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CENSORING TECHNIQUE APPLIED TO A MAP/G/1 QUEUE WITH SET-UP TIME AND MULTIPLE VACATIONS

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Abstract. Infinite state Markov chains with block-structured transition matrix find extensive applications in many areas, including telecommunications, and queueing systems, for example. In this paper, we use the censoring technique to study a MAP/G/1 queue with set-up time and multiple vacations, whose transition matrix can be transformed to a block-Toeplitz or block-repeating structured. Hence, we are able to relate the boundary conditions of the system to a Markov chain of M/G/1 type. This leads to a solution of the boundary equations, which is crucial for solving the system of differential equations. We also provide expressions for the distribution of stationary queue length, virtual waiting time and the busy period, respectively.

1. Introduction

There exists a large volume of references on Markov chains with repeating transition blocks, in which the censoring technique has always been used to study various aspects of Markov chain. For example, see Grassman and Heyman [4], Latouche and Ramaswami [7], Zhao et al. [18], or Zhao and Liu [19] and the references therein. References on the censoring technique are also plenty. Among them are Grassmann and Heyman [4, 5], Latouche [6], Latouche and Ramaswami [7], Zhao, Li and Braun [18], and Zhao, Li and Alfa [17].

In the following, let us have a review of Censoring.

Definition 1. [16]. Consider a discrete-time irreducible Markov chain $\{X_n; n = 1, 2, \cdots\}$ with state space S. Let E be a non-empty subset of S. Suppose that the successive visits of X_n to E take place at time epochs $0 < n_1 < n_2 < \cdots$. Then the process $\{X_t^E = X_{n_t}; t = 1, 2, \cdots\}$ is called the censored process with censoring set E.

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Censored Markov chains are also called the restricted, or embedded Markov chain. Using the strong Markov property, it can be proved that the censored process is also a Markov chain, called the censored Markov chain.

In the present paper, we aim to discuss a queueing system with Markovian arrival process by using censoring technique. Generally, most of the analysis on queueing in the past have been carried out assuming Poisson input. However, in many applications arrival process need not be Poisson, e.g. in modern communication systems and Asynchronous Transfer Mode (ATM) networks, arrivals to a statistical multiplexer are correlated. Then Lucantoni [11] first introduced the Markovian Arrival Process (MAP), which is a representative of correlated arrivals and includes many familiar input processes such as Markov modulated Poisson process (MMPP), PH-type renewal process, Poisson process etc.. Readers may refer to Latouche and Ramaswami [7], Lee and Jeon [8], Li and Li [10], Lucantoni [11, 12], Chapter 5 in Neuts [13], Ramaswami [14] and references therein for a detailed introduction of the MAPs.

Furthermore, queuing systems with vacations have been studied extensively recent years, because of its important application to many areas including computer-communications and manufacturing systems. For a detailed survey on queueing systems with server vacations, one can refer to References [1-3, 6, 9, 15].

The model under consideration here is described by a MAP/G/1 queue with set-up time and multiple vacations, whose transition matrices can be transformed to a block-Toeplitz or block-repeating structured. Hence, we are able to establish a connection between the solution of the boundary equations and the solution of a Markov chain of M/G/1 type. Moreover, since the solution of the Markov chain of M/G/1 type can be expressed in terms of the censoring technique and the RG-factorization, this connection makes the supplementary variable method possible for solving a type of queueing models from which the boundary equations may be complicated to be solved.

The rest of this paper is organized as follows. The model description is given in Section 2. The stationary differential equations of the model and their solutions are obtained in Section 3. The expressions for the distributions of the stationary queue length, virtual waiting time and the busy period are derived in Section 4.

2. Model Description

In this paper, we consider a single server queue with Markovian arrival of customers.

The arrival process. We assume that the arrivals of customers are MAPs with matrix descriptors (C, D) and, where the infinitesimal generator $C + D$ of sizes $m \times m$, is irreducible and positive recurrent. Let θ be the stationary probability vector of $C + D$. Then $\lambda = \theta De$ is the stationary arrival rate of customer, where e

is a column vector of ones of a suitable size.

The vacations. When the server finishes serving a customer and finds the queue empty, the server leaves for a vacation of random length V. On return from a vacation if he finds more than one customer waiting, he takes the customer from the head of the queue for service and continue to serve in this manner until the queue is empty. Otherwise, he immediately goes for another vacation. The vacation time distribution is given by

$$B_0(x) = 1 - \exp\left\{-\int_0^x \mu_0(t)\,dt\right\},$$

with mean $1/\mu_0 \in (0, +\infty)$.

The set-up time. We assume that there's a set-up time of random length U before the service of the first customer during a busy period. The set-up time distribution is given by

$$B_1(x) = 1 - \exp\left\{-\int_0^x \mu_1(t)\,dt\right\},$$

with mean $1/\mu_1 \in (0, +\infty)$.

The service time. All customers have i.i.d. service time of random length S. The service time distribution is given by

$$B_2(x) = 1 - \exp\left\{-\int_0^x \mu_2(t)\,dt\right\},$$

with mean $1/\mu_2 \in (0, +\infty)$.

The independence. We assume that all the random variables defined above are independent.

Throughout the rest of the paper, we denote by $\overline{F}(x) = 1 - F(x)$ the tail of distribution function $F(x)$. Let $V^{(E)}, U^{(E)}$ and $S^{(E)}$ denote the residual vacation time, residual set-up time and residual service time, respectively.

3. The Differential Equations and the Solutions

In this section, we first introduce several supplementary variables to construct the differential equations for the model. Then we use the censoring technique to solve these equations. The solutions to the differential equations will be used to obtain interesting performance measures of the system in a later section.

Let $N(t)$ be the number of customers in the system at time t , and let $J(t)$ be the phase of the arrivals of customers at time t. We define the states of the server as

$$I(t) = \begin{cases} 0, & \text{if the server is on vacation with vacation time distribution } B_0(x), \\ 1, & \text{if the system is setting up with set-up time distribution } B_1(x), \\ 2, & \text{if the server is working with service time distribution } B_2(x). \end{cases}$$

Correspondingly, for $t > 0$, we define the random variable $S(t)$ as follows:

$$S(t) = \begin{cases} \text{the elapsed vacation time up to t}, & if \;\; I(t) = 0; \\ \text{the elapsed set-up time up to t}, & if \;\; I(t) = 1; \\ \text{the elapsed service time up to t}, & if \;\; I(t) = 2. \end{cases}$$

Then, $\{I(t), N(t), J(t), S(t) : t \geq 0\}$ is a Markov process. The state space of the process is expressed as

$$\begin{aligned} \Omega &= \{(0, k, j, x) : k \geq 0, 1 \leq j \leq m, x \geq 0\} \\ &\cup \{(1, k, j, x) : k \geq 0, 1 \leq j \leq m, x \geq 0\} \\ &\cup \{(2, k, j, x) : k \geq 0, 1 \leq j \leq m, x \geq 0\}. \end{aligned}$$

We write:

$$p^0_{k,i}(t, x)dx = P\{I(t) = 0, N(t) = k, J(t) = i, x \leq S(t) < x + dx\},$$
$$p^1_{k,i}(t, x)dx = P\{I(t) = 1, N(t) = k, J(t) = i, x \leq S(t) < x + dx\},$$
$$p^2_{k,i}(t, x)dx = P\{I(t) = 2, N(t) = k, J(t) = i, x \leq S(t) < x + dx\};$$

$$\begin{aligned} p^0_{k,i}(x) &= \lim_{t \to +\infty} p^0_{k,i}(t, x), \\ p^1_{k,i}(x) &= \lim_{t \to +\infty} p^1_{k,i}(t, x), \\ p^2_{k,i}(x) &= \lim_{t \to +\infty} p^2_{k,i}(t, x); \\ P^0_k(x) &= (p^0_{k,1}(x), \cdots, p^0_{k,m}(x)), \\ P^1_k(x) &= (p^1_{k,1}(x), \cdots, p^1_{k,m}(x)), \\ P^2_k(x) &= (p^2_{k,1}(x), \cdots, p^2_{k,m}(x)). \end{aligned}$$

If the system is stable, then the system of stationary differential equations of the joint probability density$\{P^0_0(x), P^0_k(x), P^1_k(x), P^2_k(x), k \geq 1\}$ can be written as:

$$\frac{d}{dx} P^0_0(x) = P^0_0(x)[C - \mu_0(x)I], \tag{1}$$

$$\frac{d}{dx} P^0_k(x) = P^0_k(x)[C - \mu_0(x)I] + P^0_{k-1}(x)D, k \geq 1; \tag{2}$$

$$\frac{d}{dx} P^1_1(x) = P^1_1(x)[C - \mu_1(x)I], \tag{3}$$

$$\frac{d}{dx} P^1_k(x) = P^1_k(x)[C - \mu_1(x)I] + P^1_{k-1}(x)D, k \geq 2; \tag{4}$$

$$\frac{d}{dx}P_1^2(x) = P_1^2(x)[C - \mu_2(x)I], \tag{5}$$

$$\frac{d}{dx}P_k^2(x) = P_k^2(x)[C - \mu_2(x)I] + P_{k-1}^2(x)D, k \geq 2. \tag{6}$$

The joint probability density $\{P_0^0(x), P_k^0(x), P_k^1(x), P_k^2(x),\ k \geq 1\}$ should satisfy the boundary conditions:

$$P_0^0(0) = \int_0^{+\infty} P_1^2(x)\mu_2(x)\,dx + \int_0^{+\infty} P_0^0(x)\mu_0(x)\,dx; \tag{7}$$

$$P_k^1(0) = \int_0^{+\infty} P_k^0(x)\mu_0(x)\,dx, \quad k \geq 1; \tag{8}$$

$$P_1^2(0) = \int_0^{+\infty} P_1^1(x)\mu_1(x)\,dx + \int_0^{+\infty} P_2^2(x)\mu_2(x)\,dx; \tag{9}$$

$$P_k^2(0) = \int_0^{+\infty} P_k^1(x)\mu_1(x)\,dx + \int_0^{+\infty} P_{k+1}^2(x)\mu_2(x)\,dx, \quad k \geq 2. \tag{10}$$

and the normalization condition:

$$\left\{\sum_{k=0}^{\infty}\int_0^{+\infty} P_k^0(x)\,dx + \sum_{k=1}^{\infty}\int_0^{+\infty} P_k^1(x)\,dx + \sum_{k=1}^{\infty}\int_0^{+\infty} P_k^2(x)\,dx\right\} \boldsymbol{e} = 1. \tag{11}$$

In the remainder of this section, we solve equations (1)-(11). To solve equations (1)-(6), we define

$$Q_0^*(z,x) = \sum_{k=0}^{\infty} z^k P_k^0(x),\ Q_1^*(z,x) = \sum_{k=1}^{\infty} z^k P_k^1(x),\ Q_2^*(z,x) = \sum_{k=1}^{\infty} z^k P_k^2(x).$$

It follows from (1) and (2) that

$$\frac{\partial}{\partial x}Q_0^*(z,x) = Q_0^*(z,x)[C + zD - \mu_0(x)I],$$

which leads to

$$\begin{aligned} Q_0^*(z,x) &= Q_0^*(z,0)\exp\{(C+zD)x - \int_0^x \mu_0(t)dt\} \\ &= Q_0^*(z,0)\exp\{(C+zD)x\}\bar{B}_0(x). \end{aligned} \tag{12}$$

Similarly, it follows from (3)-(6) that

$$Q_1^*(z,x) = Q_1^*(z,0)\exp\{(C+zD)x\}\bar{B}_1(x), \tag{13}$$

$$Q_2^*(z,x) = Q_2^*(z,0)\exp\{(C+zD)x\}\bar{B}_2(x). \tag{14}$$

Let us define $\{P(n,t),\ n \geq 0,\ t \geq 0\}$ as $m \times m$ matrix whose element $(P(n,t))_{ij}$ is the probability that exactly n positive customers arrive during $[0,t)$ and the generation process passes from phase i to phase j. These matrices satisfy the following system of differential equations

$$\begin{aligned}\frac{d}{dt}P(0,t) &= P(0,t)C,\\ \frac{d}{dt}P(n,t) &= P(n,t)C + P(n-1,t)D, \quad n \geq 1,\end{aligned}$$

with $P(0,0) = I$.

We define

$$P^*(z,t) = \sum_{n=0}^{\infty} z^n P(n,t), \quad |z| \leq 1.$$

Solving the above matrix differential equation, we get

$$P^*(z,t) = e^{(C+zD)t}, \quad |z| \leq 1, t \geq 0. \tag{15}$$

Substituting (15) into (12)-(14) respectively, gives

$$P_k^0(x) = \sum_{j=0}^{k} P_j^0(0)P(k-j,x)\bar{B}_0(x) = P_0^0(0)P(k,x)\bar{B}_0(x),\ k \geq 0; \tag{16}$$

$$P_k^1(x) = \sum_{j=1}^{k} P_j^1(0)P(k-j,x)\bar{B}_1(x), \quad k \geq 1; \tag{17}$$

$$P_k^2(x) = \sum_{j=1}^{k} P_j^2(0)P(k-j,x)\bar{B}_2(x), \quad k \geq 1. \tag{18}$$

Equations (16)- (18) provide a solution for the system of differential equations (1)-(6). Furthermore, boundary equations (7)-(10) will be used to determine the vectors $P_k^0(0)$ for $k \geq 0$, $P_k^1(0)$ for $k \geq 1$ and $P_k^2(0)$ for $k \geq 1$.
We define

$$\begin{aligned}A_k &= \int_0^{+\infty} P(k,x)\, d\bar{B}_0(x),\\ B_k &= \int_0^{+\infty} P(k,x)\, d\bar{B}_1(x),\\ C_k &= \int_0^{+\infty} P(k,x)\, d\bar{B}_2(x).\end{aligned}$$

Then it follows from (7)-(10),(16)-(18) that $P = P \cdot \Pi$, where

$$P = (P_0^0(0), P_1^1(0), P_1^2(0), P_2^1(0), P_2^2(0), \cdots),$$

$$\Pi = \begin{pmatrix} A_0 & A_1 & 0 & A_2 & 0 & \cdots \\ 0 & 0 & B_0 & 0 & B_1 & \cdots \\ C_0 & 0 & C_1 & 0 & C_2 & \cdots \\ & 0 & 0 & 0 & B_0 & \cdots \\ & & C_0 & 0 & C_1 & \cdots \\ & & & 0 & 0 & \cdots \\ & & & & C_0 & \cdots \\ & & & & & \ddots \end{pmatrix}.$$

In order to transform matrix Π to be a Toeplitz type, let $P_k(0) = (P_k^1(0), P_k^2(0))$, $k \geq 1$. Then $\overline{P} = \overline{P} \cdot \overline{\Pi}$, where

$$\overline{P} = \left(P_0^0(0), P_1(0), P_2(0), \cdots\right),$$

$$\overline{\Pi} = \begin{pmatrix} A_0 & \widetilde{A}_1 & \widetilde{A}_2 & \widetilde{A}_3 & \cdots \\ \widetilde{C}_0 & \widetilde{H}_1 & \widetilde{H}_2 & \widetilde{H}_3 & \cdots \\ & \widetilde{H}_0 & \widetilde{H}_1 & \widetilde{H}_2 & \cdots \\ & & \widetilde{H}_0 & \widetilde{H}_1 & \cdots \\ & & & \widetilde{H}_0 & \cdots \\ & & & & \ddots \end{pmatrix}$$

with

$$\widetilde{A}_k = (A_k, 0),\ k \geq 1; \quad B_{-1} = 0;\ \widetilde{C}_0 = \begin{pmatrix} 0 \\ C_0 \end{pmatrix}; \quad \widetilde{H}_k = \begin{pmatrix} 0 & B_{k-1} \\ 0 & C_k \end{pmatrix}, \quad k \geq 0.$$

Theorem 1. *The matrix $\overline{\Pi}$ is irreducible, stochastic and positive recurrent.*

Proof. According to the definition of A_0, $\widetilde{C}_0$, $\widetilde{A}_k$ and $\widetilde{H_k}$, for $k \geq 0$, it is easy to see that $\bar{\Pi}$ is irreducible and positive recurrent.

To prove that $\bar{\Pi}$ is stochastic, we only need to check that $\sum\limits_{k=0}^{\infty} A_k e = e$, $\sum\limits_{k=0}^{\infty} B_k e = e$, $\sum\limits_{k=0}^{\infty} C_k e = e$. Clearly, We have

$$\sum_{k=0}^{\infty} A_k e = \sum_{k=0}^{\infty} \int_0^{+\infty} P(k,x) \bar{B}_0(x) \mu_0(x)\, dx e = e,$$
$$\sum_{k=0}^{\infty} B_k e = \sum_{k=0}^{\infty} \int_0^{+\infty} P(k,x) \bar{B}_1(x) \mu_1(x)\, dx e = e,$$

$$\sum_{k=0}^{\infty} C_k e = \sum_{k=0}^{\infty} \int_0^{+\infty} P(k,x)\bar{B}_2(x)\mu_2(x)\,dxe = e.$$

This completes the proof.

Let $X = (x_0, x_1, x_2, x_3, \cdots)$ be the stationary probability vector of the matrix $\overline{\Pi}$. And let $x_k = (x_k^1, x_k^2)$ for $k \geq 1$, then we have

$$\overline{P} = \beta X = \beta(x_0, x_1, x_2, x_3, \cdots),$$

where β is determined by (11) as

$$\beta = \frac{1}{x_0 L_0 e + \sum\limits_{k=1}^{\infty} x_k^1 L_1 e + \sum\limits_{k=1}^{\infty} x_k^2 L_2 e},$$

with

$$L_0 = \int_0^{+\infty} \exp\{(C+D)x\}\bar{B}_0(x)\,dx,$$

$$L_1 = \int_0^{+\infty} \exp\{(C+D)x\}\bar{B}_1(x)\,dx,$$

$$L_2 = \int_0^{+\infty} \exp\{(C+D)x\}\bar{B}_2(x)\,dx.$$

Now we use the censoring technique and the RG-factorization to solve the equation $X = X\overline{\Pi}$ for $X = (x_0, x_1, x_2, x_3, \cdots)$. For details, readers may refer to Grassmann and Heyman [4] and Zhao [16].

Note that the matrix $\overline{\Pi}$ is $M/G/1$ type, let

$$Q = \begin{pmatrix} \widetilde{H}_1 & \widetilde{H}_2 & \widetilde{H}_3 & \cdots \\ \widetilde{H}_0 & \widetilde{H}_1 & \widetilde{H}_2 & \cdots \\ & \widetilde{H}_0 & \widetilde{H}_1 & \cdots \\ & & \widetilde{H}_0 & \cdots \\ & & & \ddots \end{pmatrix}, \quad \hat{Q} = \sum_{n=0}^{\infty} Q^n;$$

(19) $\qquad \Phi_0 = \widetilde{H}_1 + (\widetilde{H}_2, \widetilde{H}_3, \cdots)\hat{Q}(\widetilde{H}_0^T, 0, \cdots)^T = I - [\hat{Q}(1,1)]^{-1};$

(20) $\qquad G_1 = [I - \Phi_0]^{-1}\widetilde{H}_0 = \hat{Q}(1,1)\widetilde{H}_0, \quad G_k = 0,\ k \geq 2;$

(21) $\qquad G_{1,0} = \hat{Q}(1,\cdot)(\widetilde{C}_0^T, 0, \cdots)^T = \hat{Q}(1,1)\widetilde{H}_0,$

(22) $\qquad G_{i,0} = 0, \quad i \geq 2;$

(23) $\qquad R_j = (\widetilde{H}_{j+1}, \widetilde{H}_{j+2}, \cdots)\hat{Q}(\cdot,1) = \sum\limits_{i=1}^{\infty} \widetilde{H}_{i+j} G_1^{i-1}[I - \Phi_0]^{-1}, j \geq 1;$

(24) $\qquad R_{0,j} = (\widetilde{A}_j, \widetilde{A}_{j+1}, \cdots)\hat{Q}(\cdot,1) = \sum\limits_{i=0}^{\infty} \widetilde{A}_{i+j} G_1^{i}[I - \Phi_0]^{-1}, j \geq 1.$

where $\hat{Q}(1,1), \hat{Q}(1,\cdot)$ and $\hat{Q}(\cdot,1)$ denote the (1,1)th block, the first block-row and the first block-column of $\hat{Q}$, respectively.

Remark 1. It is obvious from the above expressions that $\Phi_0 = \widetilde{H}_1 + \sum_{i=1}^{\infty} \widetilde{H}_{i+1} G_1^i$. From Neuts [13], we know that the matrix G_1 is the minimal nonnegative solution to the matrix equation $G_1 = \sum_{i=0}^{\infty} \widetilde{H}_i G_1^i$, which can be numerically computed. Since the matrix Φ_0 and the R-measure R_j and $R_{0,j}$ for $j \geq 1$ can be expressed in terms of G_1, this illustrates that all the matrices Φ_0, R_j and $R_{0,j}$ for $j \geq 1$ can be numerically computed in principle.

Let $R^*(z) = \sum_{k=1}^{\infty} z^k R_k$, $R_0^*(z) = \sum_{k=1}^{\infty} z^k R_{0,k}$ and $X^*(z) = \sum_{k=1}^{\infty} z^k x_k$. Then it follows from (28) in Grassmann and Heyman [4] that

$$X^*(z) = \frac{1 - x_0 e}{x_0 R_0^*(1)[I - R^*(1)]^{-1} e} x_0 R_0^*(z)[I - R^*(z)]^{-1},$$

or

$$x_k = \frac{1 - x_0 e}{x_0 R_0^*(1)[I - R^*(1)]^{-1} e} x_0 R_{0,k} * \sum_{n=0}^{\infty} R_k^{n*}, \quad k \geq 1. \tag{25}$$

We denote by $F(x) * G(x)$ the convolution of two functions $F(x)$ and $G(x)$ given by $F(x) * G(x) = \int_0^x F(x-u)\, dG(u)$. We write $F^{n*}(x) = F(x) * F^{(n-1)*}(x)$ for $n \geq 2$ and define $F^{0*}(x) = 1$.

Lemma 1. *The transition probability matrix Ψ_0 of the censored Markov chain of $\overline{\Pi}$ to level 0 is given by*

$$\Psi_0 = A_0 + \sum_{i=0}^{\infty} \widetilde{A}_{i+1} G_1^i [I - \sum_{i=1}^{\infty} \widetilde{H}_i G_1^{i-1}]^{-1} \widetilde{C}_0. \tag{26}$$

Proof. It follows from (20) in Zhao [16] that

$$\Psi_0 = A_0 + \sum_{j=1}^{\infty} R_{0,j} (I - \Phi_0) G_{j,0},$$

which leads to

$$\Psi_0 = A_0 + \left(\sum_{i=0}^{\infty} \widetilde{A}_{i+1} G_1^i G_{1,0} \right)$$

by using (21),(22) and (24). It follows from (19) of Zhao [16] that

$$(I-\Phi_0)G_{1,0}=\widetilde{C}_0+\sum_{j=1}^{\infty}R_j(I-\Phi_0)G_{2,0}=\widetilde{C}_0,$$

Hence

$$G_{1,0}=(I-\Phi_0)^{-1}\widetilde{C}_0=[I-\sum_{i=1}^{\infty}\widetilde{H}_iG_1^{i-1}]^{-1}\widetilde{C}_0.$$

This completes the proof.

We now summarize the above discussion into the following theorem.

Theorem 2. *For the stable system, let π_0 be the stationary probability vector of the censored Markov chain Ψ_0 given in (26). Then*

(27) $$P_k^0(x)=P_0^0(0)P(k,x)\bar{B}_0(x),k\geq 0;$$

(28) $$P_k^1(x)=\sum_{j=1}^{k}P_j^1(0)P(k-j,x)\bar{B}_1(x),k\geq 1;$$

(29) $$P_k^2(x)=\sum_{j=1}^{k}P_j^2(0)P(k-j,x)\bar{B}_2(x),k\geq 1.$$

where

$$P_0^0(0)=\beta x_0=\frac{x_0}{x_0L_0e+\sum_{j=1}^{\infty}(x_j^1L_1+x_j^2L_2)e},$$

and

$$P_k^1(0)=\beta x_k^1,\quad k\geq 1;\quad P_k^2(0)=\beta x_k^2,\quad k\geq 1.$$

where $x_k^1,x_k^2,k\geq 1$ are determined by (25), and

(30) $$x_0=\frac{\pi_0}{1+\pi_0R_0^*(1)[I-R^*(1)]^{-1}e},$$

$$\pi_0=\pi_0\Psi_0,\quad \pi_0e=1.$$

Remark 2. For the MAP/G/1 queue with set-up time and multiple vacations, performance measures such as the queue length, the virtual waiting time and the busy period (see the next section) can be numerically computed in a standard manner, once the vector sequence $\{x_k,k\geq 0\}$ has been numerically obtained. We illustrate the steps for computing $\{x_k,k\geq 0\}$ as follows.

1. Compute the two matrix sequences $\{\widetilde{A}_k,\ k\geq 0\}$, $\{\widetilde{H}_k,k\geq 0\}$ and the two matrixes A_0, $\widetilde{C}_0$ of the matrix $\overline{\Pi}$.
2. Compute the minimal nonnegative solution G_1 to the nonlinear matrix equation

$$G_1 = \sum_{k=0}^{\infty} \widetilde{H}_k G_1^k$$

using one of the available algorithms in the literature. Readers may refer to Neuts [13] for details.

3. Compute the R-measure $\{R_k\}$ and$\{R_{0,k}\}$ according to (23) and (24), respectively.
4. Compute the censored transition probability matrix Ψ_0 according to (26), and compute the stationary probability vector π_0 by solving the system of equations $\pi_0 \Psi_0 = \pi$ and $\pi e = 1$.
5. Compute the vector x_0 according to (30).
6. Compute the vector x_k for $k \geq 1$ according to (25).

4. Performance Measures of the Model

In this section, we consider three performance measures for the model: the stationary queue length, the virtual waiting time and the busy period.

4.1. The stationary queue length

We write

$$\begin{aligned}
P_k &= \lim_{t\to\infty} P\{N(t) = k\}, \quad k \geq 0; \\
P_k^0 &= \lim_{t\to\infty} P\{N(t) = k, I(t) = 0\}, \ k \geq 0; \\
P_k^1 &= \lim_{t\to\infty} P\{N(t) = k, I(t) = 1\}, \ k \geq 1; \\
P_k^2 &= \lim_{t\to\infty} P\{N(t) = k, I(t) = 2\}, \ k \geq 1.
\end{aligned}$$

Obviously,

$$P_0 = P_0^0; \qquad P_k = P_k^0 + P_k^1 + P_k^2, \quad k \geq 1.$$

Theorem 3. *If the model is stable, then*

$$\begin{cases} P_0 = \beta x_0 M_0^0 e, \\ P_k = \beta x_0 M_k^0 e + \beta \displaystyle\sum_{j=1}^{k} x_j^1 M_{k-j}^1 e + \beta \sum_{j=1}^{k} x_j^2 M_{k-j}^2 e, \quad k \geq 1. \end{cases}$$

where x_j^1, x_j^2 *are determined by* x_j, $P(k,x)$ *is determined by* (15), *and*

$$M_k^0 = \int_0^{+\infty} P(k,x)\overline{B}_0(x)\,dx, \quad k \geq 0;$$
$$M_k^1 = \int_0^{+\infty} P(k,x)\overline{B}_1(x)\,dx, \quad k \geq 1;$$
$$M_k^2 = \int_0^{+\infty} P(k,x)\overline{B}_2(x)\,dx, \quad k \geq 1.$$

Proof. It follows from (27) that

$$P_k^0 = \int_0^{+\infty} P_k^0(x)\,dx\ e = \beta x_0 M_k^0 e, \ \ k \geq 0;$$

and from (28) and (29) that

$$P_k^1 = \int_0^{+\infty} P_k^1(x)\,dx\ e = \beta \sum_{j=1}^{k} x_j^1 M_{k-j}^1 e, \ \ k \geq 1;$$
$$P_k^2 = \int_0^{+\infty} P_k^2(x)\,dx\ e = \beta \sum_{j=1}^{k} x_j^2 M_{k-j}^2 e \ \ k \geq 1.$$

This completes the proof.

4.2. The virtual waiting time

Let $\xi(t)$ denote the virtual waiting time at instant t, which is the time that a customer would have to wait for service, provided he had arrived at the instant t. Obviously, $\xi(t), t \geq 0$ is a random process with state set $[0,\infty)$. For the sake of simplicity, we assume that $\xi(0) = 0$, i.e., the system is empty at the initial instant 0. Let

$$W(x,t) = P\{\xi(t) < x\}.$$

denote the distribution function of the process $\xi(t)$.

By the definition in Section 2, we can deduce that

$$B^{(0,E)}(x) = P(V^{(E)} \leq x) = \mu_0 \int_0^x \overline{B}_0(x)dx,$$
$$B^{(1,E)}(x) = P(U^{(E)} \leq x) = \mu_1 \int_0^x \overline{B}_1(x)dx,$$
$$B^{(2,E)}(x) = P(S^{(E)} \leq x) = \mu_2 \int_0^x \overline{B}_2(x)dx.$$

We write

$$F_0(x) = P\{V^{(E)} + U \leq x\},$$

and for $n \geq 1$,

$$F_n^0(x) = P\{V^{(E)} + U + \sum_{j=1}^{n} S_i \leq x\},$$
$$F_n^1(x) = P\{U^{(E)} + \sum_{j=1}^{n} S_i \leq x\},$$
$$F_n^2(x) = P\{S_1^{(E)} + \sum_{j=2}^{n} S_i \leq x\}.$$

Lemma 2.
$$F_0(x) = B^{(0,E)}(x) * B_1(x),$$
and for $n \geq 1$,
$$F_n^0(x) = B^{(0,E)}(x) * B_1(x) * B_2^{n*}(x),$$
$$F_n^1(x) = B^{(1,E)}(x) * B_2^{n*}(x),$$
$$F_n^2(x) = B^{(2,E)}(x) * B_2^{(n-1)*}(x).$$

Proof. We can prove it easily from the mutual independency of U, V, S.

Theorem 4. *If the model is stable, then the distribution of the virtual waiting time is*

$$\begin{aligned} W(t,x) = {} & P_0 B^{(0,E)}(x) * B_1(x) + \sum_{n=1}^{\infty} P_n^0 B^{(0,E)}(x) * B_1(x) * B_2^{n*}(x) \\ & + \sum_{n=1}^{\infty} P_n^1 B^{(1,E)}(x) * B_2^{n*}(x) + \sum_{n=1}^{\infty} P_n^2 B^{(2,E)}(x) * B_2^{(n-1)*}(x). \end{aligned}$$

Proof. A customer finds that the system is in one of the following four states:

1. there are no customers in the system;
2. There are $n \geq 1$ customers in the system, and the system is on vacation;
3. There are $n \geq 1$ customers in the system, and the system is on set-up;
4. There are $n \geq 1$ customers in the system, and the system is on service.

Combining the above four cases, we obtain

$$W(t,x) = P_0 F_0(x) + \sum_{n=1}^{\infty} [P_n^0 F_n^0(x) + P_n^1 F_n^1(x) + P_n^2 F_n^2(x)].$$

Simple manipulations to the above equation complete the proof.

4.3. The busy period

We now provide an analysis of the busy period for the model. From the description in section 2, for the random variable V of the vacation time, we have

$$B_0(x) = P\{V \leq x\}.$$

We denote by T be the random variable of the inter-arrival time between two customers, and $T^{(E)}$ the random variable for the equilibrium excess distributions with respect to T. It is clear that

$$A(x) := P\{T \leq x\} = \theta \int_0^x \exp\{Ct\}\, dt\ De,$$

and

$$A^{(E)}(x) = P\{T^{(E)} \leq x\} = \frac{1}{\theta(-C)^{-1}e} \int_0^x \overline{A}(t)\, dt.$$

Let V_i be the random variable of the i-th vacation, and $\hat{V}$ be the random variable of the number of vacations during the vacation time. Then

$$\begin{aligned} P\{\hat{V} = n\} &= P\{\sum_{i=1}^{n-1} V_i < T^{(E)} \leq \sum_{i=1}^{n} V_i\} \\ &= \int_0^{+\infty} [B_0^{(n-1)*}(t) - B_0^{n*}(t)]\, dA^{(E)}(t). \end{aligned}$$

Lemma 3. *Let $\overline{V}$ be the random variable of the multiple vacations time, then*

$$E\overline{V} = \sum_{n=1}^{\infty} n \frac{1}{\mu_0} \int_0^{+\infty} [B_0^{(n-1)*}(t) - B_0^{n*}(t)]\, dA^{(E)}(t).$$

Theorem 5. *Let ξ be the random variable of the busy period of the system, then*

$$E\xi = \frac{(1 - \beta x_0 L_0 e)E\overline{V}}{\beta x_0 L_0 e}.$$

Proof. According to the renewal theory, we can obtain

$$\sum_{k=0}^{\infty} p_k^0 = \frac{E\overline{V}}{E\xi + E\overline{V}},$$

or

$$E\xi = \frac{(1 - \sum\limits_{k=0}^{\infty} p_k^0)E\overline{V}}{\sum\limits_{k=0}^{\infty} p_k^0} = \frac{(1 - \beta x_0 L_0 e)E\overline{V}}{\beta x_0 L_0 e}.$$

This completes the proof.

5. Conclusion

In the foregoing analysis, a MAP/G/1 queue with set-up time and multiple vacations is considered to obtain analytical expressions for various performance measures of interest. In principle, the performance measures obtained in this paper can be numerically computed based on the matrix-analytic method.

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