2 \in S_2(x_2)$ is a solution of the SGEP (3.8). This completes the proof.

Remark 3.1. Lemma 3.1 generalizes Lemma 2.3 of Huang et al. [22] and Lemma 3.1 of Kazmi and Khan [23] in the following ways: (1) from Hilbert spaces to Reflexive Banach spaces; (2) from a generalized mixed equilibrium problem to the more general system of generalized mixed implicit equilibrium problems.

By using similar argument as in the proof of Lemma 3.1, we can prove the following.

Lemma 3.2. $x \in K$ is a solution of the GEP (3.9) if and only if x is a solution of the GMEP (3.6).

By using Lemma 3.1 and Nadler's theorem [32], we can construct the following iterative algorithm for solving the SGMIEP (3.1).

Algorithm 3.1. Suppose that for $i = 1, 2$, $g_i(K_i) = K_i$. For given $(x_1^0, x_2^0) \in K_1 \times K_2$ and constants $\rho_1, \rho_2 > 0$, take $u_1^0 \in T_1(x_1^0)$, $v_1^0 \in S_1(x_2^0)$, $u_2^0 \in T_2(x_1^0)$, $v_2^0 \in S_2(x_2^0)$. Then $g_1(x_1^0) - \rho_1(G_1(x_1^0, x_2^0) + N_1(u_1^0, v_1^0)) \in B_1$ and $g_2(x_2^0) - \rho_2(G_2(x_1^0, x_2^0) + N_2(u_2^0, v_2^0)) \in B_2$. By Lemma 2.2, we have $J_{\rho_1}^{F_1,\varphi_1}(g_1(x_1^0) - \rho_1(G_1(x_1^0, x_2^0)+N_1(u_1^0, v_1^0))) \in K_1$ and $J_{\rho_2}^{F_2,\varphi_2}(g_2(x_2^0)-\rho_2(G_2(x_1^0, x_2^0)+N_2(u_2^0, v_2^0))) \in K_2$. Note that $g_i(K_i) = K_i,\ i = 1, 2$, there exist $x_1^1 \in K_1$ and $x_2^1 \in K_2$ such that

$$g_1(x_1^1) = J_{\rho_1}^{F_1,\varphi_1}(g_1(x_1^0) - \rho_1(G_1(x_1^0, x_2^0) + N_1(u_1^0, v_1^0))),$$

$$g_2(x_2^1) = J_{\rho_2}^{F_2,\varphi_2}(g_2(x_2^0) - \rho_2(G_2(x_1^0, x_2^0) + N_2(u_2^0, v_2^0))).$$

By Nadler's theorem [32], there exist $u_1^1 \in T_1(x_1^1)$, $v_1^1 \in S_1(x_2^1)$, $u_2^1 \in T_2(x_1^1)$, $v_2^1 \in S_2(x_2^1)$ such that

$$(3.15)\qquad \begin{cases} \|u_1^1 - u_1^0\|_1 \le (1+1)\tilde{H}_1(T_1(x_1^1), T_1(x_1^0)), \\ \|v_1^1 - v_1^0\|_2 \le (1+1)\tilde{H}_2(S_1(x_2^1), S_1(x_2^0)), \\ \|u_2^1 - u_2^0\|_1 \le (1+1)\tilde{H}_1(T_2(x_1^1), T_2(x_1^0)), \\ \|v_2^1 - v_2^0\|_2 \le (1+1)\tilde{H}_2(S_2(x_2^1), S_2(x_2^0)), \end{cases}$$

By induction, we can define the iterative sequences $\{x_1^n\}, \{x_2^n\}, \{u_1^n\}, \{v_1^n\}, \{u_2^n\}, \{v_2^n\}$ as follows:

$$
(3.16) \qquad \begin{cases}
g_1(x_1^{n+1}) = J_{\rho_1}^{F_1,\varphi_1}(g_1(x_1^n) - \rho_1(G_1(x_1^n, x_2^n) + N_1(u_1^n, v_1^n))), \\
g_2(x_2^{n+1}) = J_{\rho_2}^{F_2,\varphi_2}(g_2(x_2^n) - \rho_2(G_2(x_1^n, x_2^n) + N_2(u_2^n, v_2^n))), \\
u_1^n \in T_1(x_1^n), \\
\|u_1^{n+1} - u_1^n\|_1 \le (1 + (n+1)^{-1})\tilde{H}_1(T_1(x_1^{n+1}), T_1(x_1^n)), \\
\quad v_1^n \in B_1(x_2^n), \\
\|v_1^{n+1} - v_1^n\|_2 \le (1 + (n+1)^{-1})\tilde{H}_2(S_1(x_2^{n+1}), S_1(x_2^n)), \\
\quad u_2^n \in T_2(x_1^n),\ \|_1 \\
\|u_2^{n+1} - u_2^n \le (1 + (n+1)^{-1})\tilde{H}_1(T_2(x_1^{n+1}), T_2(x_1^n)), \\
\quad v_2^n \in B_2(x_2^n),\ \|_2 \\
\|v_2^{n+1} - v_2^n \le (1 + (n+1)^{-1})\tilde{H}_2(S_2(x_2^{n+1}), S_2(x_2^n)),\ n \ge 0.
\end{cases}
$$

By using Lemma 3.2 and Nadler's theorem [32], we can construct the following iterative algorithm for solving the GMEP (3.6).

Algorithm 3.2. Suppose that $g(K) = K$. For given $x_0 \in K$, we have $g(x_0) \in K \subseteq B$. By Lemma 2.2 and Remark 2.4, there exist $J_\rho^{F,\varphi}(g(x_0)) \in K$ and hence there exists $x_1 \in K$ such that

$$g(x_1) = J_\rho^{F,\varphi}(g(x_0)).$$

By induction, we can define the iterative sequences $\{x_n\}$ as follows:

$$(3.17) \qquad g(x_{n+1}) = J_\rho^{F,\varphi}(g(x_n)),\ \forall\ n \ge 0.$$

Definition 3.1. For $i = 1, 2$, $N_i : B_1 \times B_2 \to B_i$ is said to be $(\sigma_i, \varepsilon_i)$-mixed Lipschitz continuous, if there exist constants $\sigma_i, \varepsilon_i > 0$ such that

$$\|N_i(u_1, v_1) - N_i(u_2, v_2)\|_i \le \sigma_i\|u_1 - u_2\|_1 + \varepsilon_i\|v_1 - v_2\|_2,$$
$$\forall\ (u_1, v_1), (u_2, v_2) \in B_1 \times B_2;$$

Definition 3.2. For $i \in \{1, 2\}$, $T_i : B_1 \to CB(B_1)$ is said to be $\widetilde{H}_1$-μ_i-Lipschitz continuous, if there exists a constant $\mu_i > 0$ such that

$$\widetilde{H}_1(T_i(x_1), T_i(\bar{x}_1)) \le \mu_i\|x_1 - \bar{x}_1\|_1,\ \forall\ x_1, \bar{x}_1 \in B_1.$$

Similarly, we can define the Lipschitz continuity of the mappings S_i.

Definition 3.3. For $i \in \{1, 2\}$, a mapping $g_i : K_i \to K_i$ is said to be γ_i-strongly accretive if, for any $x, y \in K_i$, there exist $j_i(x - y) \in J_i(x - y)$ and a constant $\gamma_i > 0$ such that

$$\langle g_i(x) - g_i(y), j_i(x-y)\rangle_i \geq \gamma_i \|x-y\|_i^2,$$

where $J_i : B_i \to 2^{B_i^*}$ is the normalized duality mapping defined by

$$J_i(x) = \{f \in B_i^* : \langle x, f\rangle_i = \|f\|_i \cdot \|x\|_i, \|f\|_i = \|x\|_i\},\ \forall\, x \in B_i.$$

Theorem 3.1. *For each $i \in \{1,2\}$, let $F_i : K_i \times K_i \to R$, $\varphi_i : B_i \times B_i \to R$, and $\eta_i : B_i \times B_i \to B_i$ satisfy the conditions (i)-(v) of Theorem 2.1 where F_i is α_i-strongly monotone, and η_i is δ_i-strongly monotone and τ_i-Lipschitz continuous. Let $G_i : B_1 \times B_2 \to B_i$ be (m_i, n_i)-mixed Lipschitz continuous and $N_i : B_1 \times B_2 \to B_i$ be $(\sigma_i, \varepsilon_i)$-mixed Lipschitz continuous, $T_i : B_1 \to CB(B_1)$ be $\tilde{H}_1$-μ_i-Lipschitz continuous, $S_i : B_2 \to CB(B_2)$ be $\tilde{H}_2$-s_i-Lipschitz continuous and $g_i : K_i \to K_i$ satisfy $g_i(K_i) = K_i$ and be γ_i-strongly accretive and β_i-lipschitz continuous. If the following conditions hold for $\rho_1, \rho_2 > 0$:*

$$(3.18)\qquad \begin{cases} \dfrac{\tau_1}{\gamma_1(\delta_1+\rho_1\alpha_1)}(\beta_1+\rho_1(m_1+\sigma_1\mu_1)) + \dfrac{\tau_2\rho_2}{\gamma_2(\delta_2+\rho_2\alpha_2)}(n_1+\sigma_2\mu_2) < 1, \\[2ex] \dfrac{\tau_1\rho_1}{\gamma_1(\delta_1+\rho_1\alpha_1)}(m_2+\varepsilon_1 s_1) + \dfrac{\tau_2}{\gamma_2(\delta_2+\rho_2\alpha_2)}(\beta_2+\rho_2(n_2+\varepsilon_2 s_2)) < 1. \end{cases}$$

Then the sequences $\{x_1^n\}$, $\{x_2^n\}$, $\{u_1^n\}$, $\{v_1^n\}$, $\{u_2^n\}$ and $\{v_2^n\}$ generated by Algorithm 3.1 strongly converge to $x_1 \in K_1, x_2 \in K_2$, $u_1 \in T_1(x_1)$, $v_1 \in S_1(x_2)$, $u_2 \in T_2(x_1)$ and $v_2 \in S_2(x_2)$ respectively, and $(x_1, x_2, u_1, v_1, u_2, v_2)$ is a solution of SGMIEP (3.1).

Proof. For $i = 1, 2$, since g_i is γ_i-strongly accretive, there exists $j_i(x_i^{n+1} - x_i^n) \in J_i(x_i^{n+1} - x_i^n))$ such that

$$\begin{aligned} &\|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i \|x_i^{n+1} - x_i^n\|_i \\ &\geq \langle g_i(x_i^{n+1}) - g_i(x_i^n), j_i(x_i^{n+1} - x_i^n)\rangle_i \geq \gamma_i \|x_i^{n+1} - x_i^n\|_i^2. \end{aligned}$$

This implies that

$$(3.19)\qquad \|x_i^{n+1} - x_i^n\|_i \leq \frac{1}{\gamma_i}\|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i.$$

It follows from Algorithm 3.1 and Theorem 2.1 that

$$
\begin{aligned}
&\|x_1^{n+2}-x_1^{n+1}\|_1\\
&\le \frac{1}{\gamma_1}\|g_1(x_1^{n+2})-g_1(x_1^{n+1})\|_1\\
&= \frac{1}{\gamma_1}\|J_{\rho_1}^{F_1,\varphi_1}(g_1(x_1^{n+1})-\rho_1(G_1(x_1^{n+1},x_2^{n+1})+(N_1(u_1^{n+1},v_1^{n+1})))\\
&\quad -[J_{\rho_1}^{F_1,\varphi_1}(g_1(x_1^{n})-\rho_1(G_1(x_1^{n},x_2^{n})+N_1(u_1^{n},v_1^{n})))]\|_1\\
&\le \frac{\tau_1}{\gamma_1(\delta_1+\rho_1\alpha_1)}\|g_1(x_1^{n+1})-g_1(x_1^{n})-\rho_1(G_1(x_1^{n+1},x_2^{n+1})-G_1(x_1^{n},x_2^{n}))\\
&\quad -\rho_1(N_1(u_1^{n+1},v_1^{n+1}))-N_1(u_1^{n},v_1^{n})))\|_1\\
&\le \frac{\tau_1}{\gamma_1(\delta_1+\rho_1\alpha_1)}[\|g_1(x_1^{n+1})-g_1(x_1^{n})\|_1+\rho_1\|G_1(x_1^{n+1},x_2^{n+1})-G_1(x_1^{n},x_2^{n})\|_1\\
&\quad +\rho_1\|N_1(u_1^{n+1},v_1^{n+1})-N_1(u_1^{n},v_1^{n})\|_1].
\end{aligned}
\tag{3.20}
$$

Since g_1 is β_1-Lipschitz continuous, we have

$$
\|g_1(x_1^{n+1})-g_1(x_1^{n})\|_1\le\beta_1\|x_1^{n+1}-x_1^{n}\|_1. \tag{3.21}
$$

Since G_1 is (m_1,n_1)-mixed Lipschitz continuous, we have

$$
\|G_1(x_1^{n+1},x_2^{n+1})-G_1(x_1^{n},x_2^{n})\|_1\le m_1\|x_1^{n+1}-x_1^{n}\|_1+n_1\|x_2^{n+1}-x_2^{n}\|_2. \tag{3.22}
$$

Since N_1 is (σ_1,ε_1)-mixed Lipschitz continuous, T_1 is $\tilde{H}_1$-μ_1-Lipschitz continuous and S_1 is $\tilde{H}_2$-s_1-Lipschitz continuous, by Algorithm 3.1, we have

$$
\begin{aligned}
&\|N_1(u_1^{n+1},v_1^{n+1})-N_1(u_1^{n},v_1^{n})\|_1\le\sigma_1\|u_1^{n+1}-u_1^{n}\|_1+\varepsilon_1\|v_1^{n+1}-v_1^{n}\|_2\\
&\le(1+\frac{1}{n+1})(\sigma_1\tilde{H}_1(T_1(x_1^{n+1}),T_1(x_1^{n}))+\varepsilon_1\tilde{H}_2(S_1(x_2^{n+1}),S_1(x_2^{n})))\\
&\le(1+\frac{1}{n+1})(\sigma_1\mu_1\|x_1^{n+1}-x_1^{n}\|_1+\varepsilon_1 s_1\|x_2^{n+1}-x_2^{n}\|_2).
\end{aligned}
\tag{3.23}
$$

By (3.20))-(3.23), we have

$$
\begin{aligned}
&\|x_1^{n+2}-x_1^{n+1}\|_1\\
&\le \frac{\tau_1}{\gamma_1(\delta_1+\rho_1\alpha_1)}(\beta_1+\rho_1(m_1+\sigma_1\mu_1(1+\frac{1}{n+1})))\|x_1^{n+1}-x_1^{n}\|_1\\
&\quad+\frac{\tau_1\rho_1}{\gamma_1(\delta_1+\rho_1\alpha_1)}(n_1+\varepsilon_1 s_1(1+\frac{1}{n+1}))\|x_2^{n+1}-x_2^{n}\|_2\\
&\le\Gamma_1^n\|x_1^{n+1}-x_1^{n}\|_1+\Theta_1^n\|x_2^{n+1}-x_2^{n}\|_2,
\end{aligned}
\tag{3.24}
$$

where

$$\Gamma_1^n = \frac{\tau_1}{\gamma_1(\delta_1+\rho_1\alpha_1)}(\beta_1+\rho_1(m_1+\sigma_1\mu_1(1+\frac{1}{n+1}))),$$

$$\Theta_1^n = \frac{\tau_1\rho_1}{\gamma_1(\delta_1+\rho_1\alpha_1)}(n_1+\varepsilon_1 s_1(1+\frac{1}{n+1})).$$

Similarly, we can obtain

$$\begin{aligned}
&\|x_2^{n+2}-x_2^{n+1}\|_2 \le \frac{1}{\gamma_2}\|g_2(x_2^{n+2})-g_2(x_2^{n+1})\|_2\\
&\le \frac{1}{\gamma_2}\|J_{\rho_2}^{F_2}(g_2(x_2^{n+1})-\rho_2(G_2(x_1^{n+1},x_2^{n+1})+N_2(u_2^{n+1},v_2^{n+1})))\\
&\quad - J_{\rho_2}^{F_2}(g_2(x_2^n)-\rho_2(G_2(x_1^n,x_2^n)+N_2(u_2^n,v_2^n)))\|_2\\
&\le \frac{\tau_2}{\gamma_2(\delta_2+\rho_2\alpha_2)}\|g_2(x_2^{n+1})-\rho_2(G_2(x_1^{n+1},x_2^{n+1})+N_2(u_2^{n+1},v_2^{n+1}))\\
&\quad -(g_2(x_2^n)-\rho_2(G_2(x_1^n,x_2^n)+N_2(u_2^n,v_2^n)))\|_2\\
&\le \frac{\tau_2}{\gamma_2(\delta_2+\rho_2\alpha_2)}[\|g_2(x_2^{n+1})-g_2(x_2^n)\|_2+\rho_2\|G_2(x_1^{n+1},x_2^{n+1})-G_2(x_1^n,x_2^n)\|\\
&\quad +\rho_2\|N_2(u_2^{n+1},v_2^{n+1})-N_2(u_2^n,v_2^n)\|_2]\\
&\le \frac{\tau_2}{\gamma_2(\delta_2+\rho_2\alpha_2)}[\beta_2\|x_2^{n+1}-x_2^n\|_2+\rho_2(m_2\|x_1^{n+1}-x_1^n\|_1+n_2\|x_2^{n+1}-x_2^n\|_2)\\
&\quad +\rho_2(1+\frac{1}{n+1})(\sigma_2\mu_2\|x_1^{n+1}-x_1^n\|_1+\varepsilon_2 s_2\|x_2^{n+1}-x_2^n\|_2)]\\
&\le \frac{\tau_2\rho_2}{\gamma_2(\delta_2+\rho_2\alpha_2)}(m_2+\sigma_2\mu_2(1+\frac{1}{n+1}))\|x_1^{n+1}-x_1^n\|_1\\
&\quad +\frac{\tau_2}{\gamma_2(\delta_2+\rho_2\alpha_2)}(\beta_2+\rho_2(n_2+\varepsilon_2 s_2(1+\frac{1}{n+1})))\|x_2^{n+1}-x_2^n\|_2\\
&\le \Gamma_2^n\|x_1^{n+1}-x_1^n\|_1+\Theta_2\|x_2^{n+1}-x_2^n\|_2,
\end{aligned} \tag{3.25}$$

where

$$\Gamma_2^n = \frac{\tau_2\rho_2}{\gamma_2(\delta_2+\rho_2\alpha_2)}(m_2+\sigma_2\mu_2(1+\frac{1}{n+1})),$$

$$\Theta_2^n = \frac{\tau_2}{\gamma_2(\delta_2+\rho_2\alpha_2)}(\beta_2+\rho_2(n_2+\varepsilon_2 s_2(1+\frac{1}{n+1}))).$$

It follows from (3.24) and (3.25) that

$$\begin{aligned}
&\|x_1^{n+2}-x_1^{n+1}\|_1+\|x_2^{n+2}-x_2^{n+1}\|_2\\
&\le(\Gamma_1^n+\Gamma_2^n)\|x_1^{n+1}-x_1^n\|_1+(\Theta_1^n+\Theta_2^n)\|x_2^{n+1}-x_2^n\|_2\\
&\le\Delta_n(\|x_1^{n+1}-x_1^n\|_1+\|x_2^{n+1}-x_2^n\|_2),\ \forall\ n=1,2,\cdots,
\end{aligned}\tag{3.26}$$

where $\Delta_n=\max\{\Gamma_1^n+\Gamma_2^n,\Theta_1^n+\Theta_2^n\}$.

Let

$$\Gamma_1=\frac{\tau_1}{\gamma_1(\delta_1+\rho_1\alpha_1)}(\beta_1+\rho_1(m_1+\sigma_1\mu_1)),\ \Gamma_2=\frac{\tau_2\rho_2}{\gamma_2(\delta_2+\rho_2\alpha_2)}(m_2+\sigma_2\mu_2)$$

$$\Theta_1=\frac{\tau_1\rho_1}{\gamma_1(\delta_1+\rho_1\alpha_1)}(n_1+\varepsilon_1 s_1),\ \Theta_2=\frac{\tau_2}{\gamma_2(\delta_2+\rho_2\alpha_2)}(\beta_2+\rho_2(n_2+\varepsilon_2 s_2))$$

$$\text{and } \Delta=\max\{\Gamma_1+\Gamma_2,\Theta_+\Theta_2\}.$$

Then, we have $\Gamma_1^n\to\Gamma_1$, $\Gamma_2^n\to\Gamma_2$, $\Theta_1^n\to\Theta_1$, $\Theta_2^n\to\Theta_2$ and $\Delta_n\to\Delta$ as $n\to\infty$. Now, define a norm $\|\cdot\|_*$ on $B_1\times B_2$ by

$$\|(x_1,x_2)\|_*=\|x_1\|_1+\|x_2\|_2,\ \forall\ (x_1,x_2)\in B_1\times B_2.$$

Then $(B_1\times B_2,\|\cdot\|_*)$ is a Banach space. It follows from (3.26) that

$$\|(x_1^{n+2},x_2^{n+2})-(x_1^{n+1},x_2^{n+1})\|_*\le\Delta_n\|(x_1^{n+1},x_2^{n+1})-(x_1^n,x_2^n)\|_*.\tag{3.27}$$

By the condition (3.18), we know that $0<\Delta<1$. Hence there exist $\Delta_0\in(0,1)$ and $n_0>0$ such that $\Delta_n\le\Delta_0$ for all $n\ge n_0$. Therefore, it follows from (3.27) that

$$\|(x_1^{n+2},x_2^{n+2})-(x_1^{n+1},x_2^{n+1})\|_*\le\Delta_0\|(x_1^{n+1},x_2^{n+1})-(x_1^n,x_2^n)\|_*.\forall n\ge n_0.\tag{3.28}$$

This implies that $\{(x_1^n,x_2^n)\}$ is a Cauchy sequence in $B_1\times B_2$. Thus, there exist $(x_1,x_2)\in B_1\times B_2$ such that $(x_1^n,x_2^n)\to(x_1,x_2)$ as $n\to\infty$. Since $K_1\times K_2$ is closed, we have $(x_1,x_2)\in K_1\times K_2$. By Algorithm 3.1 and the Lipschitz continuity of T_1, T_2, S_1, and S_2, we get

$$\begin{cases}
\|u_1^{n+1}-u_1^n\|_1\le(1+(n+1)^{-1})\tilde{H}_1(T_1(x_1^{n+1}),T_1(x_1^n))\\
\qquad\le\mu_1\|x_1^{n+1}-x_1^n\|_1,\\
\|v_1^{n+1}-v_1^n\|_2\le(1+(n+1)^{-1})\tilde{H}_2(S_1(x_2^{n+1}),S_1(x_2^n))\\
\qquad\le s_1\|x_2^{n+1}-x_2^n\|_2,\\
\|u_2^{n+1}-u_2^n\|_1\le(1+(n+1)^{-1})\tilde{H}_1(T_2(x_1^{n+1}),T_2(x_1^n))\\
\qquad\le\mu_2\|x_1^{n+1}-x_1^n\|_1,\\
\|v_2^{n+1}-u_2^n\|_2\le(1+(n+1)^{-1})\tilde{H}_2(S_2(x_2^{n+1}),S_2(x_2^n))\\
\qquad\le s_2\|x_2^{n+1}-x_2^n\|_2.
\end{cases}\tag{3.29}$$

It follows that $\{u_1^n\}$, $\{v_1^n\}$, $\{u_2^n\}$ and $\{v_2^n\}$ are all Cauchy sequences. Thus, there exist $u_1, u_2 \in B_1$ and $v_1, v_2 \in B_2$ such that $u_1^n \to u_1$, $v_1^n \to v_1$, $u_2^n \to u_2$ and $v_2^n \to v_2$, as $n \to \infty$. Now, we show that $u_1 \in T_1(x_1)$. Noting $u_1^n \in T_1(x_1^n)$, we have

$$\begin{aligned} d(u_1, T_1(x_1)) &\leq \|u_1 - u_1^n\|_1 + d(u_1^n, T(x_1)) \\ &\leq \|u_1 - u_1^n\|_1 + \tilde{H}_1(T_1(x_1^n), T_1(x_1)) \\ &\leq \|u_1 - u_1^n\|_1 + \mu_1\|x_1^n - x_1\|_1 \to 0 \qquad (n \to \infty). \end{aligned}$$

Since $T_1(x_1)$ is closed, it implies $u_1 \in T_1(x_1)$. Similarly, one can show that $v_1 \in B_1(x_2)$, $u_2 \in T_2(x_1)$ and $v_2 \in B_2(x_2)$. By Algorithm 3.1, we have

$$\begin{cases} g(x_1^{n+1}) = J_{\rho_1}^{F_1,\varphi_1}(g(x_1^n) - \rho_1(G_1(x_1^n, x_2^n) + N_1(u_1^n, v_1^n))), \\ g(x_2^{n+1}) = J_{\rho_2}^{F_2,\varphi_2}(g(x_2^n) - \rho_2(G_2(x_1^n, x_2^n) + N_2(u_2^n, v_2^n))), \end{cases}$$

It follows from the continuity of g_i, $J_{\rho_i}^{F_i,\varphi_i}$, G_i, N_i, T_i and S_i that $(x_1, x_2, u_1, v_1, u_2, v_2)$ satisfies the following relation,

$$\begin{cases} g_1(x_1) = J_{\rho_1}^{F_1,\varphi_1}(g_1(x_1) - \rho_1(G_1(x_1, x_2) + N_1(u_1, v_1))), \\ g_2(x_2) = J_{\rho_2,\varphi_2}^{F_2}(g_2(x_2) - \rho_2(G_2(x_1, x_2) + N_2(u_2, v_2))), \end{cases} \tag{3.30}$$

By Lemma 3.1, $(x_1, x_2, u_1, v_1, u_2, v_2)$ is a solution of the SGMIEP (3.1). This completes the proof.

Theorem 3.2. *Let $F : K \times K \to R$, $\varphi : B \times B \to R$, and $\eta : B \times B \to B^*$ satisfy the conditions (i)-(v) of Theorem 2.1 where F is α-strongly monotone, and η is δ-strongly monotone and τ-Lipschitz continuous. Let $g : K \to K$ satisfy $g(K) = K$ and be γ-strongly accretive and β-Lipschitz continuous. If the following conditions hold for $\rho > 0$:*

$$\frac{\tau\beta}{\gamma(\delta + \rho\alpha)} < 1. \tag{3.31}$$

Then the sequences $\{x_n\}$ generated by Algorithm 3.2 strongly converge to $x^* \in K$ and x^* is a solution of GMEP (3.6).

Proof. Since g is γ-strongly accretive, there exists $j(x_{n+1} - x_n) \in J(x_{n+1} - x_n))$ such that

$$\|g(x_{n+1}) - g(x_n)\|\|x_{n+1} - x_n\| \geq \langle g(x_{n+1}) - g(x_n), j(x_{n+1} - x_n)\rangle \geq \gamma\|x_{n+1} - x_n\|^2.$$

This implies that

$$\|x_{n+1} - x_n\| \le \frac{1}{\gamma}\|g(x_{n+1}) - g(x_n)\|. \tag{3.32}$$

It follows from Algorithm 3.2, Theorem 2.1 and β-Lipschitz continuity of g that

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &\le \frac{1}{\gamma}\|g(x_{n+2}) - g(x_{n+1})\| \\ &= \frac{1}{\gamma}\|J_\rho^{F,\varphi}(g(x_{n+1})) - J_\rho^{F,\varphi}(g(x_n))\| \\ &\le \frac{\tau}{\gamma(\delta+\rho\alpha)}\|g(x_{n+1})-g(x_n)\| \\ &\le \frac{\tau\beta}{\gamma(\delta+\rho\alpha)}\|x_{n+1} - x_n\|. \end{aligned} \tag{3.33}$$

By the condition (3.31) and (3.33), $\{x_n\}$ is a Cauchy sequence in K. Note that K is closed, there exists $x^* \in K$ such that $x_n \to x^*$ as $n \to \infty$.

By Algorithm 3.2, we have

$$g(x_{n+1}) = J_\rho^{F,\varphi}(g(x_n)).$$

It follows from the continuity of g and $J_\rho^{F,\varphi}$ that,

$$g(x^*) = J_\rho^{F,\varphi}(g(x^*)).$$

Hence, x^* is a solution of GEP (3.9). By Lemma 3.2, x^* is also a solution of the GMEP (3.6).

Remark 3.2. Theorem 3.2 is a new existence result for mixed equilibrium problem which is different from that in [3, 13, 14, 22, 23].

References

1. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student*, **63** (1994), 123-145.
2. A. Moudafi and M. Théra, Proximal and dynamical approaches to equilibrium problems, in: *Lecture Notes in Economics and Mathematical Systems*, vol. 477, Springer, Berlin, pp. 187-201.
3. A. Moudafi, Mixed equilibrium problems: sensitivity analysis and algorithmic aspects, *Comput. Math. Appl.*, **44** (2002), 1099-1108.

4. F. Giannessi, A. Maugeri and M. Pardalos, *Equilibrium problems: Nonsmooth Optimization and Variational inequality Models*, Kluwer Academic, Dordrecht, 2001.

5. F. Giannessi and A. Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum, New York, 1995.

6. M. A. Noor, Multivalued general equilibrium problems, *J. Math. Anal. Appl.*, **283** (2003), 140-149.

7. M. A. Noor, Auxiliary principle technique for equilibrium problem, *J. Optim. Theory Appl.*, **122** (2004), 371-386.

8. M. A. Noor, Generalized mixed quasi-equilibrium problems with trifunction, *Appl. Math. Lett.*, **18** (2005), 695-700.

9. X. P. Ding, Iterative algorithm of solutions for generalized mixed implicit equilibrium-like problems, *Appl. Math. Comput.*, **162(2)** (2005), 799-809.

10. X. P. Ding, Y. C. Lin and J. C. Yao, Predictor-corrector algorithms for solving generalized mixed implicit quasi-equilibrium problems, *Appl. Math. Mech.*, **27(9)** (2006), 1157-1164.

11. F. Q. Xia and X. P. Ding, Predictor-corrector Algorithms for Solving Generalized Mixed Implicit Quasi-equilibrium problems, *Appl. Cpmput. Math.*, **188(1)** (2007), 173-179.

12. A. Tada and W. Takahashi, Weak and strong convergence for a a nonexpansive mapping and an equilibrium problem, *J. Optim. Theory Appl.*, **133** (2007), 359-370.

13. L. C. Ceng and J. C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, *J. Comput. Appl. Math.*, **214** (2008), 186-201.

14. G. Bigi, M. Castellani and G. Kassay, A dual view of equilibrium problems, *J. Math. Anal. Appl.*, **342** (2008), 17-26.

15. V. Colao, G. Marino and H.-K. Xu, An iterative method for finding common solutions of equilibrium and fixed point problems, *J. Math. Anal. Appl.*, **344** (2008), 340-352.

16. S. Plubtieng and R. Punpaeng, A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings, *Appl. Math. Comput.*, **197** (2008), 548-558.

17. S. S. Chang, H. W. J. Lee and C. K. Chan, A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization, *Nonlinear Anal.*, doi:10.1016/j.na.2008.04.035.

18. L. C. Ceng, S. Al-Homidan, Q. H. Ansari and J. C. Yao, An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings, *J. Comput. Appl. Math.*, **223** (2009), 967-974.

19. X. L. Qin, Y. J. Cho and S. M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, *J. Comput. Appl. Math.*, doi:10.1016/j.cam. 2008.06.011.

20. J. S. Jung, Strong convergence of composite iterative methods for equilibrium problems and fixed point problems, *Appl. Math. Comput.*, doi:10.1016/j.amc.2009.03.048.

21. V. Colao, G. L. Acedo and G. Marino, An implicit method for finding common solutions of variational inequalities and system o f equilibrium problems and fixed points of infinite family of nonexpansive mappings, *Nonlinear Anal.*, doi:10.1016/j.na.2009.01.115.

22. N. J. Huang, H. Y. Lan and Y. J. Cho, Sensitivity analysis for nonlinear generalized mixed implicit equilibrium problems with non-monotone set-valued mappings, *J. Comput. Appl. Math.*, **196** (2006), 608-618.

23. K. R. Kazmi and F. A. Khan, Existence and iterative approximation of solutions of generalized mixed equilibrium problems, *Comput. Math. Appl.*, **56** (2008), 1314-1321.

24. A. S. Antipin, Iterative gradient prediction-type methods for computing fixed-point of extremal mappings, in: *Parametric Optimization and Related Topics IV[C].*, (J. Guddat, H. Th. Jonden, F. Nizicka, G. Still, F. Twitt, eds. by), Peter Lang, Frankfurt Main, 1997, pp. 11-24.

25. S. S. Chang, *Variational Inequalities and Complementary Problems: Theory and Applications*, Shanghai Scientific and Technical Press, Shanghai, 1991, (In Chinese).

26. X. P. Ding, Generalized quasi-variational-like inclusions with nonconvex functions, *Appl. Math. Comput.*, **122** (2001), 267-282.

27. Q. H. Ansari and J. C. Yao, Iterative schemes for solving mixed variational-like inequalities, *J. Optim. Theory Appl.*, **108(3)** (2001), 527-541.

28. X. P. Ding, Existence and algorithm of solutions for mixed variational-like inequalities in Banach spaces, *J. Optim. Theory Appl.*, **127(2)** (2005), 285-302.

29. X. P. Ding and J. C. Yao, Existence and algorithm of solutions for mixed quasi-variational-like inclusions in Banach spaces, *J. Comput. Math. Appl.*, **49(5-6)** (2005), 857-869.

30. X. P. Ding, J. C. Yao and L. C. Zeng, Existence and algorithm of solutions for generalized strongly nonlinear mixed variational-like inequalities in Banach spaces, *J. Comput. Math. Appl.*, **55** (2008), 669-679.

31. L. C. Zeng, S. M. Guu and J. C. Yao, Three-step iterative algorithms for solving the system of generalized mixed quasi-variational-like inclusions, *J. Comput. Math. Appl.*, **53** (2007), 1572-1581.

32. L. C. Zeng, S. Schaible and J. C. Yao, Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities, *J. Optim. Theory Appl.*, **124(3)** (2005), 725-738.

33. L. C. Zeng, S. M. Guu and J. C. Yao, Iterative Algorithm for Completely Generalized Set-Valued Strongly Nonlinear Mixed Variational-like Inequalities, *J. Comput. Math. Appl.*, **50(5-6)** (2005), 935-945.

34. S. B. Nadler, Multivalued contraction mapping, *Pacific J. Math.*, **30** (1969), 475-488.

35. L. C. Zeng and J. C. Yao, Modified Combined Relaxation Method for General Monotone Equilibrium Problems in Hilbert Spaces, *Journal of Optimization Theory and Applications*, **131** (2006), 469-483.

36. L. C. Ceng, S. Schaible and J. C. Yao, Implicit Iteration Scheme with Perturbed Mapping for Equilibrium Problems and Fixed Point Problems of Finitely Many Nonexpansive Mappings, *Journal of Optimization Theory and Applications*, **139** (2008), 403-418.

37. L. C. Ceng and J. C. Yao, Hybrid Viscosity Approximation Schemes for Equilibrium Problems and Fixed Point Problems of Infinitely Many Nonexpansive Mappings, *Applied Mathematics and Computation*, **198** (2008), 729-741.

38. D. R. Sahu, N. C. Wong and J. C. Yao, On convergence analysis of an iterative algorithm for finding common solution of generalized mixed equilibrium problems and fixed point problems, *Mathematical Inequalities & Applications*, **12** (2009), 625-649.

39. J. W. Peng and J. C. Yao, A modified CQ method for equilibrium problems, fixed points and variational inequality, *Fixed Point Theory*, **9** (2008), 515-531.

40. L. C. Ceng, Q. H. Ansari and J. C. Yao, Viscosity Approximation Methods for Generalized Equilibrium Problems and Fixed Point Problems, *Journal of Global Optimization*, **43** (2009), 487-502.

41. L. C. Ceng, A. Petruşel and J. C. Yao, Weak convergence theorem by a modified extragradient method for nonexpansive mappings and monotone mappings, *Fixed Point Theory*, **9** (2008), 73-88.

42. J. W. Peng and J. C. Yao, A new hybrid-extragradient method for generalized mixed equilibrium problems and fixed point problems and variational inequality problems, *Taiwanese Journal of Mathematics*, **12** (2008), 1401-1433.

43. J. W. Peng and J. C. Yao, Some new iterative algorithms for generalized mixed equilibrium problems with strict pseudo-contractions and monotone mappings, *Taiwanese Journal of Mathematics*, **13** (2009), 1537-1582.

44. J. W. Peng, Y. C. Liou and J. C. Yao, An Iterative Algorithm Combining Viscosity Method with Parallel Method for a Generalized Equilibrium Problem and Strict Pseudo-contractions, *Fixed Point Theory and Applications*, 2009, to appear.

45. L. C. Ceng, A. Petruşel and J. C. Yao, Iterative Approaches to Solving Equilibrium Problems and Fixed Point Problems of Infinitely Many Nonexpansive Mappings, *Journal of Optimization Theory and Applications*, **143** (2009), 37-58.

46. J. W. Peng and J. C. Yao, A New Extragradient Method for Mixed Equilibrium Problems, Fixed Point Problems and Variational Inequality Problems, *Mathematical and Computer Modelling*, **49** (2009), 1816-1828.

47. J. W. Peng, Y. Wang, David S. Shyu and J. C. Yao, Common Solutions of An Iterative Scheme for Variational Inclusions, Equilibrium Problems and Fixed Point Problems, *Journal of Inequalities and Applications*, Volume 2008 (2008), Article ID 720371, 15 pages.

48. L. C. Ceng, G. Mastroeni and J. C. Yao, Hybrid Proximal-Point Method for Common Solutions of Equilibrium Problems and Zeros of Maximal Monotone Operators, *Journal of Optimization Theory and Applications*, **142** (2009), 431-449.

49. G. Mastroeni, B. Panicucci, M. Passacantandoand and J. C. Yao, A separation approach to vector quasi-equilibrium problems: Saddle point and gap function, *Taiwanese Journal of Mathematics*, **13** (2009), 567-673.

50. J. W. Peng and J. C. Yao, Two Extragradient Methods for Generalized Mixed Equilibrium Problems, Nonexpansive Mappings and Monotone Mappings, *Computers and Mathematics with Applications*, 2010, to appear.

51. L. C. Ceng and J. C. Yao, A Relaxed Extragradient-like Method for a Generalized Mixed Equilibrium Problem, a General System of Generalized Equilibria and a Fixed Point Problem, *Nonlinear Analysis Series A: Theory, Methods & Applications*, 2010, to appear.

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