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COMPACTNESS FOR COMMUTATORS OF MARCINKIEWICZ INTEGRALS IN MORREY SPACES

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Abstract. In this paper the authors give a characterization of the compactness for the commutator $[b,\mu_\Omega]$ in the Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$, where μ_Ω denotes the Marcinkiewicz integral. More precisely, the authors prove that if $b\in \mathrm{VMO}(\mathbb{R}^n)$, the $\mathrm{BMO}(\mathbb{R}^n)$-closure of $C_c^\infty(\mathbb{R}^n)$, then the commutators $[b,\mu_\Omega]$ is a compact operator in the Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ for $1<p<\infty$ and $0<\lambda<n$. Conversely, if $b\in\mathrm{BMO}(\mathbb{R}^n)$ and $[b,\mu_\Omega]$ is a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$ for some $p\in(1,\infty)$ and $\lambda\in(0,n)$, then $b\in\mathrm{VMO}(\mathbb{R}^n)$. In the above results, the kernel function Ω of the operator μ_Ω is assumed to satisfy a very weak condition on S^{n-1}.

1. Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. In 1958, Stein [14] defined the Marcinkiewicz integral of high dimension. Suppose that Ω satisfies the following conditions:

(a) Ω is homogeneous function of degree zero on $\mathbb{R}^n\backslash\{0\}$, i.e.

$$\Omega(tx)=\Omega(x)\quad \text{for any } t>0 \text{ and } \ x\in\mathbb{R}^n\backslash\{0\}. \tag{1.1}$$

(b) Ω has mean zero on S^{n-1}, i.e.

$$\int_{S^{n-1}}\Omega(x')\,d\sigma(x')=0. \tag{1.2}$$

(c) $\Omega\in\mathrm{Lip}(S^{n-1})$, i.e.

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$$(1.3)\qquad |\Omega(x')-\Omega(y')|\le |x'-y'| \quad \text{for any} \quad x',y'\in S^{n-1}.$$

Then the Marcinkiewicz integral μ_Ω is defined by

$$\mu_\Omega(f)(x)=\left(\int_0^\infty |F_{\Omega,t}(x)|^2\frac{dt}{t^3}\right)^{\frac{1}{2}},$$

where

$$F_{\Omega,t}(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\,f(y)\,dy.$$

The Marcinkiewicz integral μ_Ω is essentially a Littlewood-Paley g-function. In fact, if taking

$$\varphi(x)=\Omega(x)|x|^{-n+1}\chi_{\{|x|\le 1\}}(x)$$

and $\varphi_t(x)=t^{-n}\varphi(\frac{x}{t})$ for $t>0$, then

$$\mu_\Omega f(x)=\left(\int_0^\infty |\varphi_t * f(x)|^2\frac{dt}{t}\right)^{1/2}.$$

In 1958, Stein [14] gave the weak (1,1) boundedness and L^p boundedness of μ_Ω for $1<p\le 2$. In 1961, Benedeck, Calderón and Panzone [1] proved that if replacing the Lipschitz condition (1.3) by $\Omega\in C^1(S^{n-1})$, then μ_Ω is bounded operator in L^p for $1<p<\infty$. In 2000, Ding, Fan and Pan [8] showed further that the smoothness assumed on Ω is not necessary for the L^p $(1<p<\infty)$ boundedness of μ_Ω.

Theorem A. ([8]). *If $\Omega\in H^1(S^{n-1})$ satisfies (1.1) and (1.2), then μ_Ω is of type (p,p) for $1<p<\infty$, where $H^1(S^{n-1})$ denotes the Hardy space on S^{n-1}.* See [6] for the definition and properties of $H^1(S^{n-1})$.

Remark 1.1. There are the following including relationship on S^{n-1}:

$$C^1(S^{n-1})\subset \mathrm{Lip}(S^{n-1})\subset L^q(S^{n-1})\,(1<q\le\infty)\subset L\log^+ L(S^{n-1})\subset H^1(S^{n-1}),$$

and all inclusions above are proper. On the other hand, In 1976, in their famous paper [5], Coifman, Rochberg and Weiss gave an L^p-boundedness characterization of the commutator $[b,T]$ of the Calderón-Zygmund singular integral operator T.

Theorem B. ([5]). *Suppose that Ω satisfies (1.1), (1.2) and (1.3).*

(*i*) *If $b\in \mathrm{BMO}(\mathbb{R}^n)$, then $[b,T]$ is bounded in $L^p(\mathbb{R}^n)$ for all $1<p<\infty$.*

(*ii*) *If $\{[b,R_j]\}_{j=1}^n$ are bounded in $L^p(\mathbb{R}^n)$ for some p, $1<p<\infty$, then $b\in \mathrm{BMO}(\mathbb{R}^n)$, where $R_j\,(j=1,\cdots,n,)$ denotes the jth Reisz transform.*

In 1978, Uchiyama [18] and Janson [11] extended Theorem B independently. The results in [18] and [11] show that the Reisz transform R_j in the conclusion (ii) of Theorem B may be replaced by the Calderón-Zygmund singular integral operator T.

In the same paper [18], Uchiyama considered also the characterization of the commutator $[b, T]$ is a compact operator in the Lebesgue space $L^p(\mathbb{R}^n)$. Denote by $\mathrm{VMO}(\mathbb{R}^n)$ the BMO-closure of $C_c^\infty(\mathbb{R}^n)$, where $C_c^\infty(\mathbb{R}^n)$ is the set of $C^\infty(\mathbb{R}^n)$ functions with compact support. Uchiyama proved the following conclusions:

Theorem C. ([18]). *Suppose that* Ω *satisfies* (1.1), (1.2) *and* (1.3).

(*i*) *If* $b \in \mathrm{VMO}(\mathbb{R}^n)$, *then* $[b, T]$ *is compact in* $L^p(\mathbb{R}^n)$ *for all* $1 < p < \infty$.

(*ii*) *If* $[b, T]$ *is a compact operator in* $L^p(\mathbb{R}^n)$ *for some* p, $1 < p < \infty$, *then* $b \in \mathrm{VMO}(\mathbb{R}^n)$.

Naturally, one may ask the question whether hold still the conclusions in Theorems B and C if replacing $[b, T]$ by the commutator $[b, \mu_\Omega]$ of the Marcinkiewicz integral and the space $L^p(\mathbb{R}^n)$ by the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$, respectively. In this paper and the forthcoming paper, we will give a positive answer to this question. In fact, we will get similar conclusions to those of Theorems B and C under more weaker conditions than those in Theorems B and C.

Before stating some results, let us recall some definitions. For $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$, the commutator $[b, \mu_\Omega]$ formed by b and the Marcinkiewicz integral μ_Ω is defined by

$$[b, \mu_\Omega]f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} (b(x) - b(y)) f(y)\, dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

For $\Omega \in L^q(S^{n-1})$, $q \geq 1$, the *integral modulus* $\omega_q(\delta)$ *of continuity of order* q of Ω is defined by

$$\omega_q(\delta) = \sup_{\|\tau\| \le \delta} \left(\int_{S^{n-1}} |\Omega(\tau x') - \Omega(x')|^q d\sigma(x') \right)^{1/q},$$

where τ denotes the rotation in $\mathbb{R}^n$ and $\|\tau\| = \sup_{x' \in S^{n-1}} |\tau x' - x'|$. The function Ω is said to satisfy the L^q-Dini condition if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta < \infty.$$

Recently, the first and second authors of this paper gave a characterization of the compactness of the commutators for μ_Ω in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Theorem D. ([3]). *Suppose that* Ω *satisfies* (1.1) *and* (1.2).

(i) *If there exist two constants* $C_1 > 0$ *and* $\gamma > 1$ *such that*

$$|\Omega(x') - \Omega(y')| \leq \frac{C_1}{\left(\log \frac{2}{|x'-y'|}\right)^{\gamma}} \quad \textit{for any} \quad x', y' \in S^{n-1}, \tag{1.4}$$

and the commutator $[b, \mu_\Omega]$ *is a compact operator in* $L^p(\mathbb{R}^n)$ *for some* $p\,(1 < p < \infty)$, *then* $b \in \mathrm{VMO}(\mathbb{R}^n)$.

(ii) *If* $\Omega \in L^q(S^{n-1})$ $(q > 1)$ *satisfying the following condition:*

$$\int_0^1 \frac{\omega_q(\delta)}{\delta}(1 + |\log \delta|)\, d\delta < \infty, \tag{1.5}$$

then for $b \in \mathrm{VMO}(\mathbb{R}^n)$, *the commutator* $[b, \mu_\Omega]$ *is a compact operator in* $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty$.

The purpose of this paper is to give a characterization of the compactness of the commutators $[b, \mu_\Omega]$ in the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$. For $1 \leq p < \infty$ and $0 < \lambda < n$, the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{loc} : \|f\|_{p,\lambda} < \infty\},$$

where

$$\|f\|^p_{p,\lambda} = \sup_{\substack{y\in\mathbb{R}^n \\ r>0}} \frac{1}{r^\lambda} \int_{B(y,r)} |f(x)|^p\, dx$$

and $B(y, r)$ denotes the ball centered at y and with radius $r > 0$. The spaces $L^{p,\lambda}(\mathbb{R}^n)$ becomes a Banach space with norm $\|\cdot\|_{p,\lambda}$. Moreover, if $\lambda = 0$ and $\lambda = n$, then $L^{p,0}(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n)$ coincide (with equality of norms) with the space $L^p(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$, respectively.

The main results in this paper are as follows.

Theorem 1. *Suppose that* Ω *satisfies* (1.1), (1.2) *and* (1.4). *If* $0 < \lambda < n$, $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and the commutator* $[b, \mu_\Omega]$ *is a compact operator from* $L^{p,\lambda}(\mathbb{R}^n)$ *to itself for some* p $(1 < p < \infty)$, *then* $b \in \mathrm{VMO}(\mathbb{R}^n)$.

Theorem 2. *Suppose that* $0 < \lambda < n$ *and* Ω *satisfies* (1.1), (1.2) *and* (1.5) *with* $q > n/(n - \lambda)$. *If* $b \in \mathrm{VMO}(\mathbb{R}^n)$, *then the commutator* $[b, \mu_\Omega]$ *is a compact operator in* $L^{p,\lambda}(\mathbb{R}^n)$ *for* $1 < p < \infty$.

It is easy to see that the condition (1.4) implies (1.5), so we may get the following corollary immediately.

Corollary 1. *Suppose that* Ω *satisfies* (1.1), (1.2) *and* (1.4). *If* $0 < \lambda < n$, $1 < p < \infty$ *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$, *then the commutator* $[b, \mu_\Omega]$ *is a compact operator in* $L^{p,\lambda}(\mathbb{R}^n)$ *if and only if* $b \in \mathrm{VMO}(\mathbb{R}^n)$.

Throughout, the letter "C" will denote (possibly different) the constants that are independent of the essential variables.

2. Proof of Theorem 1

Let us begin with recalling some known conclusions.

Lemma 2.1. ([16]). *If $b \in \mathrm{BMO}(\mathbb{R}^n)$, $\alpha_2 > \alpha_1 > 2$, Q is a cube centered at x_0 and of diameter r, then exist positive constants α_3, α_4, α_5 (depend on α_1, α_2 and b), such that*

$$|\{\alpha_1 r < |x - x_0| < \alpha_2 r : \ |b(x) - b_Q| > v + \alpha_3\}| \leq \alpha_4 |Q| e^{-\alpha_5 v} \quad (0 < v < \infty).$$

Lemma 2.2. ([17]). *Suppose that $f(x)$ is a measurable function on $\mathbb{R}^n$. For $s > 0$, let*

$$\lambda_f(s) = \big|\{x \in \mathbb{R}^n : |f(x)| > s\}\big|$$

and

$$f^*(t) = \inf\{s : \lambda_f(s) \leq t\} \quad \textit{for} \quad t > 0.$$

Then for any measurable set E and $1 \leq p < \infty$,

$$\int_E |f(x)|^p \, dx \leq \int_0^{|E|} |f^*(t)|^p \, dt.$$

Lemma 2.3. ([18]). *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then $b \in \mathrm{VMO}(\mathbb{R}^n)$ if and only if b satisfies the following three conditions:*

(*i*) $\lim\limits_{a \to 0} \sup\limits_{|Q| = a} M(b, Q) = 0$;

(*ii*) $\lim\limits_{a \to \infty} \sup\limits_{|Q| = a} M(b, Q) = 0$;

(*iii*) $\lim\limits_{|x| \to \infty} M(b, Q + x) = 0$ *for each* Q.

Lemma 2.4. ([4]). *Let $0 < \lambda < n$. Suppose that Ω satisfies (1.1), (1.2) and $\Omega \in L^q(S^{n-1})$ for $q > n/(n - \lambda)$, T is a linear or sublinear operator satisfying*

$$|Tf(x)| \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x - y)|}{|x - y|^n} |f(y)| \, dy.$$

(*i*) *If the operator T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, then T is also bounded on $L^{p,\lambda}(\mathbb{R}^n)$.*

(*ii*) *For $b \in \mathrm{BMO}(\mathbb{R}^n)$, if the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, then $[b, T]$ is also bounded on $L^{p,\lambda}(\mathbb{R}^n)$.*

Lemma 2.5. (see [7]) *If* Ω *satisfies conditions* (1.1), (1.2) *and* (1.4). *Let* $\beta > 0$. *Then for* $|x| > 2|y|$

$$\left|\frac{\Omega(x-y)}{|x-y|^\beta} - \frac{\Omega(x)}{|x|^\beta}\right| \le \frac{C}{|x|^\beta(\log\frac{|x|}{|y|})^\gamma}.$$

Now let us return to the proof of Theorem 1. Suppose that $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $[b, \mu_\Omega]$ is a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$. By Lemma 2.3, to prove that $b \in \mathrm{VMO}(\mathbb{R}^n)$, it suffices to show that b must satisfy the conditions (i), (ii) and (iii) in Lemma 2.3.

First, we show that if b does not satisfy the condition (i) of Lemma 2.3, then $[b, \mu_\Omega]$ is not a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$. By the assumption, there exist a $\zeta > 0$ and a sequence of cubes $\{Q_j(y_j, q_j)\}_{j=1}^\infty$ with $\lim\limits_{j\to\infty} q_j = 0$ such that for every j

$$M(b, Q_j) = |Q_j|^{-1}\int_{Q_j} |b(y) - b_{Q_j}|\,dy > \zeta. \tag{2.1}$$

Without loss of the generality, we may assume $\|b\|_* = 1$. Define the function sequence $\{f_j\}_{j=1}^\infty$ by

$$f_j(y) = |Q_j|^{-(n-\lambda)/(np)}\left[\mathrm{sgn}(b(y) - b_{Q_j}) - c_0\right]\chi_{Q_j}(y), \qquad j = 1, 2, \cdots,$$

where $c_0 = |Q_j|^{-1}\int_{Q_j}\mathrm{sgn}(b(y) - b_{Q_j})\,dy$. Since $\int_{Q_j}(b(y) - b_{Q_j})\,dy = 0$, it is easy to check that $|c_0| < 1$ and $\{f_j\}$ satisfies the following properties:

$$\mathrm{supp} f_j \subset Q_j, \tag{2.2}$$

$$f_j(y)(b(y) - b_{Q_j}) \ge 0, \tag{2.3}$$

$$\int_{\mathbb{R}^n} f_j(y)\,dy = 0, \tag{2.4}$$

$$|f_j(y)| \le 2|Q_j|^{-(n-\lambda)/(np)}, \quad \text{for} \quad y \in Q_j. \tag{2.5}$$

Moreover, $\{\|f_j\|_{p,\lambda}\}_{j=1}^\infty$ is bounded uniformly. In fact, denote by $B(t, r)$ any a ball in $\mathbb{R}^n$. If $0 < r \le q_j$, then

$$\left(\frac{1}{r^\lambda}\int_{B(t,r)} |f_j(x)|^p\,dx\right)^{\frac{1}{p}} \le C\left(\frac{r}{q_j}\right)^{(n-\lambda)/p} \le C.$$

If $r > q_j > 0$, then

$$\left(\frac{1}{r^\lambda}\int_{B(t,r)}|f_j(x)|^p\,dx\right)^{\frac{1}{p}} \le \left(\frac{1}{r^\lambda}\int_{Q_j}|f_j(x)|^p\,dx\right)^{\frac{1}{p}} \le C\left(\frac{q_j}{r}\right)^{\frac{\lambda}{p}} \le C.$$

Using the Minkowski inequality, it is easy to see that

$$|\mu_\Omega(f)(x)| \le C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|\,dy.$$

Moreover, μ_Ω and its commutator $[b,\mu_\Omega]$ are both bounded on $L^p\,(1<p<\infty)$ by Theorem A and Theorem 1 in [10], respectively. Then, by Lemma 2.4, we obtain

$$\|\mu_\Omega(f)\|_{p,\lambda} \le C\|f\|_{p,\lambda} \tag{2.6}$$

and

$$\|[b,\mu_\Omega]f\|_{p,\lambda} \le C\|b\|_*\|f\|_{p,\lambda}. \tag{2.7}$$

Thus $\{[b,\mu_\Omega]f_j\}_{j=1}^\infty$ is also a bounded set in $L^{p,\lambda}(\mathbb{R}^n)$. Hence, if $\{[b,\mu_\Omega]f_j\}_{j=1}^\infty$ is not a pre-compact set in $L^{p,\lambda}(\mathbb{R}^n)$ then $[b,\mu_\Omega]$ is not compact operator in $L^{p,\lambda}(\mathbb{R}^n)$. (See [2] for the definition of the compact operator in Banach space.) To do this, it suffices to show that there exists a subsequence $\{[b,\mu_\Omega]f_{j_k}\}_{k=1}^\infty$, which has no any convergence subsequence in $L^{p,\lambda}(\mathbb{R}^n)$.

From now on, for $1\le i\le 19$, A_i denotes the positive constant depending only on $\Omega, p, n, \lambda, \zeta$ and $A_k(1\le k<i)$. Since Ω satisfies (1.2), then there exists an A_1 such that $0<A_1<1$ and

$$\sigma\left(\left\{x'\in S^{n-1}:\ \Omega(x')\ge\frac{2C_1}{\left(\log\frac{2}{A_1}\right)^\gamma}\right\}\right)>0.$$

By the condition (1.4), it is easy to see that

$$\Lambda := \left\{x'\in S^{n-1}:\ \Omega(x')\ge\frac{2C_1}{\left(\log\frac{2}{A_1}\right)^\gamma}\right\}$$

is a closed set. We now claim that

$$\text{if } x'\in\Lambda \text{ and } y'\in S^{n-1} \text{ satisfying } |x'-y'|\le A_1, \quad \text{then } \Omega(y')\ge\frac{C_1}{(\log\frac{2}{A_1})^\gamma}\ . \tag{2.8}$$

In fact, since

$$|\Omega(x')-\Omega(y')| \le \frac{C_1}{(\log\frac{2}{|x'-y'|})^\gamma} \le \frac{C_1}{(\log\frac{2}{A_1})^\gamma}\ ,$$

and note that $\Omega(x')\ge 2\dfrac{C_1}{(\log\frac{2}{A_1})^\gamma}$, we therefore get $\Omega(y')\ge\dfrac{C_1}{(\log\frac{2}{A_1})^\gamma}$. Taking $A_2>2/A_1$, if $y\in Q_j$

$$|x-y_j|>A_2|y-y_j| \quad \text{for} \quad x\in(A_2Q_j)^c\cap\{x:(x-y_j)'\in\Lambda\}.$$

Thus by [16],

$$|(x-y_j)'-(x-y)'|\le \frac{2|y-y_j|}{|x-y_j|}\le\frac{2}{A_2}<A_1.$$

Applying (2.8), we get $\Omega((x-y)')\ge\frac{C_1}{(\log\frac{2}{A_1})^\gamma}$. Hence, for $x\in(A_2Q_j)^c\cap\{x:(x-y_j)'\in\Lambda\}$, by $(2.1)-(2.3)$ and the Hölder inequality, and noting that $|x-y_j|\simeq|x-y|$, we have

$$\begin{aligned}
&|\mu_\Omega((b-b_{Q_j})f_j)(x)|\\
&\ge\frac{C_1}{\left(\log\frac{2}{A_1}\right)^\gamma}\left\{\int_0^\infty\left(\int_{Q_j}\frac{(b(y)-b_{Q_j})f_j(y)}{|x-y|^{n-1}}\,\chi_{\{|x-y|\le t\}}\,dy\right)^2\frac{dt}{t^3}\right\}^{1/2}\\
&\ge C\left\{\int_{|x-y_j|}^\infty\left(\int_{Q_j}\frac{(b(y)-b_{Q_j})f_j(y)}{|x-y|^{n-1}}\,\chi_{\{|x-y|\le t\}}\,dy\right)^2\frac{dt}{t^3}\right\}^{1/2}\\
&\ge C|x-y_j|\int_{Q_j}|x-y|^{1-n}(b(y)-b_{Q_j})f_j(y)\int_{\substack{|x-y_j|\le t\\|x-y|\le t}}\frac{dt}{t^3}\,dy\\
&\ge C|x-y_j|^{-n}\int_{Q_j}(b(y)-b_{Q_j})f_j(y)\,dy\\
&=C|x-y_j|^{-n}|Q_j|^{-1/p+\lambda/(np)}\int_{Q_j}|b(y)-b_{Q_j}|\,dy\\
&\ge C\zeta|Q_j|^{1/p'+\lambda/(np)}|x-y_j|^{-n}.
\end{aligned}\tag{2.9}$$

On the other hand, for $x\in(A_2Q_j)^c$, by $\Omega\in L^\infty(S^{n-1})$, (2.2), (2.5), the Minkowski inequality and the Hölder inequality, we obtain

$$\begin{aligned}
&|\mu_\Omega((b-b_{Q_j})f_j)(x)|\\
&\le\int_{Q_j}|b(y)-b_{Q_j}||f_j(y)|\frac{|\Omega(x-y)|}{|x-y|^n}\,dy\\
&\le C|Q_j|^{1/p'}\left(\frac{1}{|Q_j|}\int_{Q_j}|b(y)-b_{Q_j}|^{p'}\,dy\right)^{\frac{1}{p'}}\left(\int_{Q_j}\frac{|f_j(y)|^p}{|x-y|^{pn}}dy\right)^{\frac{1}{p}}\\
&\le C|Q_j|^{1/p'}\left(\int_{Q_j}|f_j(y)|^p|x-y|^{-pn}\,dy\right)^{1/p}\\
&\le C|Q_j|^{1/p'+\lambda/(np)}\,|x-y_j|^{-n}.
\end{aligned}\tag{2.10}$$

By (2.4) and Lemma 2.5, we have

$$
\begin{aligned}
&|(b(x)-b_{Q_j})\mu_\Omega(f_j)(x)| \\
&= |b(x)-b_{Q_j}|\bigg\{\int_0^\infty \bigg|\int_{\mathbb{R}^n} f_j(y)\bigg(\frac{\Omega(x-y)}{|x-y|^{n-1}}\,\chi_{\{|x-y|\le t\}} \\
&\quad -\frac{\Omega(x-y_j)}{|x-y_j|^{n-1}}\,\chi_{\{|x-y_j|\le t\}}\bigg)\,dy\bigg|^2\frac{dt}{t^3}\bigg\}^{\frac12} \\
&\le |b(x)-b_{Q_j}|\bigg\{\bigg(\int_0^\infty\bigg(\int_{|x-y|\le t<|x-y_j|}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\,|f_j(y)|\,dy\bigg)^2\frac{dt}{t^3}\bigg)^{\frac12} \\
&\quad +\bigg(\int_0^\infty\bigg(\int_{|x-y_j|\le t<|x-y|}\frac{|\Omega(x-y_j)|}{|x-y_j|^{n-1}}\,|f_j(y)|\,dy\bigg)^2\frac{dt}{t^3}\bigg)^{\frac12} \\
&\quad +\bigg(\int_0^\infty\bigg(\int_{\substack{|x-y_j|\le t\\|x-y|\le t}}\bigg|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-y_j)}{|x-y_j|^{n-1}}\bigg|\,|f_j(y)|\,dy\bigg)^2\frac{dt}{t^3}\bigg)^{\frac12}\bigg\} \\
&\le |b(x)-b_{Q_j}|\int_{Q_j}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f_j(y)|\bigg(\int_{\substack{|x-y|\le t\\|x-y_j|>t}}\frac{dt}{t^3}\bigg)^{1/2}dy \\
&\quad +|b(x)-b_{Q_j}|\int_{Q_j}\frac{|\Omega(x-y_j)|}{|x-y_j|^{n-1}}|f_j(y)|\bigg(\int_{\substack{|x-y|>t\\|x-y_j|\le t}}\frac{dt}{t^3}\bigg)^{1/2}dy \\
&\quad +|b(x)-b_{Q_j}|\int_{Q_j}|f_j(y)|\bigg|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-y_j)}{|x-y_j|^{n-1}}\bigg|\bigg(\int_{\substack{|x-y|\le t\\|x-y_j|\le t}}\frac{dt}{t^3}\bigg)^{1/2}dy \\
&\le C|b(x)-b_{Q_j}|\bigg(\int_{Q_j}\frac{|f_j(y)|}{|x-y_j|^n(\log\frac{2|x-y_j|}{q_j})^\gamma}\,dy+q_j^{1/2}\int_{Q_j}\frac{|f_j(y)|}{|x-y_j|^{n+1/2}}\,dy\bigg) \\
&\le C|Q_j|^{1/p'+\lambda/(np)}\frac{|b(x)-b_{Q_j}|}{|x-y_j|^n(\log\frac{2|x-y_j|}{q_j})^\gamma}\,.
\end{aligned}
\tag{2.11}
$$

Since $|b_{2Q}-b_Q|\le C\|b\|_*=C$ by $\|b\|_*=1$, we have

$$
\bigg(\int_{2^sq_j<|x-y_j|\le 2^{s+1}q_j}|b(x)-b_{Q_j}|^p\,dx\bigg)^{\frac1p}\le C2^{sn/p}s|Q_j|^{1/p}.
$$

For $v>A_2$, using (2.11) and the above inequality, we obtain

$$
\begin{aligned}
&\bigg(\int_{|x-y_j|>vq_j}|(b(x)-b_{Q_j})\mu_\Omega(f_j)(x)|^p\,dx\bigg)^{\frac1p} \\
&\le C|Q_j|^{1/p'+\lambda/(np)}\bigg(\int_{|x-y_j|>vq_j}\frac{|b(x)-b_{Q_j}|^p}{|x-y_j|^{np}\big(\log\frac{2|x-y_j|}{q_j}\big)^{\gamma p}}\,dx\bigg)^{\frac1p} \\
&\le C|Q_j|^{1/p'+\lambda/(np)}\sum_{s=[\log_2 v]}^\infty\bigg(\int_{2^sq_j<|x-y_j|\le 2^{s+1}q_j}\frac{|b(x)-b_{Q_j}|^p}{|x-y_j|^{np}\big(\log\frac{2|x-y_j|}{q_j}\big)^{\gamma p}}dx\bigg)^{\frac1p} \\
&\le C|Q_j|^{\lambda/(np)}\sum_{s=[\log_2 v]}^\infty 2^{-s(n-n/p)}s^{1-\gamma} \\
&\le C|Q_j|^{\lambda/(np)}(\log v)^{1-\gamma}v^{-n(1-1/p)}.
\end{aligned}
\tag{2.12}
$$

Then for $u > v > A_2$, using (2.9) and (2.12) we get

$$\begin{aligned}
&\Big(\int_{\{vq_j<|x-y_j|\le uq_j\}} |[b,\mu_\Omega]f_j(x)|^p\,dx\Big)^{\frac{1}{p}} \\
&\ge \Big(\int_{\{vq_j<|x-y_j|\le uq_j\}\bigcap\{x:(x-y_j)'\in\Lambda\}} |\mu_\Omega((b-b_{Q_j})f_j)(x)|^p\,dx\Big)^{\frac{1}{p}} \\
&\quad -\Big(\int_{|x-y_j|>vq_j} |(b(x)-b_{Q_j})\mu_\Omega(f_j)(x)|^p\,dx\Big)^{\frac{1}{p}} \qquad (2.13)\\
&\ge C\zeta|Q_j|^{1/p'+\lambda/(np)}\Big(\int_{\{vq_j<|x-y_j|\le uq_j\}\bigcap\{x:(x-y_j)'\in\Lambda\}} \frac{1}{|x-y_j|^{pn}}\,dx\Big)^{\frac{1}{p}} \\
&\quad -C|Q_j|^{\lambda/(np)}(\log v)^{1-\gamma}v^{-n(1-1/p)} \\
&\ge A_3\zeta|Q_j|^{\lambda/(np)}(v^{-np+n}-u^{-np+n})^{1/p}-A_4|Q_j|^{\lambda/(np)}(\log v)^{1-\gamma}v^{-n+n/p}.
\end{aligned}$$

From (2.10) and (2.12), it follows that

$$\begin{aligned}
&\Big(\int_{|x-y_j|>uq_j} |[b,\mu_\Omega]f_j(x)|^p\,dx\Big)^{\frac{1}{p}} \\
&\le \Big(\int_{|x-y_j|>uq_j} |\mu_\Omega((b-b_{Q_j})f_j)(x)|^p\,dx\Big)^{\frac{1}{p}} \qquad (2.14)\\
&\quad +\Big(\int_{|x-y_j|>uq_j} |(b(x)-b_{Q_j})\mu_\Omega(f_j)(x)|^p\,dx\Big)^{\frac{1}{p}} \\
&\le A_5|Q_j|^{\lambda/(np)}u^{-n+n/p}+A_6|Q_j|^{\lambda/(np)}(\log u)^{1-\gamma}u^{-n+n/p}.
\end{aligned}$$

By (2.13) and (2.14), there exist A_7, $B = B(\Omega, p, n, \lambda, \zeta, A_3, A_4, A_5, A_6) > 1$ and A_9 such that $A_2 < A_7$,

$$\Big(\int_{A_7q_j<|x-y_j|\le BA_7q_j} |[b,\mu_\Omega]f_j(y)|^p\,dy\Big)^{1/p} \ge A_9|Q_j|^{\lambda/(np)} \qquad (2.15)$$

and

$$\Big(\int_{|x-y_j|>BA_7q_j} |[b,\mu_\Omega]f_j(y)|^p\,dy\Big)^{1/p} \le \frac{A_9}{4}\,|Q_j|^{\lambda/(np)}. \qquad (2.16)$$

Let $A_8 = BA_7$ and let $E \subset \{x : A_7q_j < |x-y_j| < A_8q_j\}$ be an arbitrary measurable set. Then by (2.10) and (2.11), we have

$$
\begin{aligned}
&\Big(\int_E |[b,\mu_\Omega]f_j(x)|^p\,dx\Big)^{\frac{1}{p}}\\
&\le \Big(\int_E |\mu_\Omega((b-b_{Q_j})f_j)(x)|^p\,dx\Big)^{\frac{1}{p}}\\
&\quad+\Big(\int_E |(b(x)-b_{Q_j})\mu_\Omega(f_j)(x)|^p\,dx\Big)^{\frac{1}{p}}\\
&\le C|Q_j|^{1/p'+\lambda/(np)}\Big(\int_E |x-y_j|^{-pn}\,dx\Big)^{\frac{1}{p}}\\
&\quad+C|Q_j|^{1/p'+\lambda/(np)}\Big(\int_E \frac{|b(x)-b_{Q_j}|^p}{|x-y_j|^{np}\big(\log\frac{2|x-y_j|}{q_j}\big)^{\gamma p}}\,dx\Big)^{\frac{1}{p}}\\
&\le A_{10}|Q_j|^{\lambda/(np)}\Big\{\frac{|E|^{1/p}}{|Q_j|^{1/p}}+\Big(\frac{1}{|Q_j|}\int_E|b(x)-b_{Q_j}|^p\,dx\Big)^{\frac{1}{p}}\Big\}.
\end{aligned}
\tag{2.17}
$$

Let $h_j(x)=b(x)-b_{Q_j}$, and for $0<\omega<\infty$, denote by $\lambda_{h_j}(\omega)$ the measure of the following set:

$$\{A_7q_j<|x-y_j|<A_8q_j:\ |h_j(x)|>\omega\}.$$

Then by Lemma 2.1, there exist positive constants A_{11}, A_{12} and A_{13}, such that

$$\lambda_{h_j}(\omega+A_{11})\le A_{12}|Q_j|e^{-A_{13}\omega}.$$

Hence, $\lambda_{h_j}(\omega)\le A_{12}|Q_j|e^{-A_{13}(\omega-A_{11})}$. For $t>0$, let $h_j^*(t)=\inf\{\omega:\ \lambda_{h_j}(\omega)\le t\}$. Then when $0<t<A_{12}|Q_j|$,

$$h_j^*(t)\le\frac{1}{A_{13}}\log\frac{A_{12}|Q_j|}{t}+A_{11}. \tag{2.18}$$

Recall $E\subset\{x: A_7q_j<|x-y_j|<A_8q_j\}$, applying Lemma 2.2 and (2.18), if $|E|\ll A_{12}|Q_j|$, we have

$$
\begin{aligned}
\frac{1}{|Q_j|}\int_E|b(x)-b_{Q_j}|^p\,dx &\le \frac{1}{|Q_j|}\int_0^{|E|}|h_j^*(t)|^p\,dt\\
&\le \frac{1}{|Q_j|}\int_0^{|E|}\Big(A_{11}-\frac{1}{A_{13}}\log\frac{t}{A_{12}|Q_j|}\Big)^p\,dt\\
&= A_{12}\int_0^{|E|/(A_{12}|Q_j|)}\big(A_{11}-\frac{1}{A_{13}}\log t\big)^p\,dt\\
&\le A_{14}\frac{|E|}{|Q_j|}\Big(1+\log\frac{A_{12}|Q_j|}{|E|}\Big)^{[p]+1}.
\end{aligned}
\tag{2.19}
$$

Combining (2.17) with (2.19), there exists a positive constant $A_{15} < \min\{A_{12}^{1/n}, A_8\}$, such that

$$\left(\int_E |[b,\mu_\Omega]f_j(y)|^p\,dy\right)^{\frac{1}{p}} \le \frac{A_9}{4}\,|Q_j|^{\lambda/(np)} \tag{2.20}$$

for every measurable set E satisfying $E \subset \{x:\ A_7 q_j < |x-y_j| < A_8 q_j\}$ and $|E|/|Q_j| < A_{15}^n$. Now we choose a subsequence $\{Q_{j_k}\}$ satisfying

$$q_{j_{k+1}}/q_{j_k} < A_{15}/A_8. \tag{2.21}$$

For $m > 0$, we have

$$\begin{aligned}&\left(\int_{B(y_{j_k},\,A_8 q_{j_k})} |[b,\mu_\Omega]f_{j_k} - [b,\mu_\Omega]f_{j_{k+m}}|^p\,dx\right)^{\frac{1}{p}}\\ &\ge \left(\int_{G_1} |[b,\mu_\Omega]f_{j_k} - [b,\mu_\Omega]f_{j_{k+m}}|^p\,dx\right)^{\frac{1}{p}}\\ &\ge \left(\int_{G_1} |[b,\mu_\Omega]f_{j_k}|^p\,dx\right)^{\frac{1}{p}} - \left(\int_{G_2} |[b,\mu_\Omega]f_{j_{k+m}}|^p\,dx\right)^{\frac{1}{p}},\end{aligned}$$

where

$$G_1 = \{x: A_7 q_{j_k} < |x-y_{j_k}| < A_8 q_{j_k}\} \backslash \{x: |x-y_{j_{k+m}}| \le A_8 q_{j_{k+m}}\} \subset B(y_{j_k}, A_8 q_{j_k})$$

and $G_2 = \{x: |x-y_{j_{k+m}}| > A_8 q_{j_{k+m}}\}$. Let

$$G = \{x: A_7 q_{j_k} < |x-y_{j_k}| < A_8 q_{j_k}\},$$

then $G_1 = G - (G_2^c \cap G)$. Thus by (2.15) and (2.16), we get

$$\begin{aligned}&\left(\int_{B(y_{j_k},\,A_8 q_{j_k})} |[b,\mu_\Omega]f_{j_k} - [b,\mu_\Omega]f_{j_{k+m}}|^p\,dx\right)^{\frac{1}{p}}\\ &\ge \left(\int_G |[b,\mu_\Omega]f_{j_k}|^p\,dx - \int_{G_2^c \bigcap G} |[b,\mu_\Omega]f_{j_k}|^p\,dx\right)^{\frac{1}{p}} - \left(\int_{G_2} |[b,\mu_\Omega]f_{j_{k+m}}|^p\,dx\right)^{\frac{1}{p}}\\ &\ge \left(A_9^p |Q_{j_k}|^{\lambda/n} - \int_{G_2^c \cap G} |[b,\mu_\Omega]f_{j_k}|^p\,dx\right)^{\frac{1}{p}} - \frac{A_9}{4}\,|Q_{j_{k+m}}|^{\lambda/(np)}.\end{aligned}$$

Since $(G_2^c \cap G) \subset G$ and by (2.21), we have

$$\frac{|G_2^c \cap G|}{|Q_{j_k}|} \le \frac{A_8^n q_{j_{k+m}}^n}{q_{j_k}^n} < A_8^n \left(\frac{A_{15}^n}{A_8^n}\right)^m < A_{15}^n.$$

By (2.20), we get

$$\int_{G_2^c \cap G} |[b, \mu_\Omega] f_{j_k}|^p \, dx \le (\frac{A_9}{4})^p |Q_{j_k}|^{\lambda/n}.$$

So

$$\left(\int_{B(y_{j_k},\, A_8 q_{j_k})} |[b, \mu_\Omega] f_{j_k} - [b, \mu_\Omega] f_{j_{k+m}}|^p \, dx \right)^{\frac{1}{p}} \ge \frac{A_9}{4} \, |Q_{j_k}|^{\lambda/(np)}.$$

Then

$$\left(\frac{1}{q_{j_k}^\lambda} \int_{B(y_{j_k},\, A_8 q_{j_k})} |[b, \mu_\Omega] f_{j_k} - [b, \mu_\Omega] f_{j_{k+m}}|^p \, dx \right)^{\frac{1}{p}} \ge A_{16}.$$

Therefore

$$\| [b, \mu_\Omega] f_{j_k} - [b, \mu_\Omega] f_{j_{k+m}} \|_{p,\lambda} \ge A_{17}.$$

Thus the sequence $\{[b, \mu_\Omega] f_{j_k}\}_{k=1}^\infty$ has no any convergence subsequence in $L^{p,\lambda}(\mathbb{R}^n)$, i.e., $[b, \mu_\Omega]$ is not a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$. This contradiction shows that b must satisfy the condition (i) of Lemma 2.3.

Similarly, we may show that if b does not satisfy the conditions (ii) or (iii) in Lemma 2.3, then $[b, \mu_\Omega]$ is also not a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$. For simplicity, we give only the outline of the proofs. In fact, if b does not satisfy the condition (ii) of Lemma 2.3, we can select a sequence $\{Q_j\}$ such that (2.1) holds and $\lim_{j \to \infty} q_j = \infty$, where q_j is the diameters of Q_j, and y_j is the center of Q_j. Similarly, we select a sequence $\{f_j\} \subset L^{p,\lambda}(\mathbb{R}^n)$ such that (2.15), (2.16) and (2.20) hold. Hence, if we choose a subsequence $\{Q_{j_k}\}$ such that $q_{j_k} > 1$ and

$$q_{j_k} / q_{j_{k+1}} < A_{15}/A_8, \tag{2.22}$$

then for $m > 0$, we have

$$\begin{aligned} &\left(\int_{B(y_{j_{k+m}},\, A_8 q_{j_{k+m}})} |[b, \mu_\Omega] f_{j_k} - [b, \mu_\Omega] f_{j_{k+m}}|^p \, dx \right)^{\frac{1}{p}} \\ \ge &\left(\int_{G_1} |[b, \mu_\Omega] f_{j_k} - [b, \mu_\Omega] f_{j_{k+m}}|^p \, dx \right)^{\frac{1}{p}} \\ \ge &\left(\int_{G_1} |[b, \mu_\Omega] f_{j_{k+m}}|^p \, dx \right)^{\frac{1}{p}} - \left(\int_{G_2} |[b, \mu_\Omega] f_{j_k}|^p \, dx \right)^{\frac{1}{p}}, \end{aligned}$$

where

$$\begin{aligned} G_1 &= \{x : A_7 q_{j_{k+m}} < |x - y_{j_{k+m}}| < A_8 q_{j_{k+m}}\} \backslash \{x : |x - y_{j_k}| \le A_8 q_{j_k}\} \\ &\subset B(y_{j_{k+m}},\, A_8 q_{j_{k+m}}) \end{aligned}$$

and $G_2 = \{x : |x - y_{j_k}| > A_8 q_{j_k}\}$. Set

$$G = \{x : A_7 q_{j_{k+m}} < |x - y_{j_{k+m}}| < A_8 q_{j_{k+m}}\},$$

then $G_1 = G - \big(G_2^c \bigcap G\big)$. Thus by (2.15) and (2.16) we get

$$\begin{aligned}
&\left(\int_{B(y_{j_{k+m}}, A_8 q_{j_{k+m}})} |[b,\mu_\Omega]f_{j_k} - [b,\mu_\Omega]f_{j_{k+m}}|^p\,dx\right)^{\frac{1}{p}} \\
&\geq \left(\int_G |[b,\mu_\Omega]f_{j_{k+m}}|^p\,dx - \int_{G_2^c \bigcap G} |[b,\mu_\Omega]f_{j_{k+m}}|^p\,dx\right)^{\frac{1}{p}} - \left(\int_{G_2} |[b,\mu_\Omega]f_{j_k}|^p\,dx\right)^{\frac{1}{p}} \\
&\geq \left(A_9^p |Q_{j_{k+m}}|^{\lambda/n} - \int_{G_2^c \cap G} |[b,\mu_\Omega]f_{j_{k+m}}|^p\,dx\right)^{\frac{1}{p}} - \frac{A_9}{4}\,|Q_{j_k}|^{\lambda/(np)}.
\end{aligned}$$

Since $G_2^c \bigcap G \subset G$, by (2.22) we have

$$\frac{|G_2^c \cap G|}{|Q_{j_{k+m}}|} \leq \frac{A_8^n q_{j_k}^n}{q_{j_{k+m}}^n} < A_8^n \Big(\frac{A_{15}^n}{A_8^n}\Big)^m < A_{15}^n.$$

Thus, by (2.20) we get

$$\int_{G_2^c \cap G} |[b,\mu_\Omega]f_{j_{k+m}}|^p\,dx \leq \left(\frac{A_9}{4}\right)^p |Q_{j_{k+m}}|^{\lambda/n}.$$

Hence

$$\left(\int_{B(y_{j_{k+m}}, A_8 q_{j_{k+m}})} |[b,\mu_\Omega]f_{j_k} - [b,\mu_\Omega]f_{j_{k+m}}|^p\,dx\right)^{\frac{1}{p}} \geq \frac{A_9}{4}\,|Q_{j_{k+m}}|^{\lambda/(np)}.$$

and

$$\|[b,\mu_\Omega]f_{j_k} - [b,\mu_\Omega]f_{j_{k+m}}\|_{p,\lambda} \geq \ A_{18}.$$

Thus $\{[b,\mu_\Omega]f_{j_k}\}_{k=1}^{\infty}$ has no any convergence subsequence in $L^{p,\lambda}(\mathbb{R}^n)$. But this is contrary to the assumption that $[b,\mu_\Omega]$ is a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$. Hence, b should satisfy the condition (ii) of Lemma 2.3.

Finally, if b does not satisfy the condition (iii) of Lemma 2.3, then there exist a cube Q and sequence $\{y_j\}$ with $\lim\limits_{j\to\infty} |y_j| = \infty$ such that (2.1) holds for $\{Q_j = Q + y_j\}$. Let

$$E_j = \{x \in \mathbb{R}^n :\ |x - y_j| < A_8 q'\},$$

where q' is the diameter of Q. We select a sequence $\{f_j\} \subset L^{p,\lambda}(\mathbb{R}^n)$ such that (2.15) and (2.16) hold. Now, we choose a subsequence $\{E_{j_k}\}$ such that

$$E_{j_k} \bigcap E_{j_l} = \emptyset \quad \text{for} \quad l \neq k.$$

Then for $m > 0$, we have

$$\begin{aligned}
&\left(\int_{B(y_{j_k},\, A_8 q')} |[b, \mu_\Omega] f_{j_k} - [b, \mu_\Omega] f_{j_{k+m}}|^p \, dx \right)^{\frac{1}{p}} \\
\geq &\left(\int_{G_1} |[b, \mu_\Omega] f_{j_k} - [b, \mu_\Omega] f_{j_{k+m}}|^p \, dx \right)^{\frac{1}{p}} \\
\geq &\left(\int_{G_1} |[b, \mu_\Omega] f_{j_k}|^p \, dx \right)^{\frac{1}{p}} - \left(\int_{G_2} |[b, \mu_\Omega] f_{j_{k+m}}|^p \, dx \right)^{\frac{1}{p}},
\end{aligned}$$

where

$$G_1 = \{x : A_7 q' < |x - y_{j_k}| < A_8 q'\} \backslash \{x : |x - y_{j_{k+m}})| \leq A_8 q'\} \subset B(y_{j_k},\, A_8 q')$$

and $G_2 = \{x : |x - y_{j_{k+m}}| > A_8 q'\}$. Let

$$G = \{x : A_7 q' < |x - y_{j_k}| < A_8 q'\},$$

then $G_1 = G - G_2^c = G$. Thus by (2.15) and (2.16) we get

$$\begin{aligned}
&\left(\int_{B(y_{j_k},\, A_8 q')} |[b, \mu_\Omega] f_{j_k} - [b, \mu_\Omega] f_{j_{k+m}}|^p \, dx \right)^{\frac{1}{p}} \\
\geq &\left(\int_{G} |[b, \mu_\Omega] f_{j_k}|^p \, dx \right)^{\frac{1}{p}} - \left(\int_{G_2} |[b, \mu_\Omega] f_{j_{k+m}}|^p \, dx \right)^{\frac{1}{p}} \\
\geq &A_9 |Q|^{\lambda/(np)} - \frac{A_9}{4} |Q|^{\lambda/(np)} \geq \frac{A_9}{4} |Q|^{\lambda/(np)}.
\end{aligned}$$

Hence

$$\|[b, \mu_\Omega] f_{j_k} - [b, \mu_\Omega] f_{j_{k+m}}\|_{p,\lambda} \geq A_{19}.$$

This is inconsistent with the compactness of $[b, \mu_\Omega]$ in $L^{p,\lambda}(\mathbb{R}^n)$. So, b satisfies also the condition (iii) of Lemma 2.3.

3. Proof of Theorem 2

First we give some lemmas, which will be used in the proof of Theorem 2.

Lemma 3.1. ([4]). *Suppose that* $1 \leq p < \infty$ *and* $0 < \lambda < n$. *If the subset* G *in* $L^{p,\lambda}(\mathbb{R}^n)$ *satisfies the following conditions:*

(3.1) $$\sup_{f \in G} \|f\|_{p,\lambda} < \infty,$$

(3.2) $$\lim_{y\to 0}\|f(\cdot+y)-f(\cdot)\|_{p,\lambda}=0 \quad \textit{uniformly in } f\in G,$$

(3.3) $$\lim_{\alpha\to\infty}\|f\chi_{E_\alpha}\|_{p,\lambda}=0 \quad \textit{uniformly in } \ f\in G,$$

where $E_\alpha=\{x\in\mathbb{R}^n:|x|>\alpha\}$, *then* G *is strongly pre-compact set in* $L^{p,\lambda}(\mathbb{R}^n)$.

Lemma 3.2. ([9]) *Suppose that* $0\le\beta<n$, Ω *satisfies* (1.1) *and the* L^q*-Dini condition*

$$\int_0^1\frac{\omega_q(\delta)}{\delta}\,d\delta<\infty$$

for $q\ge 1$. *If there exists a positive constant* $0<\theta<1/2$ *such that* $|x|<\theta R$, *then we have the following inequality*

(3.4) $$\begin{aligned}&\left(\int_{R<|y|<2R}\left|\frac{\Omega(y-x)}{|y-x|^{n-\beta}}-\frac{\Omega(y)}{|y|^{n-\beta}}\right|^q dy\right)^{1/q}\\ &\le CR^{n/q-(n-\beta)}\left\{\frac{|x|}{R}+\int_{|x|/2R}^{|x|/R}\frac{\omega_q(\delta)}{\delta}d\delta\right\},\end{aligned}$$

where the constant $C>0$ *is independent of* R *and* x.

Lemma 3.3. ([12]). *Suppose that* $1<p<\infty$ *and* $1\le r<p<\infty$, *then the maximal operator* $\mathcal{M}_r$ *and Calderón-Zygmund singular integral operator* T *are bounded operators on* $L^{p,\lambda}(\mathbb{R}^n)$, *where* $\mathcal{M}_rf(x)=\{\mathcal{M}(|f|^r)(x)\}^{1/r}$ *and* $\mathcal{M}$ *is the Hardy-Littlewood maximal operator.*

Now let us return to the proof of Theorem 2. Suppose that F is an arbitrary bounded set in $L^{p,\lambda}(\mathbb{R}^n)$, that is, there exists a constant $D>0$ such that $\|f\|_{p,\lambda}\le D$ for every $f\in F$. Let $G=\{[b,\mu_\Omega]f:\ f\in F\}$ if $b\in C_c^\infty(\mathbb{R}^n)$ and $\widetilde{G}=\{[b,\mu_\Omega]f:\ f\in F\}$ if $b\in \mathrm{VMO}(\mathbb{R}^n)$. For $b\in\mathrm{VMO}(\mathbb{R}^n)$, by (2.7), we can easily get $[b,\mu_\Omega]$ is continuous in $L^{p,\lambda}(\mathbb{R}^n)$. So, by the definition of the compact operator (see [2], for example), it suffices to prove that for any bounded set F in $L^{p,\lambda}(\mathbb{R}^n)$, $\widetilde{G}$ is strongly pre-compact in $L^{p,\lambda}(\mathbb{R}^n)$. We first show that if (3.1)-(3.3) hold uniformly in G, then (3.1)-(3.3) hold also uniformly in $\widetilde{G}$ and thus $[b,\mu_\Omega]$ is a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$.

In fact, suppose that $b\in\mathrm{VMO}(\mathbb{R}^n)$, then for any $\varepsilon>0$ there exists $b^\varepsilon\in C_c^\infty(\mathbb{R}^n)$ such that $\|b-b^\varepsilon\|_*<\varepsilon$. By

$$\begin{aligned}&|[b,\mu_\Omega]f(x)-[b^\varepsilon,\mu_\Omega]f(x)|\\ &\le\left\{\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[(b-b^\varepsilon)(x)-(b-b^\varepsilon)(y)]f(y)\,dy\right|^2\frac{dt}{t^3}\right\}^{1/2}\end{aligned}$$

and (2.7), we obtain

$$\|[b,\mu_\Omega]-[b^\varepsilon,\mu_\Omega]\|_{L^{p,\lambda}\to L^{p,\lambda}}\le \|[b-b^\varepsilon,\mu_\Omega]\|_{L^{p,\lambda}\to L^{p,\lambda}}\le C\varepsilon. \tag{3.5}$$

For any $f\in F$, by (3.1) and (3.5) we get

$$\sup_{f\in F}\|[b,\mu_\Omega]f\|_{p,\lambda}\le \sup_{f\in F}\|[b^\varepsilon,\mu_\Omega]f\|_{p,\lambda}+CD\varepsilon<\infty.$$

On the other hand, by (3.2) and (3.5), for any $f\in F$

$$\begin{aligned}&\lim_{|y|\to 0}\|[b,\mu_\Omega]f(\cdot+y)-[b,\mu_\Omega]f(\cdot)\|_{p,\lambda}\\ \le &\lim_{|y|\to 0}\|[b^\varepsilon,\mu_\Omega]f(\cdot+y)-[b^\varepsilon,\mu_\Omega]f(\cdot)\|_{p,\lambda}+2\|[b-b^\varepsilon,\mu_\Omega]f\|_{p,\lambda}\\ < &\,2CD\varepsilon.\end{aligned}$$

Therefore (3.2) holds uniformly for $\widetilde{G}$. Similarly, by (3.3) and (3.5), we see that

$$\lim_{\alpha\to+\infty}\|[b,\mu_\Omega]f\chi_{E_\alpha}\|_{p,\lambda}\le \lim_{\alpha\to+\infty}\|[b^\varepsilon,\mu_\Omega]f\chi_{E_\alpha}\|_{p,\lambda}+\|[b-b^\varepsilon,\mu_\Omega]f\|_{p,\lambda}\le CD\varepsilon.$$

Thus (3.3) holds also for $\widetilde{G}$ uniformly. Therefore, by Lemma 3.1, we know $\widetilde{G}$ is a strongly pre-compact set in $L^{p,\lambda}(\mathbb{R}^n)$ and then $[b,\mu_\Omega]$ is a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$.

Thus, it suffices to prove that (3.1)-(3.3) hold uniformly in G. Recalling $\|f\|_{p,\lambda}\le D$ for every $f\in F$, and noticing that $b\in C_c^\infty(\mathbb{R}^n)$, by (2.7), we have

$$\sup_{f\in F}\|[b,\mu_\Omega]f\|_{p,\lambda}\le C\|b\|_*\sup_{f\in F}\|f\|_{p,\lambda}\le CD\|b\|_*<\infty. \tag{3.6}$$

Suppose that $\operatorname{supp} b\subset\{x:\ |x|\le\beta\}$. For any $0<\varepsilon<1$, we take $\alpha>\max\{1,\beta\}$ such that $(\alpha-\beta)^{n(1-q)}<\varepsilon^q$. If $q\le p$, then for any x satisfying $|x|>\alpha$ and every $f\in F$, we have

$$\begin{aligned}|[b,\mu_\Omega]f(x)|&=\left\{\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\,(b(x)-b(y))f(y)\,dy\right|^2\frac{dt}{t^3}\right\}^{1/2}\\ &\le C\int_{|y|\le\beta}\frac{|b(y)||\Omega(x-y)|}{|x-y|^{n-1}}|f(y)|\left\{\int_{|x-y|\le t}\frac{dt}{t^3}\right\}^{1/2}dy\\ &\le C\int_{|y|\le\beta}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|\,dy\\ &\le C\left(\int_{|y|\le\beta}\frac{|\Omega(x-y)|^q}{|x-y|^{qn}}|f(y)|^q\,dy\right)^{1/q}.\end{aligned}$$

For every $s \in \mathbb{R}^n$ and $r > 0$, by the Minkowski inequality and the choice of α, we get

$$\begin{aligned} &\left(\frac{1}{r^\lambda}\int_{B(s,r)} |[b,\mu_\Omega]f(x)|^p \chi_{E_\alpha}(x)\,dx\right)^{1/p} \\ &\le C\|f\|_{p,\lambda}\left(\int_{|y|>\alpha-\beta} \frac{|\Omega(y)|^q}{|y|^{nq}}\,dy\right)^{1/q} \\ &\le C\|f\|_{p,\lambda}\left(\int_{\alpha-\beta}^{\infty}\int_{S^{n-1}} |\Omega(y')|^q\,d\sigma(y')\frac{dr}{r^{nq-n+1}}\right)^{1/q} \\ &\le CD\|\Omega\|_{L^q(S^{n-1})}\varepsilon \le CD\varepsilon. \end{aligned} \tag{3.7}$$

If $q > p$, we choose q_0 such that $1 < q_0 \le p < q$. Notice that $\Omega \in L^{q_0}(S^{n-1})$ and $\|\Omega\|_{L^{q_0}(S^{n-1})} \le C\|\Omega\|_{L^q(S^{n-1})}$, by (3.7), for every $s \in \mathbb{R}^n$ and $r > 0$, we still get

$$\left(\frac{1}{r^\lambda}\int_{B(s,r)} |[b,\mu_\Omega]f(x)|^p \chi_{E_\alpha}(x)\,dx\right)^{1/p} \le CD\|\Omega\|_{L^{q_0}(S^{n-1})}\varepsilon \le CD\varepsilon. \tag{3.8}$$

The estimates (3.7) and (3.8) show that (3.3) holds for the commutator $[b,\mu_\Omega]$ in G uniformly. Finally, to finish the proof of Theorem 2, it remains to show (3.2) holds for the commutator $[b,\mu_\Omega]$ in G uniformly. We need to prove that for any $\varepsilon > 0$, if $|z|$ is sufficiently small depended only on ε, then for every $f \in F$,

$$\|[b,\mu_\Omega]f(\cdot + z) - [b,\mu_\Omega]f(\cdot)\|_{p,\lambda} \le C\varepsilon. \tag{3.9}$$

Then for $z \in \mathbb{R}^n$, we write

$$\begin{aligned} &|[b,\mu_\Omega]f(x+z) - [b,\mu_\Omega]f(x)| \\ &\le \left\{\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)\,dy \right.\right. \\ &\qquad \left.\left. - \int_{|x+z-y|\le t} \frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}(b(x+z)-b(y))f(y)\,dy\right|^2 \frac{dt}{t^3}\right\}^{1/2} \\ &:= \left\{\int_0^\infty |I(x,t)|^2 \frac{dt}{t^3}\right\}^{1/2}. \end{aligned} \tag{3.10}$$

We take ε such that $0 < \varepsilon < \frac{1}{2}$. Then for $z \in \mathbb{R}^n$, decompose $I(x,t)$ as

$$
\begin{aligned}
&I(x,t)\\
&=\int_{\substack{|x-y|>e^{\frac{1}{\varepsilon}}|z|,|x-y|\le t\\ |x+z-y|\ge t}}\frac{\Omega(x-y)}{|x-y|^{n-1}}\,(b(x+z)-b(y))f(y)\,dy\\
&\quad+\int_{\substack{|x-y|>e^{\frac{1}{\varepsilon}}|z|,|x-y|\ge t,\\ |x+z-y|\le t}}\frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}\,(b(y)-b(x+z))f(y)\,dy\\
&\quad+\int_{\substack{|x-y|>e^{\frac{1}{\varepsilon}}|z|,|x-y|\le t,\\ |x+z-y|\le t}}\left(\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}\right)(b(x+z)-b(y))f(y)\,dy\\
&\quad+\int_{|x-y|>e^{\frac{1}{\varepsilon}}|z|,\,|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\,(b(x)-b(x+z))f(y)\,dy\\
&\quad+\int_{|x-y|\le e^{\frac{1}{\varepsilon}}|z|,|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\,(b(x)-b(y))f(y)\,dy\\
&\quad+\int_{\substack{|x-y|\le e^{\frac{1}{\varepsilon}}|z|\\ |x+z-y|\le t}}\frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}\,(b(y)-b(x+z))f(y)\,dy\\
&:=J_1(x,t)+J_2(x,t)+J_3(x,t)+J_4(x,t)+J_5(x,t)+J_6(x,t).
\end{aligned}\tag{3.11}
$$

By $|b(x+z)-b(y)|<C$ and the Minkowski inequality, we have

$$
\begin{aligned}
&\left(\int_0^\infty |J_1(x,t)|^2\frac{dt}{t^3}\right)^{1/2}\\
&=\left(\int_0^\infty\left|\int_{\substack{|x-y|>e^{\frac{1}{\varepsilon}}|z|\\ |x-y|\le t,|x+z-y|\ge t}}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x+z)-b(y))f(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}\\
&\le C\int_{|x-y|>e^{\frac{1}{\varepsilon}}|z|}\frac{|f(y)||\Omega(x-y)|}{|x-y|^{n-1}}\left\{\int_{\substack{|x-y|\le t\\ |x+z-y|\ge t}}\frac{dt}{t^3}\right\}^{1/2}dy\\
&\le C\int_{|x-y|>e^{\frac{1}{\varepsilon}}|z|}\frac{|z|^{1/2}|\Omega(x-y)|}{|x-y|^{n+1/2}}|f(y)|\,dy.
\end{aligned}\tag{3.12}
$$

Since $\Omega\in L^1(S^{n-1})$, for every $s\in\mathbb{R}^n$ and $r>0$ we get

$$
\begin{aligned}
&\left\{\frac{1}{r^\lambda}\int_{B(s,r)}\left(\int_0^\infty |J_1(x,t)|^2\frac{dt}{t^3}\right)^{p/2}dx\right\}^{\frac{1}{p}}\\
&\le C\left\{\frac{1}{r^\lambda}\int_{B(s,r)}\left(\int_{|y|>e^{\frac{1}{\varepsilon}}|z|}\frac{|z|^{1/2}|\Omega(y)|}{|y|^{n+1/2}}|f(x-y)|\,dy\right)^p dx\right\}^{1/p}\\
&\le C\|f\|_{p,\lambda}\int_{|y|>e^{\frac{1}{\varepsilon}}|z|}\frac{|z|^{1/2}|\Omega(y)|}{|y|^{n+1/2}}dy\\
&=C\|f\|_{p,\lambda}|z|^{1/2}\int_{e^{\frac{1}{\varepsilon}}|z|}^\infty\frac{dr}{r^{1+1/2}}\int_{S^{n-1}}|\Omega(y')|\,d\sigma(y')\\
&\le Ce^{-\frac{1}{2\varepsilon}}\|f\|_{p,\lambda}\\
&\le CD\varepsilon.
\end{aligned}
$$

Hence

$$(3.13)\qquad \left\|\left\{\int_0^\infty |J_1(\cdot,t)|^2\frac{dt}{t^3}\right\}^{1/2}\right\|_{p,\lambda} \le CD\varepsilon.$$

Similar to the estimate of $J_1(x,t)$, we can get

$$\left\{\int_0^\infty |J_2(x,t)|^2\frac{dt}{t^3}\right\}^{1/2} \le C\int_{|x-y|>e^{\frac{1}{\varepsilon}}|z|}\frac{|z|^{1/2}|\Omega(x+z-y)|}{|x+z-y|^{n+1/2}}|f(y)|\,dy.$$

Thus for any $s\in\mathbb{R}^n$ and $r>0$, we have

$$\begin{aligned}
&\left\{\frac{1}{r^\lambda}\int_{B(s,r)}\left(\int_0^\infty |J_2(x,t)|^2\frac{dt}{t^3}\right)^{p/2}dx\right\}^{\frac{1}{p}}\\
&\le C\left\{\frac{1}{r^\lambda}\int_{B(s,r)}\left(\int_{|y|>(e^{\frac{1}{\varepsilon}}-1)|z|}\frac{|z|^{1/2}|\Omega(y)|}{|y|^{n+1/2}}|f(x+z-y)|\,dy\right)^p dx\right\}^{1/p}\\
&\le C\|f\|_{p,\lambda}|z|^{1/2}\int_{|y|>(e^{\frac{1}{\varepsilon}}-1)|z|}\frac{|\Omega(y)|}{|y|^{n+1/2}}\,dy\\
&\le C(e^{\frac{1}{\varepsilon}}-1)^{-1/2}\|f\|_{p,\lambda}\\
&\le CD\varepsilon.
\end{aligned}$$

Therefore

$$(3.14)\qquad \left\|\left\{\int_0^\infty |J_2(\cdot,t)|^2\frac{dt}{t^3}\right\}^{1/2}\right\|_{p,\lambda} \le CD\varepsilon.$$

About J_3. By the Minkowski inequality and $|b(x+z)-b(y)|<C$, we have

$$\begin{aligned}
&\left\{\int_0^\infty |J_3(x,t)|^2\frac{dt}{t^3}\right\}^{1/2}\\
&=\left\{\int_0^\infty\left|\int_{\substack{|x-y|>e^{\frac{1}{\varepsilon}}|z|,|x-y|\le t\\ |x+z-y|\le t}}\left(\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}\right)\right.\right.\\
&\qquad\left.\left.(b(x+z)-b(y))f(y)\,dy\right|^2\frac{dt}{t^3}\right\}^{1/2}\\
&\le C\int_{|x-y|>e^{\frac{1}{\varepsilon}}|z|}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}\right||f(y)|\left\{\int_{\substack{|x-y|\le t\\ |x+z-y|\le t}}\frac{dt}{t^3}\right\}^{1/2}dy\\
&\le C\int_{|x-y|>e^{\frac{1}{\varepsilon}}|z|}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}\right|\frac{|f(y)|}{|x-y|}\,dy.
\end{aligned}$$

Using Lemma 3.2, we get

$$
\begin{aligned}
&\left\{\frac{1}{r^{\lambda}}\int_{B(s,r)}\left(\int_0^{\infty}|J_3(x,t)|^2\frac{dt}{t^3}\right)^{p/2}dx\right\}^{\frac{1}{p}}\\
&\le C\left\{\frac{1}{r^{\lambda}}\int_{B(s,r)}\left(\int_{|x-y|>e^{\frac{1}{\varepsilon}}|z|}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x+z-y)}{|x+z-y|^{n-1}}\right|\frac{|f(y)|}{|x-y|}\,dy\right)^p dx\right\}^{1/p}\\
&\le C\|f\|_{p,\lambda}\int_{|y|>e^{\frac{1}{\varepsilon}}|z|}\left|\frac{\Omega(y)}{|y|^{n-1}}-\frac{\Omega(y+z)}{|y+z|^{n-1}}\right|\frac{1}{|y|}\,dy\\
&\le C\|f\|_{p,\lambda}\sum_{k=0}^{\infty}\int_{2^k e^{\frac{1}{\varepsilon}}|z|\le|y|<2^{k+1}e^{\frac{1}{\varepsilon}}|z|}\left|\frac{\Omega(y)}{|y|^{n-1}}-\frac{\Omega(y+z)}{|y+z|^{n-1}}\right|\frac{1}{|y|}\,dy\\
&\le C\|f\|_{p,\lambda}\sum_{k=0}^{\infty}\left\{\frac{|z|}{2^k e^{\frac{1}{\varepsilon}}|z|}+\int_{\frac{|z|}{2^{k+1}e^{\frac{1}{\varepsilon}}|z|}}^{\frac{|z|}{2^k e^{\frac{1}{\varepsilon}}|z|}}\frac{\omega(\delta)}{\delta}d\delta\right\}\\
&\le C\|f\|_{p,\lambda}\sum_{k=0}^{\infty}\left\{\frac{1}{2^k e^{\frac{1}{\varepsilon}}}+\frac{1}{1+k+1/\varepsilon}\int_{\frac{1}{2^{k+1}e^{\frac{1}{\varepsilon}}}}^{\frac{1}{2^k e^{\frac{1}{\varepsilon}}}}\frac{\omega(\delta)}{\delta}(1+|\log\delta|)\,d\delta\right\}\\
&\le C(e^{-\frac{1}{\varepsilon}}+\varepsilon)\|f\|_{p,\lambda}\\
&\le CD\varepsilon.
\end{aligned}
$$

Thus

$$
\left\|\left\{\int_0^{\infty}|J_3(\cdot,t)|^2\frac{dt}{t^3}\right\}^{1/2}\right\|_{p,\lambda}\le CD\varepsilon. \tag{3.15}
$$

Now we give the estimate of J_4. Since $b\in C_c^{\infty}(\mathbb{R}^n)$, we have $|b(x)-b(x+z)|\le C|z|$. If set $\eta=e^{\frac{1}{\varepsilon}}|z|$ and

$$
\mu_{\Omega,\eta}(f)(x)=\left\{\int_0^{\infty}\left|\int_{\substack{|x-y|>\eta\\|x-y|\le t}}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)\,dy\right|^2\frac{dt}{t^3}\right\}^{1/2},
$$

then

$$
\left(\int_0^{\infty}|J_4(x,t)|^2\frac{dt}{t^3}\right)^{1/2}\le C|z|\mu_{\Omega,\eta}(f)(x).
$$

We now claim that

$$
\|\mu_{\Omega,\eta}(f)\|_{p,\lambda}\le C\|f\|_{p,\lambda}\,,\quad 1<p<\infty, \tag{3.16}
$$

where C is independent of η and f. In fact, if B is a ball center at $x\in\mathbb{R}^n$ and of radius $\eta/2$. Let $f_1(y)=f_{\chi_{2B}}(y)$ and $f_2(y)=f(y)-f_1(y)$, then

$$\begin{aligned}\mu_{\Omega,\eta}(f)(x) &\le \frac{1}{|B|}\int_B |\mu_\Omega(f)(\xi)|\,d\xi + \frac{1}{|B|}\int_B |\mu_\Omega(f_1)(\xi)|\,d\xi \\ &\quad + \frac{1}{|B|}\int_B |\mu_\Omega(f_2)(\xi) - \mu_{\Omega,\eta} f(x)|\,d\xi \\ &\le \mathcal{M}(\mu_\Omega(f))(x) + I(f)(x) + II(f)(x).\end{aligned} \tag{3.17}$$

By (2.6) and Lemma 3.3, we can get

$$\|\mathcal{M}(\mu_\Omega(f))\|_{p,\lambda} \le C\|\mu_\Omega(f)\|_{p,\lambda} \le C\|f\|_{p,\lambda}.$$

Applying Theorem A, for any $1 < u < \infty$

$$I(f)(x) \le \frac{C}{|B|^{1/u}}\|\mu_\Omega(f_1)\|_u \le \frac{C}{|B|^{1/u}}\|f_1\|_u \le C(\mathcal{M}(|f|^u)(x))^{1/u}.$$

Taking $1 < u < p$ and using Lemma 3.3 again, we have

$$\|I(f)\|_{p,\lambda} \le C\|f\|_{p,\lambda}.$$

Regarding $II(f)(x)$, let $\xi \in B$. By the Minkowski inequality, we have

$$\begin{aligned}&|\mu_\Omega(f_2)(\xi) - \mu_{\Omega,\eta}(f)(x)| \\ &\le \left\{\int_0^\infty \left|\int_{\substack{|\xi-y|\ge t\\ |x-y|\le t}} \frac{\Omega(x-y)}{|x-y|^{n-1}}\, f_2(y)\,dy\right|^2 \frac{dt}{t^3}\right\}^{1/2} \\ &\quad + \left\{\int_0^\infty \left|\int_{\substack{|\xi-y|\le t\\ |x-y|\ge t}} \frac{\Omega(\xi-y)}{|\xi-y|^{n-1}}\, f_2(y)\,dy\right|^2 \frac{dt}{t^3}\right\}^{1/2} \\ &\quad + \left\{\int_0^\infty \left|\int_{\substack{|\xi-y|\le t\\ |x-y|\le t}} \left(\frac{\Omega(\xi-y)}{|\xi-y|^{n-1}} - \frac{\Omega(x-y)}{|x-y|^{n-1}}\right) f_2(y)\,dy\right|^2 \frac{dt}{t^3}\right\}^{1/2} \\ &:= H_1(\xi,x) + H_2(\xi,x) + H_3(\xi,x).\end{aligned}$$

Since $\xi \in B,\ y \in (2B)^c$, similar to the estimate of (3.12), we may get

$$\frac{1}{|B|}\int_B H_1(\xi,x)\,d\xi \le C\eta^{1/2}\int_{(2B)^c} \frac{|f(y)||\Omega(x-y)|}{|x-y|^{n+1/2}}\ dy.$$

By the Minkowski inequality, for $s \in \mathbb{R}^n$ and $r > 0$, we have

$$\begin{aligned}&\left\{\frac{1}{r^\lambda}\int_{B(s,r)} \left|\frac{1}{|B|}\int_B H_1(\xi,x)\,d\xi\right|^p dx\right\}^{\frac{1}{p}} \\ &\le C\eta^{1/2}\left\{\frac{1}{r^\lambda}\int_{B(s,r)} \left(\int_{|y|>\eta} \frac{|f(x-y)||\Omega(y)|}{|y|^{n+1/2}}\ dy\right)^p dx\right\}^{\frac{1}{p}} \\ &\le C\eta^{1/2}\|f\|_{p,\lambda}\int_{|y|>\eta} \frac{|\Omega(y)|}{|y|^{n+1/2}}\ dy \\ &\le C\|f\|_{p,\lambda}.\end{aligned}$$

Thus

$$\left\|\frac{1}{|B|}\int_B H_1(\xi,\cdot)\,d\xi\right\|_{p,\lambda} \le C\|f\|_{p,\lambda}.$$

For $H_2(\xi,x)$, we can get

$$H_2(\xi,x) \le C\eta^{1/2}\int_{(2B)^c}\frac{|f(y)||\Omega(\xi-y)|}{|\xi-y|^{n+1/2}}\,dy.$$

Then

$$\begin{aligned}\frac{1}{|B|}\int_B H_2(\xi,x)\,d\xi &\le C\eta^{1/2}\frac{1}{|B|}\int_B\int_{(2B)^c}\frac{|f(y)||\Omega(\xi-y)|}{|\xi-y|^{n+1/2}}\,dy\,d\xi\\ &= C\eta^{1/2}\sum_{k=1}^{\infty}\frac{1}{|B|}\int_B\int_{2^{k+1}B\setminus 2^kB}\frac{|f(y)||\Omega(\xi-y)|}{|\xi-y|^{n+1/2}}\,dy\,d\xi\\ &\le C\mathcal{M}(|f|)(x).\end{aligned}$$

Thus, by Lemma 3.3, we have

$$\left\|\frac{1}{|B|}\int_B H_2(\xi,\cdot)\,d\xi\right\|_{p,\lambda} \le C\|f\|_{p,\lambda}.$$

Since $\xi\in B$, we get

$$\begin{aligned}&\frac{1}{|B|}\int_B H_3(\xi,x)\,d\xi\\ &\le \frac{1}{|B|}\int_B\int_{\mathbb{R}^n}\left|\frac{\Omega(\xi-y)}{|\xi-y|^{n-1}}-\frac{\Omega(x-y)}{|x-y|^{n-1}}\right|\frac{|f_2(y)|}{|x-y|}\,dy\,d\xi\\ &= \frac{1}{|B(0,\eta/2)|}\int_{B(0,\eta/2)}\int_{|y-x|>\eta}\left|\frac{\Omega(x-\xi-y)}{|x-\xi-y|^{n-1}}-\frac{\Omega(x-y)}{|x-y|^{n-1}}\right|\frac{|f(y)|}{|x-y|}\,dy\,d\xi\\ &= \frac{1}{|B(0,\eta/2)|}\int_{B(0,\eta/2)}\int_{|y|>\eta}\left|\frac{\Omega(y-\xi)}{|y-\xi|^{n-1}}-\frac{\Omega(y)}{|y|^{n-1}}\right|\frac{|f(x-y)|}{|y|}\,dy\,d\xi.\end{aligned}$$

By the Minkowski inequality and Lemma 3.2, for every $s\in\mathbb{R}^n$ and $r>0$, we get

$$\begin{aligned}&\left\{\frac{1}{r^\lambda}\int_{B(s,r)}\left|\frac{1}{|B|}\int_B H_3(\xi,x)\,d\xi\right|^p dx\right\}^{\frac{1}{p}}\\ &\le C\|f\|_{p,\lambda}\frac{1}{|B(0,\eta/2)|}\int_{B(0,\eta/2)}\int_{|y|>\eta}\left|\frac{\Omega(y-\xi)}{|y-\xi|^{n-1}}-\frac{\Omega(y)}{|y|^{n-1}}\right|\frac{1}{|y|}\,dy\,d\xi\\ &= C\|f\|_{p,\lambda}\frac{1}{|B(0,\eta/2)|}\int_{B(0,\eta/2)}\sum_{k=1}^{\infty}\int_{2^{k-1}\eta<|y|<2^k\eta}\left|\frac{\Omega(y-\xi)}{|y-\xi|^{n-1}}-\frac{\Omega(y)}{|y|^{n-1}}\right|\frac{1}{|y|}\,dy\,d\xi\end{aligned}$$

$$\le C\|f\|_{p,\lambda}\frac{1}{|B(0,\eta/2)|}\int_{B(0,\eta/2)}\sum_{k=1}^{\infty}\left(\frac{|\xi|}{2^{k-1}\eta}+\int_{\frac{|\xi|}{2^k\eta}}^{\frac{|\xi|}{2^{k-1}\eta}}\frac{\omega(\delta)}{\delta}\,d\delta\right)d\xi$$
$$\le C\|f\|_{p,\lambda}\frac{1}{|B(0,\eta/2)|}\int_{B(0,\eta/2)}\left(1+\int_0^1\frac{\omega(\delta)}{\delta}\,d\delta\right)d\xi$$
$$\le C\|f\|_{p,\lambda}.$$

Thus

$$\left\|\frac{1}{|B|}\int_B H_3(\xi,\cdot)\,d\xi\right\|_{p,\lambda}\le C\|f\|_{p,\lambda}.$$

Therefore

$$\|II(f)\|_{p,\lambda}\le C\|f\|_{p,\lambda}.$$

Summing up $\|M(\mu_\Omega f)\|_{p,\lambda}$, $\|I(f)\|_{p,\lambda}$, and $\|II(f)\|_{p,\lambda}$, by (3.17), we get (3.16). Then

$$\left\|\left\{\int_0^\infty|J_4(\cdot,t)|^2\frac{dt}{t^3}\right\}^{1/2}\right\|_{p,\lambda}\le C|z|\|f\|_{p,\lambda}\le CD|z|. \tag{3.18}$$

About J_5, since $|b(x)-b(y)|\le C|x-y|$, by the Minkowski inequality, we get

$$\left\{\int_0^\infty|J_5(x,t)|^2\frac{dt}{t^3}\right\}^{1/2}$$
$$\le C\int_{|x-y|\le e^{\frac{1}{\varepsilon}}|z|}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\,|b(x)-b(y)||f(y)|\left\{\int_{|x-y|\le t}\frac{dt}{t^3}\right\}^{1/2}dy$$
$$\le C\int_{|x-y|\le e^{\frac{1}{\varepsilon}}|z|}\frac{|f(y)||\Omega(x-y)|}{|x-y|^{n-1}}\,dy.$$

Then by the Minkowski inequality and $\Omega\in L^1(S^{n-1})$, for every $s\in\mathbb{R}^n$ and $r>0$, we get

$$\left\{\frac{1}{r^\lambda}\int_{B(s,r)}\left(\int_0^\infty|J_5(x,t)|^2\frac{dt}{t^3}\right)^{p/2}dx\right\}^{\frac{1}{p}}\le CDe^{\frac{1}{\varepsilon}}|z|.$$

Thus

$$\left\|\left\{\int_0^\infty|J_5(\cdot,t)|^2\frac{dt}{t^3}\right\}^{1/2}\right\|_{p,\lambda}\le CDe^{\frac{1}{\varepsilon}}|z|. \tag{3.19}$$

Similarly, using the estimate $|b(x+z)-b(y)|\le C|x+z-y|$, we have

$$\left\{\int_0^\infty |J_6(x,t)|^2\frac{dt}{t^3}\right\}^{1/2} \le C\int_{|x-y|\le e^{\frac{1}{\varepsilon}}|z|}\frac{|f(y)||\Omega(x+z-y)|}{|x+z-y|^{n-1}}\,dy.$$

Then

$$\left\|\left\{\int_0^\infty |J_6(\cdot,t)|^2\frac{dt}{t^3}\right\}^{1/2}\right\|_{p,\lambda} \le CD(e^{\frac{1}{\varepsilon}}|z|+|z|). \tag{3.20}$$

Hence, for any $\varepsilon > 0$, we may take $|z|$ to be small sufficiently, then by (3.10), (3.11), (3.13)-(3.15) and (3.18)-(3.20), we have

$$\|[b,\mu_\Omega]f(\cdot+z)-[b,\mu_\Omega]f(\cdot)\|_{p,\lambda} \le C\varepsilon.$$

Therefore, we show that (3.2) holds for the commutator $[b,\mu_\Omega]$ in G uniformly and complete the proof of Theorem 2.

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