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ON SOME SEMILINEAR ELLIPTIC PROBLEMS WITH SINGULAR POTENTIALS INVOLVING SYMMETRY

Hoang Quoc Toan and Nguyen Thanh Chung

Abstract. This paper deals with the existence and multiplicity of solutions for a class of semilinear elliptic problems of the form

$$\begin{cases} -\Delta u &= \dfrac{\mu}{|x|^2}u + f(x,u) \quad \text{in} \quad \Omega, \\ \quad u &= 0 \qquad\qquad\qquad\quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^N$ $(N \geqq 5)$ is a bounded domain having cylindrical symmetry, $\Omega_1 \subset \mathbb{R}^m$ is a bounded regular domain and Ω_2 is a $k-$dimensional ball of radius R, centered in the origin and $m + k = N$, and $m \geqq 2$, $k \geqq 3$, $0 \leqq \mu < \mu^\star = \left(\frac{N-2}{2}\right)^2$. The proofs rely essentially on the critical point theory tools combined with the Hardy inequality.

1. Introduction and Premilinaries

In this paper, we are concerned with the semilinear elliptic problems with singular potentials of the form

$$\begin{cases} -\Delta u = \dfrac{\mu}{|x|^2}u + f(x,u) \quad \text{in} \quad \Omega, \\ \quad u = 0 \qquad\qquad\qquad\quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqq 3)$ is a bounded domain containing the origin with smooth boundary $\partial\Omega$ and μ is a parameter. Such problems were intensively studied in many works, in which the authors were usually interested in the singular critical growth case (see [2, 9] and [15]) or combining a critical Sobolev-Hardy exponent with a Hardy-type term (see [11] and [12]). In [13], A. Kristály et al. considered problem

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(1.1) in the case $f(x,u) = f(u)$ is sublinear at infinity. Then they obtained at least two nontrivial weak solutions using a recent result by G. Bonanno [4]. We also find that in [7], N. T. Chung studied problem (1.1) under condition that $f(x,u)$ is a sign-changing Carathéodory function. Using the Minimum principle combined with the Mountain pass theorem by A. Ambrosetti and P.H. Rabinowitz [1] he gave some existence and nonexistence results for (1.1).

In the present paper, we shall investigate problem (1.1) in the case when the domain $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^N$ ($N \geqq 5$), $\Omega_1 \subset \mathbb{R}^m$ ($m \geqq 2$) is a bounded regular domain and Ω_2 is a $k-$dimensional ball of radius R ($k \geqq 3$), centered in the origin and $m + k = N$, the parameter μ satisfies the condition $0 \leqq \mu < \mu^\star$ with $\mu^\star = \left(\frac{N-2}{2}\right)^2$ is the best constant in the Hardy inequality, i.e.

$$\int_\Omega \frac{|u|^2}{|x|^2} dx \leqq \frac{1}{\mu^\star} \int_\Omega |\nabla u|^2 dx \tag{1.2}$$

for all $u \in H_0^1(\Omega)$. Our paper is motivated by the interesting results in [5], [6] and [10], in which the authors studied the effect of the topology of the domain on the existence of nodal solutions of elliptic problems with critical nonlinearity in a symmetric domain, but the key in our arguments is a compactness result due to W. Wang [18] (see Lemma 1.5).

Let $H_0^1(\Omega)$ be the usual Sobolev space under the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$ and define the subspace $H_{0,s}^1(\Omega)$ by

$$H_{0,s}^1(\Omega) = \left\{ u \in H_0^1(\Omega) : u(x_1, x_2) = u(x_1, |x_2|), \forall x = (x_1, x_2) \in \Omega \right\},$$

which is a closed subspace $H_0^1(\Omega)$.

First, we consider problem (1.1) in the case when $f(x,u) = h(x)|u|^{q-2}u$ with $h(x) = |x_2|^l$ for all $x = (x_1, x_2) \in \Omega_1 \times \Omega_2$, and l is a positive real number. Then the problem becomes

$$\begin{cases} -\Delta u = \dfrac{\mu}{|x|^2} u + h(x)|u|^{q-2}u & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

Throughout this paper we always assume that the constant q verifies the following inequalities

$$2 < q < 2^\star + \tau, \quad 2^\star = \frac{2N}{N-2}, \quad \tau = \frac{2}{N-2} \min\{\frac{2(k-2)}{m}, l\}. \tag{1.4}$$

This says that the problem considered here contains the subcritical, critical and supercritical cases. So, the results of our paper are better than those in [5], [6] and [10]. If $\mu = 0$ then problem (1.3) is a model of Hénon equation in a cylindrically symmetric domain. Regarding the Hénon equations, we refer the readers to some works [3], [16] and [17].

Definition 1.1. We say that a function $u \in H^1_{0,s}(\Omega)$ is a weak solution of problem (1.3) if and only if

$$\int_\Omega \nabla u \cdot \nabla \varphi dx - \int_\Omega \frac{\mu}{|x|^2} u\varphi dx - \int_\Omega h(x)|u|^{q-2}u\varphi dx = 0$$

for all $\varphi \in H^1_{0,s}(\Omega)$.

The first result of ours is given by the following theorem.

Theorem 1.2. *Assume that relation (1.4) is satisfied, then problem (1.3) has at least one nontrivial weak solution, provided that* $0 \leqq \mu < \mu^\star$.

Next, a natural question is to see what happens if the above problem is affected by a certain perturbation. For this purpose, we shall consider the perturbed problem for (1.3), in which $f(x,u) = h(x)|u|^{q-2}u + g(x)$, i.e.

$$\begin{cases} -\Delta u &= \frac{\mu}{|x|^2}u + h(x)|u|^{q-2}u + g(x) \quad \text{in} \quad \Omega, \\ \quad u &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

where g is a function which belongs to the dual space of $H^1_{0,s}(\Omega)$, denoted by $H^{-1}_{0,s}(\Omega)$.

Definition 1.3. We say that a function $u \in H^1_{0,s}(\Omega)$ is a weak solution of problem (1.5) if and only if

$$\int_\Omega \nabla u \cdot \nabla \varphi dx - \int_\Omega \frac{\mu}{|x|^2} u\varphi dx - \int_\Omega h(x)|u|^{q-2}u\varphi dx - \int_\Omega g(x)\varphi dx = 0$$

for all $\varphi \in H^1_{0,s}(\Omega)$.

We obtain a multiplicity result for problem (1.5) as follows.

Theorem 1.4. *Assume that relation (1.4) is satisfied, then for any* $0 \leqq \mu < \mu^\star$*, there exists a constant* $\overline{\epsilon}_\mu > 0$ *depending on* μ*, such that for any* $g \in H^{-1}_{0,s}(\Omega)$ *with* $0 < \|g\|_{-1} < \overline{\epsilon}_\mu$*, problem (1.5) has at least two nontrivial weak solutions. Moreover, we have* $\overline{\epsilon}_\mu \to 0$ *as* $\mu \to \mu^\star$.

Due to the special geometry of the domain Ω in our problems, in order to prove the above theorems, we need to recall a recent compactness result by W. Wang [18] which is stated in the following lemma.

Lemma 1.5. (see [18, Theorem 2.4]). *Assume that* $\Omega = \Omega_1 \times \Omega_2$ *and dim*$(\Omega_1) = m \geqq 2$, *dim*$(\Omega_2) = k \geqq 3$, *then the embedding*

$$H^1_{0,s}(\Omega) \hookrightarrow L^q(h,\Omega), \quad q \in (1, 2^\star + \tau), \quad \tau = \frac{2}{N-2}\min\{\frac{2(k-2)}{m}, l\} \tag{1.6}$$

is compact, where $L^q(h,\Omega)$ *is the usual Lebesgue's space with weighted* h.

2. Proof of Main Results

In our arguments, the proof of Theorem 1.4 contains the existence result which is stated as in Theorem 1.2. So, for the sake of brevity, we will deal with only problem (1.5) in detail. We first define the functional $J : H^1_{0,s}(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega \left[|\nabla u|^2 - \frac{\mu}{|x|^2}|u|^2\right] dx - \frac{1}{q}\int_\Omega h(x)|u|^q dx - \int_\Omega g(x) u dx, \tag{2.1}$$

for all $u \in H^1_{0,s}(\Omega)$.

By Lemma 1.5, a simple computation implies that the functional J is well-defined and of C^1 class on $H^1_{0,s}(\Omega)$ and we have

$$DJ(u)(\varphi) = \int_\Omega \left[\nabla u \cdot \nabla \varphi - \frac{\mu}{|x|^2}u\varphi\right] dx - \int_\Omega h(x)|u|^{q-2}u\varphi dx - \int_\Omega g(x)\varphi dx$$

for all $u, \varphi \in H^1_{0,s}(\Omega)$. Thus, weak solutions of problem (1.5) are exactly the critical points of the functional J. The following lemma shows that the functional J has the geometry of the Mountain pass theorem [1].

Lemma 2.1.

(i) *For any* $\mu \in [0, \mu^\star)$, *there exist* $\overline{\epsilon}_\mu > 0$ *(depends on* μ*),* $\eta > 0$, *and* $\alpha > 0$ *such that* $J(u) \geqq \alpha$ *for all* $u \in H^1_{0,s}(\Omega)$ *with* $\|u\| = \eta$, *provided that* $0 < \|g\|_{-1} < \overline{\epsilon}_\mu$. *Moreover, we have* $\overline{\epsilon}_\mu \to 0$ *as* $\mu \to \mu^\star$.

(ii) *For* $\eta > 0$ *as in* (i), *there exists* $e \in H^1_{0,s}(\Omega)$ *such that* $\|e\| > \eta$ *and* $J(e) < 0$.

Proof. (**i**) For any $\epsilon > 0$, using Young's inequality we deduce that

$$\left|\int_\Omega g(x) u dx\right| \leqq \|g\|_{-1}\|u\| \leqq \frac{\epsilon^2}{2}\|u\|^2 + \frac{1}{2\epsilon^2}\|g\|^2_{-1},$$

for all $u \in H^1_{0,s}(\Omega)$.

Hence, by Lemma 1.5 and the Hardy inequality (1.2), it follows that for any $\epsilon > 0$ and for all $u \in H^1_{0,s}(\Omega)$ we obtain that

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega \left[|\nabla u|^2 - \frac{\mu}{|x|^2}|u|^2\right] dx - \frac{1}{q}\int_\Omega h(x)|u|^q dx - \int_\Omega g(x)u dx \\ &\geqq \left(\frac{1}{2} - \frac{\mu}{2\mu^\star}\right)\|u\|^2 - \frac{1}{qC_q^q}\|u\|^q - \frac{\epsilon^2}{2}\|u\|^2 - \frac{1}{2\epsilon^2}\|g\|^2_{-1} \qquad (2.2) \\ &= \left(\frac{\mu^\star(1-\epsilon^2)-\mu}{2\mu^\star} - \frac{1}{qC_q^q}\|u\|^{q-2}\right)\|u\|^2 - \frac{1}{2\epsilon^2}\|g\|^2_{-1} \end{aligned}$$

where C_q is the best constant in the embedding $H^1_{0,s}(\Omega) \hookrightarrow L^q(h,\Omega)$ and $\mu^\star = \left(\frac{N-2}{2}\right)^2$ is the best constant in the Hardy inequality.

For each $\mu \in [0,\mu^\star)$, by fixing $\epsilon \in \left(0, \sqrt{1-\frac{\mu}{\mu^\star}}\right)$, we can find $\eta > 0$, $\overline{\epsilon}_\mu > 0$ and $\alpha > 0$ such that the conclusion of the lemma holds true. For example, we can take

$$\eta = \left(MqC_q^q\right)^{\frac{1}{q-2}}, \quad \overline{\epsilon}_\mu = \epsilon M^{\frac{q}{2(q-2)}}\left(qC_q^q\right)^{\frac{1}{q-2}}, \quad \alpha = \frac{1}{2}M^{\frac{q}{q-2}}(qC_q^q)^{\frac{2}{q-2}},$$

where

$$M = \frac{\mu^\star(1-\epsilon^2)-\mu}{4\mu^\star} > 0.$$

Now, let $\mu \to \mu^\star = \left(\frac{N-2}{2}\right)^2$ we deduce since $\epsilon \in \left(0, \sqrt{1-\frac{\mu}{\mu^\star}}\right)$ that $\epsilon \to 0$ and then $\overline{\epsilon}_\mu \to 0$.

(**ii**) Let $\varphi_0 \in C_0^\infty(\Omega) \cap H^1_{0,s}(\Omega)$ such that $\int_\Omega h(x)|\varphi_0(x)|^q dx > 0$. Then for any $t > 0$ and for all $0 \leqq \mu < \mu^\star$ we obtain that

$$\begin{aligned} J(t\varphi_0) &= \frac{t^2}{2}\int_\Omega |\nabla\varphi_0|^2 dx - \mu\frac{t^2}{2}\int_\Omega \frac{|\varphi_0|^2}{|x|^2}dx - \frac{t^q}{q}\int_\Omega h(x)|\varphi_0|^q dx - t\int_\Omega g(x)\varphi_0 dx \\ &\leqq \frac{t^2}{2}\int_\Omega |\nabla w_0|^2 dx - \frac{t^q}{q}\int_\Omega h(x)|\varphi_0|^q dx - t\int_\Omega g(x)\varphi_0 dx, \end{aligned}$$

which approaches $-\infty$ as $t \to +\infty$ since $q > 2$. Thus, the lemma holds true. ∎

Lemma 2.2. *For any $\mu \in [0,\mu^\star)$, the functional J satisfies the Palais-Smale condition in $H^1_{0,s}(\Omega)$.*

Proof. Let $\{u_n\}$ be a Palais-Smale sequence for the functional J in $H^1_{0,s}(\Omega)$, i.e.

$$J(u_n) \to \overline{c}, \quad DJ(u_n) \to 0 \text{ in } H^{-1}_{0,s}(\Omega) \text{ as } n \to \infty. \qquad (2.3)$$

We shall prove that the sequence $\{u_n\}$ is bounded in $H^1_{0,s}(\Omega)$. Indeed, we assume by contradiction that $\|u_n\| \to \infty$ as $n \to \infty$. Then, using relation (2.3) we deduce that for n large enough the following inequalities hold

$$
\begin{aligned}
\overline{c} + 1 + \|u_n\| &\geqq J(u_n) - \frac{1}{q} DJ(u_n)(u_n) \\
&\geqq \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega |\nabla u_n|^2 dx - \left(\frac{\mu}{2} - \frac{\mu}{q}\right) \int_\Omega \frac{|u_n|^2}{|x|^2} dx \\
&\quad - \left(1 - \frac{1}{q}\right) \int_\Omega g(x) u_n dx \\
&\geqq \left(\frac{1}{2} - \frac{1}{q}\right) \cdot \left(1 - \frac{\mu}{\mu^\star}\right) \|u_n\|^2 - \left(1 - \frac{1}{q}\right) \|g\|_{-1} \|u_n\|.
\end{aligned} \tag{2.4}
$$

Dividing the above inequality by $\|u_n\|$ and letting $n \to \infty$ we obtain a contradiction since $q > 2$. This implies that the sequence $\{u_n\}$ is bounded in $H^1_{0,s}(\Omega)$.

Since $H^1_{0,s}(\Omega)$ is reflexive, there exists $u \in H^1_{0,s}(\Omega)$ such that, passing to a subsequence, still denoted by $\{u_n\}$, it converges weakly to u in $H^1_{0,s}(\Omega)$. Using the Hardy inequality 1.2 again, we have

$$
\begin{aligned}
\left(1 - \frac{\mu}{\mu^\star}\right) \|u_n - u\|^2 &\leqq \|u_n - u\|^2 - \mu \int_\Omega \frac{|u_n - u|^2}{|x|^2} dx \\
&= DJ(u_n)(u_n - u) + DJ(u)(u - u_n) \\
&\quad + \int_\Omega h(x)(|u_n|^{q-2}u - |u|^{q-2}u)(u_n - u) dx.
\end{aligned} \tag{2.5}
$$

By Lemma 1.5 we deduce that the sequence $\{u_n\}$ converges strongly to u in $L^q(h, \Omega)$. Combining this with the fact that

$$
\begin{aligned}
&\int_\Omega h(x)(|u_n|^{q-2}u - |u|^{q-2}u)(u_n - u) dx \\
\leqq &\int_\Omega h(x)(|u_n|^{q-1} + |u|^{q-1})|u_n - u| dx \\
= &\int_\Omega \left(h^{\frac{1}{q}}(x)|u_n|\right)^{q-1} h^{\frac{1}{q}}(x)|u_n - u| dx + \int_\Omega \left(h^{\frac{1}{q}}(x)|u|\right)^{q-1} h^{\frac{1}{q}}(x)|u_n - u| dx \\
\leqq &(\|u_n\|^{q-1}_{L^q(h,\Omega)} + \|u\|^{q-1}_{L^q(h,\Omega)}) \|u_n - u\|_{L^q(h,\Omega)}
\end{aligned}
$$

imply that

$$
\lim_{n \to \infty} \int_\Omega h(x)(|u_n|^{q-2}u - |u|^{q-2}u)(u_n - u) dx = 0. \tag{2.6}
$$

Using relations (2.3), (2.5) and (2.6) we conclude that the sequence $\{u_n\}$ converges strongly to u in $H^1_{0,s}(\Omega)$ and hence the functional J satisfies the Palais-Smale condition in $H^1_{0,s}(\Omega)$. ■

Proof. [Proof of Theorem 1.4 completed] By Lemmas 2.1 and 2.2, all assumptions of the Mountain pass theorem in [1] are satisfied, then we deduce the existence of $u_1 \in H^1_{0,s}(\Omega)$ as a non-trivial weak solution of (1.5) and $J(u_1) = \overline{c} > 0$. We now prove that there exists a second weak solution $u_2 \in H^1_{0,s}(\Omega)$ such that $u_2 \not\equiv u_1$.

For $\eta > 0$ is given as in Lemma 2.1, we define the number $\underline{c}$ by

$$\underline{c} := \inf\{J(u) : u \in H^1_{0,s}(\Omega) \text{ with } \|u\| \leqq \eta\}. \tag{2.7}$$

Then we have $\underline{c} < J(0) = 0$ since $g \not\equiv 0$. We denote by $\overline{B}_\eta(0)$ the closed ball of radius η centered at the origin in $H^1_{0,s}(\Omega)$, i.e.

$$\overline{B}_\eta(0) := \{u \in H^1_{0,s}(\Omega) : \|u\| \leqq \eta\},$$

it follows that the set $\overline{B}_\eta(0)$ is a complete metric space with respect to the distance dist$(u,v) := \|u - v\|$ for all $u, v \in \overline{B}_\eta(0)$.

On the other hand, with the similar arguments as those used in [14, Theorem 3.2] we conclude that for all $\mu \in [0, \mu^\star)$, the functional J is weakly lower semicontinuous in $H^1_{0,s}(\Omega)$ and bounded from below since relation (2.2) holds true.

Let ϵ be such that $0 < \epsilon < \inf_{\partial B_\eta(0)} J - \inf_{B_\eta(0)} J$. Applying Ekeland's variational principle [8] for the functional $J : \overline{B}_\eta(0) \to \mathbb{R}$, there exists a function $u_\epsilon \in \overline{B}_\eta(0)$ such that

$$\begin{aligned} &J(u_\epsilon) < \inf_{\overline{B}_\eta(0)} J + \epsilon, \\ &J(u_\epsilon) < J(u) + \epsilon.\|u - u_\epsilon\|, \quad u \neq u_\epsilon. \end{aligned}$$

Since

$$J(u_\epsilon) \leqq \inf_{\overline{B}_\eta(0)} J + \epsilon \leqq \inf_{B_\eta(0)} J + \epsilon < \inf_{\partial B_\eta(0)} J$$

it follows that $u_\epsilon \in B_\eta(0)$.

We now define the functional $K : \overline{B}_\eta(0) \to \mathbb{R}$ by $K(u) = J(u) + \epsilon\|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of K and thus,

$$\frac{K(u_\epsilon + t\varphi) - K(u_\epsilon)}{t} \geq 0 \tag{2.8}$$

for $t > 0$ small enough and $\varphi \in B_\eta(0)$. Relation (2.8) yields that

$$\frac{J(u_\epsilon + t\varphi) - J(u_\epsilon)}{t} + \epsilon\|\varphi\| \geqq 0. \tag{2.9}$$

It follows from (2.9) by letting $t \to 0$ that $DJ(u_\epsilon)(\varphi) + \epsilon\|\varphi\| > 0$ and we infer that $\|DJ(u_\epsilon)\|_{-1} \leqq \epsilon$.

From above information, we deduce that there exists a sequence $\{u_n\} \subset B_\eta(0)$ such that

$$J(u_n) \to \underline{c} \text{ and } DJ(u_n) \to 0 \text{ in } H^{-1}_{0,s}(\Omega) \text{ as } n \to \infty. \tag{2.10}$$

Using Lemma 2.2, we can show that $\{u_n\}$ converges strongly to some $u_2 \in H^1_{0,s}(\Omega)$. Thus, u_2 is a weak solution of (1.5) and u_2 is non-trivial since $J(u_2) = \underline{c} < 0$. Finally, since $J(u_1) = \overline{c} > 0 > \underline{c} = J(u_2)$ we have $u_2 \not\equiv u_1$. Theorem 1.4 is completely proved. ■

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Hoang Quoc Toan
Department of Mathematics
Hanoi University of Science
334 Nguyen Trai
Thanh Xuan, Hanoi
Vietnam
E-mail: hq_toan@yahoo.com

Nguyen Thanh Chung
Department of Mathematics and Informatics
Quang Binh University
312 Ly Thuong Kiet
Dong Hoi, Quang Binh
Vietnam
E-mail: ntchung82@yahoo.com