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ITERATIVE APPROXIMATION OF FIXED POINTS FOR ASYMPTOTICALLY STRICT PSEUDOCONTRACTIVE TYPE MAPPINGS IN THE INTERMEDIATE SENSE

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Abstract. We introduced the concept of an asymptotically κ-strict pseudocontractive type mapping in the intermediate sense which is not necessarily Lipschitzian. We proved that the modified Mann iteration process: $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,\ \forall n \geq 1$ where $\{\alpha_n\}$ is a sequence in $(0,1)$ with $\delta \leq \alpha_n \leq 1-\kappa-\delta$ for $\delta \in (0,1)$ converges weakly to a fixed point of an asymptotically κ-strict pseudocontractive type mapping T in the intermediate sense. Furthermore, a CQ method which generates a strongly convergent sequence for this class of mappings is proposed and strong convergence result for this CQ method is established.

1. INTRODUCTION

We first recall the following concepts.

Definition 1.1. Let C be a nonempty subset of a normed space X and $T : C \to C$ be a mapping.

(i) T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(ii) T is asymptotically nonexpansive (cf. [7]) if there exists a sequence $\{k_n\}$ of positive numbers satisfying the property $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall n \geq 1,\ \forall x, y \in C;$$

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(iii) T is asymptotically nonexpansive in the intermediate sense [2] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0;$$

(iv) T is uniformly Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \quad \forall n \ge 1, \ \forall x, y \in C.$$

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly Lipschitzian.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [7] as an important generalization of the class of nonexpansive mappings. The existence of fixed points of asymptotically nonexpansive mappings was proved by Goebel and Kirk [7] as below:

Theorem 1.1. (Goebel and Kirk [7, Theorem 1]). If C is a nonempty closed convex bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive mapping $T : C \to C$ has a fixed point in C.

An iterative method for the approximation of fixed points of asymptotically nonexpansive mappings was developed by Schu [20] in the following interesting result:

Theorem 1.2. (Schu [20]). Let C be a nonempty closed convex bounded subset of a Hilbert space H and $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the condition $\delta \le \alpha_n \le 1 - \delta$ for all $n \ge 1$ and for some $\delta > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \ge 1, \tag{1.1}$$

converges weakly to a fixed point of T.

Iterative methods for approximation of fixed points of asymptotically nonexpansive mappings have been further studied by authors (see e.g. [3-5, 10, 14, 17-19, 21, 23, 24, 27] and references therein).

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck, Kuczumow and Reich [2] and iterative methods for the approximation of fixed points of such types of non-Lipschitzian mappings have been studied by Agarwal, O'Regan and Sahu [1], Bruck, Kuczumow and Reich [2], Chidume, Shahzad and Zegeye [6], Kim and Kim [11] and many others.

Recently, Kim and Xu [13] introduced the concept of asymptotically κ-strict pseudocontractive mappings in a Hilbert space as below:

Definition 1.2. Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ is said to be an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ if there exists a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\begin{aligned} &\|T^n x - T^n y\|^2 \\ &\leq (1+\gamma_n)\|x-y\|^2 + \kappa\|x-T^n x - (y- T^n y)\|^2, \quad \forall n \geq 1,\ \forall x, y \in C. \end{aligned} \tag{1.2}$$

They studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ is a uniformly L-Lipschitzian mapping with $L = \sup\{\frac{\kappa+\sqrt{1+(1-\kappa)\gamma_n}}{1+\kappa} : n \geq 1\}$.

Very recently, Sahu, Xu and Yao [28] considered the concept of asymptotically κ-strict pseudocontractive mappings in the intermediate sense, which are not necessarily Lipschitzian.

Definition 1.3. Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ is said to be an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\|^2 \\ &-(1+\gamma_n)\|x-y\|^2 - \kappa\|x - T^n x - (y - T^n y)\|^2) \leq 0. \end{aligned} \tag{1.3}$$

Put

$$c_n := \max\{0, \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - \kappa\|x - T^n x - (y - T^n y)\|^2)\}.$$

Then $c_n \geq 0$ $(\forall n \geq 1)$, $c_n \to 0$ $(n \to \infty)$ and (1.3) reduces to the relation

$$\begin{aligned} &\|T^n x - T^n y\|^2 \\ &\leq (1+\gamma_n)\|x-y\|^2 + \kappa\|x-T^n x-(y-T^n y)\|^2 + c_n, \quad \forall n \geq 1,\ \forall x, y \in C. \end{aligned} \tag{1.4}$$

Whenever $c_n = 0$ for all $n \geq 1$ in (1.4) then T is an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$. Sahu, Xu and Yao [28] proved the weak convergence of the modified Mann iteration process (1.1) and strong convergence of the further modification of (1.1) for asymptotically κ-strict pseudocontractive mappings in the intermediate sense, respectively.

Motivated and inspired by the concept of asymptotically κ-strict pseudocontractive mappings in the intermediate sense, we introduce the following concept of asymptotically κ-strict pseudocontractive type mappings in the intermediate sense which are not necessarily Lipschitzian.

Definition 1.4. Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ will be called an asymptotically κ-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 \\ &-\kappa \max\{\|x - T^n x - (y - T^n y)\|, \|x - T^n x + (y - T^n y)\|\}^2) \le 0. \end{aligned} \tag{1.7}$$

Throughout this paper we assume that

$$\begin{aligned} \theta_n := \max\{0, &\sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 \\ &-\kappa \max\{\|x - T^n x - (y - T^n y)\|, \|x - T^n x + (y - T^n y)\|\}^2)\}. \end{aligned}$$

Then $\theta_n \ge 0$ $(\forall n \ge 1)$, $\theta_n \to 0$ $(n \to \infty)$, and (1.7) reduces to the relation

$$\begin{aligned} &\|T^n x - T^n y\|^2 \le (1 + \gamma_n)\|x - y\|^2 \\ &+\kappa \max\{\|x - T^n x - (y - T^n y)\|, \|x - T^n x + (y - T^n y)\|\}^2 + \theta_n \end{aligned} \tag{1.8}$$

for all $x, y \in C$ and $n \ge 1$.

Remark 1.1. (1) It is easy to see that if T is an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$, then T must be an asymptotically κ-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$;

(2) Whenever T is an asymptotically κ-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$, T is not necessarily uniformly L-Lipschitzian (see Lemma 2.6).

Example 1.1. Let $X = R$ the set of real numbers, and $C = [0, \infty)$. For each $x \in C$, we define

$$Tx = \begin{cases} kx, & \text{if } x \in [0, 1], \\ 0, & \text{if } x \in (1, \infty), \end{cases}$$

where $0 < k \le \frac{1}{4}$. Then $T : C \to C$ is discontinuous at $x = 1$ and hence T is not Lipschitzian. Set $C_1 := [0, 1]$ and $C_2 := (1, \infty)$. Hence

$$|T^n x - T^n y| = k^n|x - y| \le |x - y|, \quad \forall x, y \in C_1,\ \forall n \ge 1$$

and

$$|T^n x - T^n y| = 0 \le |x - y|, \quad \forall x, y \in C_2,\ \forall n \ge 1.$$

For $x \in C_1$ and $y \in C_2$, we have

$$
\begin{aligned}
|T^n x - T^n y|^2 &= |k^n x - 0|^2 = |k^n(x-y) + k^n y|^2 \\
&\leq (k^n|x-y| + k^n|y|)^2 \\
&\leq (\tfrac{k^{n-1}|x-y| + k^{n-1}|y|}{4})^2 \\
&\leq \tfrac{1}{8}k^{2(n-1)}|x-y|^2 + \tfrac{1}{8}k^{2(n-1)}|y|^2 \\
&\leq |x-y|^2 + \tfrac{1}{8}k^{2(n-1)}|y + x - T^n x - (x - T^n x)|^2 \\
&\leq |x-y|^2 + \tfrac{1}{2}k^{2(n-1)}(\tfrac{|y+x-T^n x| + |x-T^n x|}{2})^2 \\
&\leq |x-y|^2 + \tfrac{1}{2}k^{2(n-1)}(\tfrac{1}{2}|y+x-T^n x|^2 + \tfrac{1}{2}|x - T^n x|^2) \\
&\leq |x-y|^2 + \tfrac{1}{4}|y + x - T^n x|^2 + \tfrac{1}{4}k^{2(n-1)} \\
&= |x-y|^2 + \tfrac{1}{4}|x - T^n x + y - T^n y|^2 + \tfrac{1}{4}k^{2(n-1)} \\
&\leq |x-y|^2 + \tfrac{1}{4}\max\{|x - T^n x - (y - T^n y)|, \\
&\quad\ |x - T^n x + y - T^n y|\}^2 + \tfrac{1}{4}k^{2(n-1)}.
\end{aligned}
$$

Thus for all $x, y \in C$ and $n \geq 1$ we get

$$
\begin{aligned}
&|T^n x - T^n y|^2 \\
&\leq |x-y|^2 + \tfrac{1}{4}\max\{|x-T^n x-(y-T^n y)|, |x-T^n x + y-T^n y|\}^2 + \tfrac{1}{4}k^{2(n-1)}.
\end{aligned}
$$

Therefore, T is an asymptotically $\frac{1}{4}$-strict pseudocontractive type mapping in the intermediate sense.

The paper is organized as follows: In Section 2, we will recall the useful definitions and lemmas. In Section 3, we derive the demiclosedness principle and weak convergence of the modified Mann iteration process (1.1) for the class of asymptotically κ-strict pseudocontractive type mappings in the intermediate sense. In Section 4, we apply Sahu, Xu and Yao's further modification [28] of the modified Mann iteration process (1.1) for asymptotically κ-strict pseudocontractive type mappings in the intermediate sense which generates a strongly convergent sequence. Our weak and strong convergence theorems for the class of asymptotically κ-strict pseudocontractive type mappings in the intermediate sense are more general than the known results in [28] for the class of asymptotically κ-strict pseudocontractive mappings in the intermediate sense.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively and let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H.$$

We will adopt the following notations:

(i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;

(ii) $\omega_w(\{x_n\}) = \{x \in H : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$;

(iii) $F(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of a self-mapping T on a set C.

We need some facts and tools which are listed as lemmas below:

Lemma 2.1. ([18,22]). Let $\{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of non-negative numbers satisfying the recursive inequality:

$$\delta_{n+1} \le \beta_n \delta_n + \gamma_n, \quad \forall n \ge 1.$$

If $\beta_n \ge 1,\ \sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then $\lim_{n\to\infty} \delta_n$ exists.

Lemma 2.2. (Agarwal, O'Regan and Sahu [1, Proposition 2.4]). Let $\{x_n\}$ be a bounded sequence on a reflexive Banach space X. If $\omega_w(\{x_n\}) = \{x\}$, then $x_n \rightharpoonup x$.

Lemma 2.3. Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C be the metric projection mapping from H onto C.Given $x \in H$ and $z \in C$, then

$$z = P_C x \quad \text{if and only if} \quad \langle x - z, y - z\rangle \le 0,\ \forall y \in C.$$

Lemma 2.4. Let H be a real Hilbert space. Then the following hold:

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ for all $x, y \in H$;

(ii) $\|(1-t)x + ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x - y\|^2$ for all $t \in [0,1]$ and for all $x, y \in H$;

(iii) If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H. \tag{2.1}$$

Lemma 2.5. (cf. [25]). Let H be a real Hilbert space. Given a nonempty closed convex subset C of H and points $x, y, z \in H$ and given also a real number $a \in R$, the set

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed).

Lemma 2.6. Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ be an asymptotically κ-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then

$$\|T^n x - T^n y\| \leq \frac{1}{1-\kappa}(\kappa\|x-y\| + \sqrt{(1 + (1-\kappa)\gamma_n)\|x-y\|^2 + (1-\kappa)h_n(x, y)})$$

for all $x, y \in C$ and $n \geq 1$, where $h_n(x, y) = 4\kappa\|y - T^n y\|\|x - T^n x + y - T^n y\| + \theta_n$. In particular, if $F(T) \neq \emptyset$, then the above inequality reduces to the following

$$\|T^n x - p\| \leq \frac{1}{1-\kappa}(\kappa\|x - p\| + \sqrt{(1 + (1-\kappa)\gamma_n)\|x - p\|^2 + (1-\kappa)\theta_n}),$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Proof. For every $x, y \in C$, we have

$$\begin{aligned}
&\|x - T^n x + (y - T^n y)\|^2 \\
&= \|x - T^n x - (y - T^n y) + 2(y - T^n y)\|^2 \\
&\leq \|x - T^n x - (y - T^n y)\|^2 + 2\langle 2(y - T^n y), x - T^n x + (y - T^n y)\rangle \\
&\leq \|x - T^n x - (y - T^n y)\|^2 + 4\|y - T^n y\|\|x - T^n x + (y - T^n y)\| \\
&= (\|x - y\| + \|T^n x - T^n y\|)^2 + 4\|y - T^n y\|\|x - T^n x + (y - T^n y)\| \\
&= \|x - y\|^2 + 2\|x - y\|\|T^n x - T^n y\| + \|T^n x - T^n y\|^2 \\
&\quad + 4\|y - T^n y\|\|x - T^n x + (y - T^n y)\|,
\end{aligned}$$

which hence implies that

$$\begin{aligned}
&\max\{\|x - T^n x - (y - T^n y)\|, \|x - T^n x + (y - T^n y)\|\}^2 \\
&\leq \|x - y\|^2 + 2\|x - y\|\|T^n x - T^n y\| + \|T^n x - T^n y\|^2 \\
&\quad + 4\|y - T^n y\|\|x - T^n x + (y - T^n y)\|.
\end{aligned}$$

Thus, from (1.8) we get

$$\begin{aligned}
&\|T^n x - T^n y\|^2 \\
&\leq (1 + \gamma_n)\|x - y\|^2 + \kappa \max\{\|x - T^n x - (y - T^n y)\|, \\
&\quad \|x - T^n x + (y - T^n y)\|\}^2 + \theta_n \\
&\leq (1 + \gamma_n)\|x - y\|^2 + \kappa[\|x - y\|^2 + 2\|x - y\|\|T^n x - T^n y\| + \|T^n x - T^n y\|^2 \\
&\quad + 4\|y - T^n y\|\|x - T^n x + (y - T^n y)\|] + \theta_n
\end{aligned}$$

$$= (1+\kappa+\gamma_n)\|x-y\|^2 + \kappa[2\|x-y\|\|T^nx-T^ny\| + \|T^nx-T^ny\|^2] \\ +4\kappa\|y-T^ny\|\|x-T^nx+(y-T^ny)\| + \theta_n \\ = (1+\kappa+\gamma_n)\|x-y\|^2+\kappa[2\|x-y\|\|T^nx-T^ny\|+\|T^nx-T^ny\|^2] + h_n(x,y),$$

where $h_n(x,y) = 4\kappa\|y-T^ny\|\|x-T^nx+y-T^ny\|+\theta_n$. Consequently, it gives us that

$$(1-\kappa)\|T^nx-T^ny\|^2-2\kappa\|x-y\|\|T^nx-T^ny\|-(1+\kappa+\gamma_n)\|x-y\|^2-h_n(x,y) \le 0,$$

which is a quadratic inequality in $\|T^nx-T^ny\|$. Hence the conclusion follows. In particular, for all $x \in C$, $p \in F(T)$, and $n \ge 1$, we deduce that $h_n(x,p) = \theta_n$ and hence

$$\|T^nx-p\| \le \frac{1}{1-\kappa}(\kappa\|x-p\| + \sqrt{(1+(1-\kappa)\gamma_n)\|x-p\|^2+(1-\kappa)\theta_n}).$$

This completes the proof. ∎

Lemma 2.7. Let C be a nonempty subset of a Hilbert space H and $T: C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a bounded sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^nx_n\| \to 0$ as $n \to \infty$. If $F(T) \neq \emptyset$, then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

Proof. Since T is an asymptotically κ-strict pseudocontractive type mapping in the intermediate sense, we obtain from Lemma 2.6 that

$$\begin{aligned} &\|T^{n+1}x_n - T^{n+1}x_{n+1}\| \\ &\le \frac{1}{1-\kappa}(\kappa\|x_n-x_{n+1}\| \\ &+\sqrt{(1+(1-\kappa)\gamma_{n+1})\|x_n-x_{n+1}\|^2+(1-\kappa)h_{n+1}(x_n,x_{n+1})}), \end{aligned} \tag{2.2}$$

where $h_{n+1}(x_n,x_{n+1}) = 4\kappa\|x_{n+1}-T^{n+1}x_{n+1}\|\|x_n-T^{n+1}x_n+x_{n+1}-T^{n+1}x_{n+1}\| +\theta_{n+1}$.

Now let us show that $\{T^{n+1}x_n\}$ is bounded. Indeed, again from Lemma 2.6 we conclude that for every $p \in F(T)$,

$$\|T^{n+1}x_n-p\| \le \frac{1}{1-\kappa}(\kappa\|x_n-p\|+\sqrt{(1+(1-\kappa)\gamma_{n+1})\|x_n-p\|^2+(1-\kappa)\theta_{n+1}}).$$

Thus, from the boundedness of $\{x_n\}$ it follows that $\{T^{n+1}x_n\}$ is bounded. Repeating the same argument, we can see that $\{T^nx_n\}$ is also bounded.

Next let us show that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Indeed, observe that

$$\begin{aligned} & h_{n+1}(x_n, x_{n+1}) \\ &= 4\kappa\|x_{n+1} - T^{n+1}x_{n+1}\|\|x_n - T^{n+1}x_n + x_{n+1} - T^{n+1}x_{n+1}\| + \theta_{n+1} \\ &\le 4\kappa\|x_{n+1} - T^{n+1}x_{n+1}\|[\|x_n - T^{n+1}x_n\| + \|x_{n+1} - T^{n+1}x_{n+1}\|] + \theta_{n+1}. \end{aligned}$$

Since $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$, we obtain that $h_{n+1}(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Also, since $\|x_n - x_{n+1}\| \to 0$, it follows from (2.2) and $h_{n+1}(x_n, x_{n+1}) \to 0$ that $\|T^{n+1}x_n - T^{n+1}x_{n+1}\| \to 0$. Note that

$$\begin{aligned} \|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &+\|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\|. \end{aligned} \tag{2.3}$$

By the uniform continuity of T, it follows from $x_n – T^n x_n \to 0$ that $Tx_n – T^{n+1}x_n \to 0$. Since $x_n - T^n x_n \to 0,\ x_n - x_{n+1} \to 0$ and $Tx_n - T^{n+1}x_n \to 0$, the inequality (2.3) implies that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. This completes the proof. ■

3. Weak Convergence of the Modified Mann Iteration Process

First, we give some basic properties of asymptotically κ-strict pseudocontractive type mappings in the intermediate sense.

Proposition 3.1. (Demiclosedness principle). Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a continuous asymptotically κ-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$. Then $I-T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, then $(I - T)x = 0$.

Proof. Assume that T is a continuous asymptotically κ-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x \in C$ and

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0. \tag{3.1}$$

Then $\{x_n\}$ is bounded. Now let us show that the sets $\{T^m x_n : m, n \ge 1\}$ and $\{T^m x : m \ge 1\}$ are bounded. Indeed, by Lemma 2.6 we have for every $p \in F(T)$,

$$\|T^m x_n - p\| \le \frac{1}{1-\kappa}(\kappa\|x_n - p\| + \sqrt{(1 + (1-\kappa)\gamma_m)\|x_n - p\|^2 + (1-\kappa)\theta_m}),$$

and

$$\|T^m x - p\| \le \frac{1}{1-\kappa}(\kappa\|x - p\| + \sqrt{(1 + (1-\kappa)\gamma_m)\|x - p\|^2 + (1-\kappa)\theta_m}).$$

Thus, we know that $\{T^m x_n : m, n \geq 1\}$ and $\{T^m x : m \geq 1\}$ are bounded. Consequently, there exists some constant $K' > 0$ such that $\|T^m x_n - T^m x\| \leq K'$ for all $m, n \geq 1$. Define

$$\varphi(x) := \limsup_{n\to\infty} \|x_n - x\|^2, \quad \forall x \in H.$$

Since $x_n \rightharpoonup x$, it follows from (2.1) that

$$\varphi(y) = \varphi(x) + \|x - y\|^2, \quad \forall y \in H. \tag{3.2}$$

Observe that

$$\begin{aligned}
&\|x_n - T^m x_n + (x - T^m x)\|^2 = \|x - T^m x + (x_n - T^m x_n)\|^2 \\
&= \|x - T^m x - (x_n - T^m x_n) + 2(x_n - T^m x_n)\|^2 \\
&\leq \|x - T^m x - (x_n - T^m x_n)\|^2 + 2\langle 2(x_n - T^m x_n), x_n - T^m x_n + (x - T^m x)\rangle \\
&\leq \|x - T^m x - (x_n - T^m x_n)\|^2 + 4\|x_n - T^m x_n\|\|x_n - T^m x_n + (x - T^m x)\| \\
&\leq \|x - T^m x - (x_n - T^m x_n)\|^2 + 4\|x_n - T^m x_n\|\widetilde{K}
\end{aligned}$$

for all $m, n \geq 1$ and some constant $\widetilde{K}$ satisfying $\|x_n - T^m x_n + (x - T^m x)\| \leq \widetilde{K}$ (because $\{x_n : n \geq 1\}, \{T^m x_n : m, n \geq 1\}$ and $\{T^m x : m \geq 1\}$ are bounded).

Since T is an asymptotically κ-strict pseudocontractive type mapping in the intermediate sense, by relation (1.8), we have

$$\begin{aligned}
&\varphi(T^m x) = \limsup_{n\to\infty} \|x_n - T^m x\|^2 \\
&\leq \limsup_{n\to\infty} (\|x_n - T^m x_n\| + \|T^m x_n - T^m x\|)^2 \\
&= \limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + \|T^m x_n - T^m x\|^2 + 2\|x_n - T^m x_n\|\|T^m x_n - T^m x\|) \\
&\leq \limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + \|T^m x_n - T^m x\|^2 + 2\|x_n - T^m x_n\|K') \\
&\leq \limsup_{n\to\infty} \|T^m x_n - T^m x\|^2 + \limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + 2\|x_n - T^m x_n\|K') \\
&\leq \limsup_{n\to\infty} \{(1 + \gamma_m)\|x_n - x\|^2 \\
&\quad + \kappa \max\{\|x_n - T^m x_n - (x - T^m x)\|, \|x_n - T^m x_n + (x - T^m x)\|\}^2 \\
&\quad + \theta_m\} + \limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + 2\|x_n - T^m x_n\|K') \\
&\leq \limsup_{n\to\infty} \{(1 + \gamma_m)\|x_n - x\|^2 + \kappa(\|x - T^m x - (x_n - T^m x_n)\|^2 \\
&\quad + 4\|x_n - T^m x_n\|\widetilde{K}) + \theta_m\} \\
&\quad + \limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + 2\|x_n - T^m x_n\|K') \\
&\leq \varphi(x) + \kappa \limsup_{n\to\infty} \|x - T^m x - (x_n - T^m x_n)\|^2 \\
&\quad + 4\kappa\widetilde{K} \limsup_{n\to\infty} \|x_n - T^m x_n\| + \varphi(x)\gamma_m + \theta_m \\
&\quad + \limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + 2\|x_n - T^m x_n\|K'), \quad \forall m \geq 1.
\end{aligned}$$

From (3.2), we have

$$\begin{aligned}
&\varphi(x) + \|x - T^m x\|^2 = \varphi(T^m x) \\
&\leq \varphi(x) + \kappa \limsup_{n\to\infty} \|x - T^m x - (x_n - T^m x_n)\|^2 \\
&\quad +4\kappa\widetilde{K} \limsup_{n\to\infty} \|x_n - T^m x_n\| + \varphi(x)\gamma_m + \theta_m \\
&\quad + \limsup_{n\to\infty}(\|x_n - T^m x_n\|^2 + 2\|x_n - T^m x_n\|K'),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x - T^m x\|^2 \\
&\leq \kappa \limsup_{n\to\infty} \|x - T^m x - (x_n - T^m x_n)\|^2 + 4\kappa\widetilde{K} \limsup_{n\to\infty} \|x_n - T^m x_n\| \\
&\quad +\varphi(x)\gamma_m + \theta_m + \limsup_{n\to\infty}(\|x_n - T^m x_n\|^2 + 2\|x_n - T^m x_n\|K').
\end{aligned} \tag{3.3}$$

Since $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, it follows from (3.3) that

$$\limsup_{m\to\infty} \|x - T^m x\|^2 \leq \kappa \limsup_{m\to\infty} \|x - T^m x\|^2.$$

It means that $T^m x \to x$ as $m \to \infty$. Therefore, the continuity of T implies that $(I - T)x = 0$. This completes the proof. ∎

Remark 3.1. Proposition 3.1 extends the demiclosed principles studied for certain classes of nonlinear mappings in Gornicki [8], Kim and Xu [13], Marino and Xu [15], Xu [26] and Sahu, Xu and Yao [28].

Proposition 3.2. Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a continuous asymptotically κ-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$. Then $F(T)$ is closed and convex.

Proof. Since T is continuous, $F(T)$ is closed. To see the convexity of $F(T)$, consider $x, y \in F(T)$. Let $z = (1-t)x + ty$ for $t \in (0,1)$. Note that

$$\|x - z\| = t\|x - y\| \quad \text{and} \quad \|y - z\| = (1-t)\|x - y\|.$$

By Lemma 2.4 (ii) and the relation (1.8), we have

$$\begin{aligned}
\|z - T^n z\|^2 &= \|(1-t)(x - T^n z) + t(y - T^n z)\|^2 \\
&= (1-t)\|x - T^n z\|^2 + t\|y - T^n z\|^2 - t(1-t)\|x - y\|^2 \\
&\leq (1-t)[(1+\gamma_n)\|x - z\|^2 + \kappa\|z - T^n z\|^2 + \theta_n] \\
&\quad +t[(1+\gamma_n)\|y - z\|^2 + \kappa\|z - T^n z\|^2 + \theta_n] - t(1-t)\|x - y\|^2
\end{aligned}$$

$$= (1-t)[(1+\gamma_n)t^2\|x-y\|^2 + \kappa\|z-T^nz\|^2 + \theta_n] \\ +t[(1+\gamma_n)(1-t)^2\|x-y\|^2+\kappa\|z-T^nz\|^2+\theta_n]-t(1-t)\|x-y\|^2 \\ = \kappa\|z-T^nz\|^2 + t(1-t)\gamma_n\|x-y\|^2 + \theta_n.$$

It means that $T^nz \to z$ as $n \to \infty$. The continuity of T implies that $z \in F(T)$. This completes the proof. ∎

We now prove the weak convergence of (1.1) for asymptotically κ-strict pseudocontractive type mappings in the intermediate sense.

Theorem 3.1. Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $0 < \delta \leq \alpha_n \leq 1 - \kappa - \delta < 1$ and $\sum_{n=1}^{\infty} \alpha_n\theta_n < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the modified Mann iteration process:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_nT^nx_n, \quad \forall n \geq 1. \tag{3.4}$$

Then $\{x_n\}$ converges weakly to some element of $F(T)$.

Proof. Let p be an element of $F(T)$. Using Lemma 2.4 (ii), we obtain

$$\begin{aligned} &\|x_{n+1}-p\|^2 \\ &= \|(1-\alpha_n)(x_n-p) + \alpha_n(T^nx_n-p)\|^2 \\ &= (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\|T^nx_n-p\|^2 - \alpha_n(1-\alpha_n)\|x_n-T^nx_n\|^2 \\ &\leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n[(1+\gamma_n)\|x_n-p\|^2+\kappa\|x_n-T^nx_n\|^2+\theta_n] \qquad (3.5) \\ &\quad -\alpha_n(1-\alpha_n)\|x_n-T^nx_n\|^2 \\ &\leq (1+\gamma_n)\|x_n-p\|^2 - \alpha_n(1-\alpha_n-\kappa)\|x_n-T^nx_n\|^2 + \alpha_n\theta_n \\ &\leq (1+\gamma_n)\|x_n-p\|^2 - \delta^2\|x_n-T^nx_n\|^2 + \alpha_n\theta_n \end{aligned}$$

$$\leq (1+\gamma_n)\|x_n-p\|^2 + \alpha_n\theta_n. \tag{3.6}$$

By Lemma 2.1, (3.6) and the assumption $\sum_{n=1}^{\infty} \alpha_n\theta_n < \infty$, we deduce that

$$\lim_{n\to\infty} \|x_n-p\| \text{ exists.} \tag{3.7}$$

Suppose $\lim_{n\to\infty} \|x_n-p\| = r$ for some $r > 0$. It is easy to see from (3.5) that

$$\delta^2\|x_n-T^nx_n\|^2 \leq (1+\gamma_n)\|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n\theta_n,$$

which implies that $\lim_{n\to\infty} \|x_n-T^nx_n\| = 0$. Observe that

$$\|x_{n+1}-x_n\| = \alpha_n\|x_n-T^nx_n\| \leq (1-\kappa-\delta)\|x_n-T^nx_n\| \to 0 \quad \text{as } n \to \infty.$$

Since $\|x_{n+1} - x_n\| \to 0,\ \|x_n - T^n x_n\| \to 0$ as $n \to \infty$, $F(T) \neq \emptyset$, $\{x_n\}$ is a bounded sequence in C and T is uniformly continuous, we obtain from Lemma 2.7 that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$. Note that T is uniformly continuous and $\|x_n - Tx_n\| \to 0$, we see that $\|x_n - T^m x_n\| \to 0$ for all $m \geq 1$. By Proposition 3.1, we obtain $x \in F(T)$. To complete the proof, it suffices to show that $\omega_w(\{x_n\})$ consists of exactly one point, namely, x. Suppose there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $z \neq x$. As in the case of x, we must have $z \in F(T)$. It follows from (3.7) that $\lim_{n\to\infty} \|x_n - x\|$ and $\lim_{n\to\infty} \|x_n - z\|$ exist. Since H satisfies the Opial condition, we have

$$\lim_{n\to\infty} \|x_n - x\| = \lim_{k\to\infty} \|x_{n_k} - x\| < \lim_{k\to\infty} \|x_{n_k} - z\| = \lim_{n\to\infty} \|x_n - z\|,$$

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{j\to\infty} \|x_{n_j} - z\| < \lim_{j\to\infty} \|x_{n_j} - x\| = \lim_{n\to\infty} \|x_n - x\|,$$

which leads to a contradiction. Hence $x = z$. This shows that $\omega_w(\{x_n\})$ is a singleton. Therefore, $\{x_n\}$ converges weakly to x by Lemma 2.2. This completes the proof. ■

We remark that Theorem 3.1 is more general than the results studied in Huang and Lan [9], Kim and Xu [13], Marino and Xu [15], Schu [20] and Sahu, Xu and Yao [28]. As a consequence of Theorem 3.4, we have the following results.

Corollary 3.1. Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive type mapping in the intermediate sense with $F(T) \neq \emptyset$ (in this case, $\gamma_n = 0,\ \forall n \geq 1$). Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $0 < \delta \leq \alpha_n \leq 1-\kappa-\delta < 1$ and $\sum_{n=1}^{\infty} \alpha_n \theta_n < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the modified Mann iteration process defined by (3.4). Then $\{x_n\}$ converges weakly to an element of $F(T)$.

Corollary 3.2. (Sahu, Xu and Yao [28, Theorem 3.4]). Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $0 < \delta \leq \alpha_n \leq 1 - \kappa - \delta < 1$ and $\sum_{n=1}^{\infty} \alpha_n c_n < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the modified Mann iteration process defined by (3.4). Then $\{x_n\}$ converges weakly to an element of $F(T)$.

Corollary 3.3. (Kim and Xu [13, Theorem 3.1]). Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$ and

$\sum_{n=1}^{\infty} \gamma_n < \infty$. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $0 < \delta \le \alpha_n \le 1 - \kappa - \delta < 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the modified Mann iteration process defined by (3.4). Then $\{x_n\}$ converges weakly to an element of $F(T)$.

4. The CQ Method for the Mann Iteration Process

It is proved in Theorems 1.2 and 3.1 that the modified Mann iteration method (1.1) is in general not strongly convergent for either asymptotically nonexpansive mappings or uniformly continuous asymptotically κ-strict pseudocontractive type mappings in the intermediate sense. So to get strong convergence, one has to modify the iteration method (1.1). The main result of this section is the following which is more general than Theorem 4.1 of Kim and Xu [13] and Theorem 4.1 of Sahu, Xu and Yao [28].

Theorem 4.1. Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(T)$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \delta \le \alpha_n \le 1 - \kappa$ for all $n \ge 1$. Let $\{x_n\}$ be the sequence in C generated by the following (CQ) algorithm:

$$
(4.1) \qquad \begin{cases} u = x_1 \in C \text{ chosen arbitrary}, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \Theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, u - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(u), \ \forall n \ge 1, \end{cases}
$$

where $\Theta_n = \theta_n + \gamma_n \Delta_n$ and $\Delta_n = \sup\{\|x_n - z\|^2 : z \in F(T)\} < \infty$. Then $\{x_n\}$ converges strongly to $P_{F(T)}(u)$.

Proof. We divide the proof into the following six steps:

Step 1. C_n is convex.
Indeed, the defining inequality in C_n is equivalent to the inequality

$$\langle 2(x_n - y_n), v \rangle \le \|x_n\|^2 - \|y_n\|^2 + \Theta_n,$$

it follows from Lemma 2.5 that C_n is convex.

Step 2. $F(T) \subset C_n$.

Let $p \in F(T)$. From (4.1), we have

$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\|^2 \\ &= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n((1+\gamma_n)\|x_n - p\|^2 + \kappa\|x_n - T^n x_n\|^2 + \theta_n) \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 \\ &\leq \|x_n - p\|^2 + \alpha_n(\kappa - (1-\alpha_n))\|x_n - T^n x_n\|^2 + \theta_n + \gamma_n \Delta_n \\ &\leq \|x_n - p\|^2 + \theta_n + \gamma_n \Delta_n = \|x_n - p\|^2 + \Theta_n.\end{aligned}$$

Hence $p \in C_n$.

Step 3. $F(T) \subset C_n \cap Q_n$ for all $n \geq 1$.

It is sufficient to show that $F(T) \subset Q_n$. We prove this by induction.

For $n = 1$, we have $F(T) \subset C = Q_1$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of u onto $C_n \cap Q_n$, it follows that

$$\langle x_{n+1} - z, u - x_{n+1}\rangle \geq 0, \quad \forall z \in C_n \cap Q_n.$$

As $F(T) \subset C_n \cap Q_n$, the last inequality holds, in particular for all $z \in F(T)$. By the definition of Q_{n+1},

$$Q_{n+1} = \{z \in C : \langle x_{n+1} - z, u - x_{n+1}\rangle \geq 0\},$$

it follows that $F(T) \subset Q_{n+1}$. By the principle of mathematical induction, we have

$$F(T) \subset Q_n, \quad \forall n \geq 1.$$

Step 4. $\|x_n - x_{n+1}\| \to 0$.

By the definition of Q_n, we obtain that $x_n = P_{Q_n}(u)$ and

$$\|u - x_n\| \leq \|u - y\|, \quad \forall y \in F(T) \subset Q_n.$$

Note that the boundedness of $F(T)$ implies that $\{\|u - x_n\|\}$ is bounded. Since $x_n = P_{Q_n}(u)$ which together with the fact that $x_{n+1} \in C_n \cap Q_n \subset Q_n$ implies that

$$\|u - x_n\| \leq \|u - x_{n+1}\|.$$

Thus, $\{\|u - x_n\|\}$ is nondecreasing. Since $\{\|u - x_n\|\}$ is bounded, we know that $\lim_{n\to\infty} \|x_n - u\|$ exists.

Observe that $x_n = P_{Q_n}(u)$ and $x_{n+1} \in Q_n$ which imply that

$$\langle x_{n+1} - x_n, x_n - u\rangle \geq 0.$$

Using Lemma 2.4 (i), we obtain

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &= \|x_{n+1}-u-(x_n-u)\|^2\\ &= \|x_{n+1}-u\|^2-\|x_n-u\|^2-2\langle x_{n+1}-x_n, x_n-u\rangle\\ &\le \|x_{n+1}-u\|^2-\|x_n-u\|^2 \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Step 5. $\|x_n - Tx_n\| \to 0$.

By the definition of y_n, we have

$$\begin{aligned}\|x_n-T^nx_n\| &= \alpha_n^{-1}\|x_n-y_n\|\\ &\le \alpha_n^{-1}(\|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|)\\ &\le \delta^{-1}(\|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|).\end{aligned} \tag{4.2}$$

Since $x_{n+1}\in C_n$, we have

$$\|y_n-x_{n+1}\|^2 \le \|x_n-x_{n+1}\|^2+\theta_n+\gamma_n\Delta_n \to 0.$$

It follows from (4.2) that

$$\|x_n-T^nx_n\|\to 0 \quad \text{as } n\to\infty. \tag{4.3}$$

Note that T is uniformly continuous such that $F(T)\neq\emptyset$. Since $\|x_n-x_{n+1}\|\to 0$ and $\|x_n-T^nx_n\|\to 0$ as $n\to\infty$ and $\{x_n\}$ is a bounded sequence in C, in terms of Lemma 2.7 we conclude that $x_n-Tx_n\to 0$ as $n\to\infty$.

Step 6. $x_n\to v\in F(T)$.

Since H is reflexive and $\{x_n\}$ is bounded, we obtain that $\omega_w(\{x_n\})$ is nonempty. First, let us show that $\omega_w(\{x_n\})$ is a singleton. Assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i}\rightharpoonup v\in C$. Since $x_n-Tx_n\to 0$ by Step 5, it follows from the uniform continuity of T that $x_n-T^mx_n\to 0$ for all $m\ge 1$. Note that T is uniformly continuous such that $F(T)\neq\emptyset$. Thus, by Proposition 3.1, $v\in\omega_w(\{x_n\})\subset F(T)$.

Since $x_{n+1}=P_{C_n\cap Q_n}(u)$, we get that

$$\|u-x_{n+1}\|\le\|u-P_{F(T)}(u)\|,\quad \forall n\ge 1.$$

Observe that

$$u-x_{n_i}\rightharpoonup u-v.$$

By the weak lower semicontinuity of the norm,

$$\|u-P_{F(T)}(u)\|\le\|u-v\|\le\liminf_{i\to\infty}\|u-x_{n_i}\|\le\limsup_{i\to\infty}\|u-x_{n_i}\|\le\|u-P_{F(T)}(u)\|,$$

which yields

$$\|u-P_{F(T)}(u)\|=\|u-v\|$$

and

$$\lim_{i\to\infty} \|u - x_{n_i}\| = \|u - P_{F(T)}(u)\|. \tag{4.4}$$

Hence $v = P_{F(T)}(u)$ by the uniqueness of the nearest point projection of u onto $F(T)$. Thus, $\|x_{n_i} - u\| \to \|v - u\|$. This shows that $x_{n_i} - u \to v - u$, i.e., $x_{n_i} \to v$. Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, it follows that $\omega_w(\{x_n\}) = \{v\}$ and hence from Lemma 2.2 we have $x_n \rightharpoonup v$. It is easy to see as (4.4) that $\|x_n - u\| \to \|v - u\|$. Therefore, $x_n \to v$. This completes the proof. ■

Corollary 4.1. (Sahu, Xu and Yao [28, Theorem 4.1]). Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(T)$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \delta \le \alpha_n \le 1 - \kappa$ for all $n \ge 1$. Let $\{x_n\}$ be the sequence in C generated by the following (CQ) algorithm:

$$\begin{cases} u = x_1 \in C \text{ chosen arbitrary}, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \vartheta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, u - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(u), \ \forall n \ge 1, \end{cases} \tag{4.5}$$

where $\vartheta_n = c_n + \gamma_n \Delta_n$ and $\Delta_n = \sup\{\|x_n - z\|^2 : z \in F(T)\} < \infty$. Then $\{x_n\}$ converges strongly to $P_{F(T)}(u)$.

Corollary 4.2. (Kim and Xu [12, Theorem 2.2]). Let C be a nonempty closed convex bounded subset of a real Hilbert space H and $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}$ in $[1, \infty)$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \delta \le \alpha_n \le 1$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in C by the following algorithm:

$$\begin{cases} u = x_1 \in C \text{ chosen arbitrary}, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \vartheta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, u - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(u), \ \forall n \ge 1, \end{cases} \tag{4.6}$$

where $\vartheta_n = (k_n^2 - 1)\mathrm{diam}(C)^2$ for all $n \ge 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}(u)$.

Corollary 4.3. (Nakajo and Takahashi [16, Theorem 3.4]). Let C be a nonempty closed convex bounded subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \delta \leq \alpha_n \leq 1$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in C by the following algorithm:

$$(4.7) \qquad \begin{cases} u = x_1 \in C \text{ chosen arbitrary}, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, u - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(u), \ \forall n \geq 1. \end{cases}$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}(u)$.

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