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OPTIMAL CONTROL OF HEMIVARIATIONAL INEQUALITIES WITH DELAYS

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Abstract. In this paper we prove the existence of solutions for hemivariational inequalities with delays and then investigate optimal control problems for some cost functions.

1. Introduction

Let Ω be a given bounded domain in $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$. Let r and T be constants satisfying $0 < r < T$. For $t > 0$, set $Q = (0, T) \times \Omega$, $Q_{-r} = (-r, 0) \times \Omega$ and $\Sigma = (0, T) \times \partial\Omega$. Let *B* be the Borel σ-algebra of the interval $[-r, 0]$ and $\mu(\cdot)$ be a given finite signed measure defined on $([-r, 0], B)$. We define the time-delay operator G as follows: For any $h \in L^2((-r, \infty) \times \Omega; \mathbb{R}^n)$,

$$(Gh)(t, x) \triangleq \int_{-r}^{0} h(t + \theta, x)\mu(d\theta) \quad \text{a.e.} \quad (t, x) \in (0, \infty) \times \Omega.$$

In order for the above integral to make sense, we always take the integrand to be a Borel correction of h (by which we mean a Borel measurable function that is equal to h almost everywhere). In this paper, we shall study the following optimal control problem:

$$(P) \qquad \text{Minimize} \quad J(y, u, v)$$

subject to the hemivariational inequality with delay of the form:

$$(1.1) \qquad \begin{aligned} y'(t, x) &- \triangle y(t, x) + G(\triangle y)(t, x) + \Xi(t, x) \\ &= Bu(t, x) + f(t, y(t, x)) \quad \text{a.e.} \quad (t, x) \in Q, \\ y(0, x) &= \phi_0(x) \quad \text{a.e.} \quad x \in \Omega, \\ y(t, x) &= \phi(t, x) \quad \text{a.e.} \quad (t, x) \in Q_{-r}, \\ \Xi(t, x) &\in \varphi(t, x, v(t, x), y(t, x)) \quad \text{a.e.} \quad (t, x) \in Q, \end{aligned}$$

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where φ is a discontinuous and nonlinear multi-valued mapping by filling in jumps of a locally bounded function b, u and v denote the control variables and B is a bounded linear operator. Here the cost functional $J(y,u,v)$ is given by

$$J(y,u,v) = \int_0^T \{g(y(t)) + h(u(t), v(t))\}dt,$$

where g and h are convex functionals.

Optimal control problems for variational inequalities without delays have been discussed by many authors from different aspect(see [1,3,6]). There is also an extensive literature on the optimal control of infinite-dimensional evolution equations with time-delays(see [2,5]). Pan and Yong([9]) studied the optimal control problem for an abstract parabolic equation with delays in the highest-order spatial derivative terms. Haslinger and Panagiotopoulous([7]) proved the existence of optimal controls for coercive hemivariational inequality and Migórski and Ochal([8]) showed the existence of optimal control problems for parabolic hemivariational inequalities. Zhu([10]) studied the optimal control of variational inequalities with delays in the highest order spatial derivatives.

Motivated by those works, we consider the optimal control problems for hemivariational inequalities with delays. This paper is organized as follows. In section 2, assumptions and lemmas are given. In section 3, the existence of a solution to the problem (1.1) is proved using the Faedo-Galerkin method and finally in section 4 the existence of solutions to the optimal control problem (P) is investigated.

2. Assumptions and Lemmas

Throughout this paper, we denote

$$(y,z) = \int_\Omega y(x)z(x)dx \quad \text{and} \quad \|y\|^2 = \int_\Omega |y(x)|^2 dx$$

and $(\cdot,\cdot)$ the dual pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Let U be a real Hilbert space of variable u, $L^2(Q)$ a space of variable v and $U_{\text{ad}} \times W_{\text{ad}}$ a nonempty subset of $L^2(0,T;U) \times L^2(Q)$. We denote by $\|\cdot\|_X$ the norm of a Banach space X. Now we assume the following conditions concerning (1.1).

(Hyp.b) $b : Q \times \mathbb{R}^2 \to \mathbb{R}$ is a locally bounded function satisfying the following conditions:

(i) b is continuous in η uniformly with respect to ξ, that is, there exists $\delta_0 > 0$ such that for all $(t,x,\eta,\xi) \in Q \times \mathbb{R}^2$ and for all $\varepsilon > 0$, there exists $\gamma = \gamma(\varepsilon,t,x,\eta,\xi) > 0$ such that

$$|b(t,x,\eta,\xi) - b(t,x,\eta',\xi')| < \varepsilon$$

if $|\eta - \eta'| < \gamma$ and $|\xi - \xi'| < \delta_0$.

(ii) $(t,x) \to b(t,x,\eta,\xi)$ is continuous on Q for all $\eta \in \mathbb{R}$ and a.e. $\xi \in \mathbb{R}$.

(iii) $(t,x,\xi) \to b(t,x,\eta,\xi)$ is measurable in $Q \times \mathbb{R}$ for all $\eta \in \mathbb{R}$.

(iv) $|b(t,x,\eta,\xi)| \le \nu_0(t,x) + \nu_1(1+|\eta|+|\xi|)$ for all $(t,x,\eta,\xi) \in Q \times \mathbb{R}^2$ with a nonnegative function $\nu_0 \in L^2(Q)$ and a positive constant ν_1.

The multi-valued function $\varphi : Q \times \mathbb{R}^2 \to 2^{\mathbb{R}}$ is obtained by filling in jumps of a function $b(t,x,\eta,\cdot) : \mathbb{R} \to \mathbb{R}$ by means of the functions $\underline{b}_\varepsilon, \overline{b}_\varepsilon, \underline{b}, \overline{b} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\underline{b}_\varepsilon(t,x,\eta,\xi) = \text{ess inf}_{|s-\xi|\le\varepsilon} b(t,x,\eta,s),$$

$$\overline{b}_\varepsilon(t,x,\eta,\xi) = \text{ess sup}_{|s-\xi|\le\varepsilon} b(t,x,\eta,s),$$

$$\underline{b}(t,x,\eta,\xi) = \lim_{\varepsilon\to 0^+} \underline{b}_\varepsilon(t,x,\eta,\xi),$$

$$\overline{b}(t,x,\eta,\xi) = \lim_{\varepsilon\to 0^+} \overline{b}_\varepsilon(t,x,\eta,\xi),$$

$$\varphi(t,x,\eta,\xi) = [\underline{b}(t,x,\eta,\xi), \overline{b}(t,x,\eta,\xi)].$$

Remark 2.1. Let $j : Q \times \mathbb{R}^2 \to \mathbb{R}$ be a locally Lipschitz continuous function with respect to the last variable obtained from b by integration, that is,

$$j(t,x,\eta,\xi) = \int_0^\xi b(t,x,\eta,\tau)d\tau.$$

Then the following relation holds(see [7]):

$$\varphi(t,x,\eta,\xi) = \partial j(t,x,\eta,\xi),$$

where ∂ denotes the generalized gradient of Clarke(see [4] for example of the definition and the relevant results for Clarke's generalized gradient).

We shall need a regularization of b defined by

$$b^m(t,x,\eta,\xi) = m\int_{-\infty}^{\infty} b(t,x,\eta,\xi-\tau)\rho(m\tau)d\tau,$$

where $\rho \in C_0^\infty((-1,1))$, $\rho \ge 0$ and $\int_{-1}^1 \rho(\tau)d\tau = 1$.

Remark 2.2. It is easy to show that $b^m(t,x,\eta,\xi)$ is continuous in t for all $m \in N$ and $\underline{b}_\varepsilon$, $\overline{b}_\varepsilon$, $\underline{b}$, $\overline{b}$, b^m satisfy the same condition (Hyp.b)(iv) with possibly different constants if b satisfies (Hyp.b)(iv). So, in the remainder of this paper, we denote different constants by the same symbol as original constants.

(Hyp.B) $B : L^2(0,T;U) \to L^2(0,T;L^2(\Omega))$ is a bounded linear operator.

(Hyp.*U,W*) $U_{\rm ad}$ is a closed convex subset of $L^2(0,T;U)$ and $W_{\rm ad}$ is a compact subset of $L^2(Q)$.

(Hyp.f) $(t,x) \to f(t,x,y)$ is measurable in Q for all $y \in \mathbb{R}$ and $f(t,x,\cdot)$ belong to $C^1(\mathbb{R})$. Moreover, for some constant $k>0$, we have

$$f(t,x,0)=0 \quad \text{and} \quad |f_y(t,x,y)| \le k,$$

for all $(t,x,y) \in Q \times \mathbb{R}$.

(Hyp.μ) $\lim_{s\to 0} |\mu|([-r,0])|\mu|([-s,0]) < 1$.

(Hyp.g) $g : L^2(\Omega) \to \mathbb{R}$ is proper, convex and continuous. Moreover, there exists k_1 and $k_2 \in \mathbb{R}$ such that

$$g(y) \ge k_1\|y| + k_2$$

for all $y \in L^2(\Omega)$.

(Hyp.h) $h : U \times L^2(\Omega) \to \bar{\mathbb{R}}$ is a proper, convex and lower semicontinuous functional satisfying

$$h(u,v) \ge k_3(\|u\|_U^2 + \|v\|^2) + k_4$$

for all $(u,v) \in U \times L^2(\Omega)$, where $k_3 > 0$ and $k_4 \in \mathbb{R}$.

Definition 2.1. Given $(u,v) \in L^2(0,T;U) \times L^2(Q)$, $\phi_0 \in H_0^1(\Omega)$ and $\phi \in L^2(-r,0;H_0^1(\Omega)\cap H^2(\Omega))$, y is said to be a solution of (1.1) if $y \in L^2(-r,T;H_0^1(\Omega)$ $\cap H^2(\Omega))\cap W^{1,2}([0,T];L^2(\Omega))$, there exists $\Xi \in L^2(0,T;L^2(\Omega))$ and the following relations hold:

$$\begin{aligned} &\int_0^t (y'(s),w)ds + \int_0^t (\nabla y(s),\nabla w)ds - \int_0^t (G(\nabla y),\nabla w)ds \\ &\quad + \int_0^t (\Xi(s),w)ds \end{aligned}$$

$$= \int_0^t (Bu(s),w)ds + \int_0^t (f(s,y(s)),w)ds, \quad \forall t \in [0,T], \quad \forall w \in H_0^1(\Omega), \tag{2.1}$$

$$\Xi(t,x) \in \varphi(t,x,v(t,x),y(t,x)) \quad \text{a.e.} \quad (t,x) \in Q, \tag{2.2}$$

$$y(0,x) = \phi_0(x) \quad \text{a.e.} \quad x \in \Omega,$$

$$y(t,x) = \phi(t,x), \quad \text{a.e.} \quad (t,x) \in Q_{-r}. \tag{2.3}$$

Remark 2.3. ([10]). For any $0 < s \le +\infty$, G is a bounded linear operator from $L^2([-r,s)\times\Omega;R^m)$ to $L^2((0,s)\times\Omega;R^m)$ and $\|G\| \le \mu([-r,0])$.

3. Existence Results

In this section we are going to show the existence of solutions to the problem (1.1) using the Faedo-Galerkin approximation.

Lemma 3.1. ([10]). *If (Hyp.μ) holds, $y \in L^2((-r,T)\times\Omega; R^m)$, $z \in L^2(Q)$, $\alpha > 0$ and*

$$\int_\Omega |y(t,x)|^2 dx + \alpha \int_0^t \int_\Omega |z(t,x)|^2 dxdt$$
$$\leq \gamma + \delta \int_0^t \int_\Omega |y(t,x)|^2 dxdt + \alpha \Big| \int_0^t \int_\Omega G(z(t,x))z(t,x)dxdt \Big|,$$

for any $t \in [0,T]$ and some constants $\gamma, \delta > 0$, then $y \in L^2(0,T; L^2(\Omega))$.

Furthermore, there exists a constant $C = C(r,T,\delta,\mu(\cdot)) > 0$ such that

$$\|y\|^2 + \alpha \int_0^t \|z\|^2 ds \leq C(\gamma + \alpha \int_{-r}^0 \|z\|^2 ds).$$

Theorem 3.1. Assume that (Hyp.μ), (Hyp.b), (Hyp.B) and (Hyp.f) hold. Let $(u,v) \in L^2(0,T;U)\times L^2(Q)$, $\phi_0 \in H_0^1(\Omega)$ and $\phi \in L^2(-r,0; H_0^1(\Omega)\cap H^2(\Omega))$. Then the problem (1.1) has a solution.

Proof. We represent by $\{w_j\}_{j\geq 1}$ a basis in $H_0^1(\Omega)$ which is orthogonal in $L^2(\Omega)$. Let V_m be the space generated by $w_1, w_2, \cdots, w_m$. We may choose (φ_{0m}) in V_m such that $\varphi_{0m} \to \varphi_0$ in $H_0^1(\Omega)$. Let $y_m(t) = \sum_{j=1}^m g_{jm}(t)w_j$ be the solution of the equation

$$(y_m'(t), w) + (\nabla y_m(t), \nabla w) - (G(\nabla y_m(t)), \nabla w)$$
$$+ (b^m(t, v(t), y_m(t)), w)$$
$$= (f(t, y_m(t)), w) + (Bu(t), w), \quad \forall w \in V_m, \tag{3.1}$$

$$y_m(0) = \phi_{0m}, \tag{3.2}$$

$$y_m(t) = \phi_m(t), \quad t \in [-r, 0). \tag{3.3}$$

By standard differential equation methods, we can prove the existence of a solution to (3.1)-(3.3) on some interval $[0, t_m)$. This solution can be extended to the closed interval $[0,T]$ using a priori estimates below.

Step 1. (A priori estimates).

Replacing w by $y_m(t)$ in (3.1), we obtain

$$
\begin{aligned}
&(y_m'(t), y_m(t)) + (\nabla y_m(t), \nabla y_m(t)) \\
&= (G(\nabla y_m(t)), \nabla y_m(t)) - (b^m(t, v(t), y_m(t)), y_m(t)) \\
&\quad + (f(t, y_m(t)), y_m(t)) + (Bu(t), y_m(t)).
\end{aligned} \tag{3.4}
$$

By (Hyp.b)(iv) and $v \in L^2(Q)$, there exists $c_1 > 0$ such that

$$
\begin{aligned}
&\int_0^t \|b^m(s, v(s), y_m(s))\|^2 ds \\
&\le \int_0^t \int_\Omega |b^m(s, x, v(s,x), y_m(s,x))|^2 dxdt \\
&\le 2\|\nu_0\|^2_{L^2(Q)} + 2\nu_1^2 \int_0^t \int_\Omega (1 + |v(s,x)| + |y_m(s,x)|)^2 dxdt \\
&\le c_1 + 2\nu_1^2 \int_0^t \|y_m(s)\|^2 ds
\end{aligned} \tag{3.5}
$$

and hence

$$
\begin{aligned}
&\Big| \int_0^t (b^m(s, v(s), y_m(s)), y_m(s)) ds \Big| \\
&\le \left(\int_0^t \|b^m(s, v(s), y_m(s))\|^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|y_m(s)\|^2 ds \right)^{\frac{1}{2}} \\
&\le \frac{1}{2}\{c_1 + (2\nu_1^2 + 1) \int_0^t \|y_m(s)\|^2 ds\}.
\end{aligned} \tag{3.6}
$$

From (3.4), (3.6) and integrating over (0,t), we get

$$
\begin{aligned}
&\frac{1}{2}\|y_m(t)\|^2 + \int_0^t \|\nabla y_m(s)\|^2 ds \\
&\le c_2 + \frac{1}{2}\|\phi_{0m}\|^2 + c_3 \int_0^t \|y_m(s)\|^2 ds + \int_0^t (f(s, y_m(s)), y_m(s)) ds \\
&\quad + \int_0^t (Bu(s), y_m(s)) ds + \int_0^t (G(\nabla y_m(s)), \nabla y_m(s)) ds
\end{aligned}
$$

and by (Hyp.f), (Hyp.B) and using Young's inequality, we have

$$
\begin{aligned}
&\frac{1}{2}\|y_m(t)\|^2 + \int_0^t \|\nabla y_m(s)\|^2 ds \\
&\le c_2 + \frac{1}{2}\|\phi_{0m}\|^2 + c_3 \int_0^t \|y_m(s)\|^2 ds + k \int_0^t \|y_m(s)\|^2 ds
\end{aligned}
$$

$$+\frac{M}{2}\int_0^t \|u(s)\|^2 ds + \frac{1}{2}\int_0^t \|y_m(s)\|^2 ds$$
$$+ |\int_0^t (G(\nabla y_m(s)), \nabla y_m(s))ds|$$

for some positive constants c_2, c_3 and M. Since $u \in L^2(0,T;U)$ and $\phi_0 \in H_0^1(\Omega)$, we obtain

$$\|y_m(t)\|^2 + \int_0^t \|\nabla y_m(s)\|^2 ds$$
$$\le c_4 + c_5 \int_0^t \|y_m(s)\|^2 ds + |\int_0^t (G(\nabla y_m(s)), \nabla y_m(s))ds|,$$

where c_4 and c_5 are some constants. Thus, in view of Lemma 3.1, we obtain

$$\|y_m(t)\|^2 + \int_0^t \|\nabla y_m(t)\|^2 \le C(1 + \|\nabla \phi\|^2_{L^2(Q-\gamma)}).$$

Here and in the sequel, we denote C generic positive constant. Since $\varphi \in L^2(-r, 0; H_0^1(\Omega) \cap H^2(\Omega))$, we deduce that

$$\|y_m(t)\|^2 + \int_0^t \|\nabla y_m(t)\|^2 \le C. \tag{3.7}$$

Similarly, replacing w by $\triangle y_m(t)$ in (3.1), we have

$$\begin{aligned}(y_m'(t), \triangle y_m(t)) &+ (\triangle y_m(t), \triangle y_m(t))\\ &= (G(\triangle y_m(t)), \triangle y_m(t)) - (b^m(t, v(t), y_m(t)), \triangle y_m(t))\\ &\quad + (f(t, y_m(t)), \triangle y_m(t)) + (Bu(t), \triangle y_m(t)).\end{aligned} \tag{3.8}$$

From (3.5) we have that

$$\begin{aligned}&\Big|\int_0^t (b^m(s, v(s), y_m(s)), \triangle y_m(s))ds\Big|\\ &\le \Big(\int_0^t \|b^m(s, v(s), y_m(s))\|^2 ds\Big)^{\frac{1}{2}} \Big(\int_0^t \|\triangle y_m(s)\|^2 ds\Big)^{\frac{1}{2}}\\ &\le c_1' + \nu_1^2 \int_0^t \|y_m(s)\|^2 ds + \frac{1}{2}\int_0^t \|\triangle y_m(s)\|^2 ds,\end{aligned} \tag{3.9}$$

where c_1' is a positive constant. From (3.8), (3.9) and integrating over $(0,t)$ we obtain

$$\frac{1}{2}\|\nabla y_m(t)\|^2 + \frac{1}{2}\int_0^t \|\triangle y_m(s)\|^2 ds$$

$$\leq c_2' + \frac{1}{2}\|\nabla\phi_{0m}\|^2 + c_3'\int_0^t \|y_m(s)\|^2 ds + \int_0^t (f(s, y_m(s)), \triangle y_m(s))ds$$
$$+ \int_0^t (Bu(s), \triangle y_m(s))ds + \int_0^t (G(\triangle y_m(s)), \triangle y_m(s))ds,$$

where c_2' and c_3' are some positive constants. By (Hyp.B), (Hyp.f) and imbedding theorem, we derive

$$\frac{1}{2}\|\nabla y_m(t)\|^2 + \frac{1}{2}\int_0^t \|\triangle y_m(s)\|^2 ds$$
$$\leq c_2' + \frac{1}{2}\|\nabla\phi_{0m}\|^2 + c_3'\int_0^t \|y_m(s)\|^2 ds + c_\varepsilon\int_0^t \|f(s, y_m(s))\|^2 ds$$
$$+ \varepsilon\int_0^t \|\triangle y_m(s)\|^2 ds + c_\varepsilon\int_0^t \|u(s)\|^2 ds + \varepsilon\int_0^t \|\triangle y_m(s)\|^2 ds$$
$$+ |\int_0^t (G(\triangle y_m(s)), \triangle y_m(s))ds|$$
$$\leq c_2' + \frac{1}{2}\|\nabla\phi_{0m}\|^2 + c_3'\int_0^t \|y_m(s)\|^2 ds + c_\varepsilon k\int_0^t \|y_m(s)\|^2 ds$$
$$+ \varepsilon\int_0^t \|\triangle y_m(s)\|^2 ds + c_\varepsilon\int_0^t \|u(s)\|^2 ds + \varepsilon\int_0^t \|\triangle y_m(s)\|^2 ds$$
$$+ |\int_0^t (G(\triangle y_m(s)), \triangle y_m(s))ds|$$
$$\leq c_4' + \frac{1}{2}\|\nabla\phi_{0m}\|^2 + 2\varepsilon\int_0^t \|\triangle y_m(s)\|^2 ds + c_5'\int_0^t \|\nabla y_m(s)\|^2 ds$$
$$+ c_\varepsilon\int_0^t \|u(s)\|^2 ds + |\int_0^t (G(\triangle y_m(s)), \triangle y_m(s))ds|,$$

where c_4' and c_5' are some positive constants. Since $u \in L^2(0, T; U)$ and $\phi_0 \in H_0^1(\Omega)$, for a sufficiently small $\varepsilon > 0$, we obtain

$$\|\nabla y_m(t)\|^2 + \int_0^t \|\triangle y_m(s)\|^2 ds$$
$$\leq c_6' + c_7'\int_0^t \|\nabla y_m(s)\|^2 ds + |\int_0^t (G(\triangle y_m(s)), \triangle y_m(s))ds|$$

for some constants c_6' and c_7'. In View of Lemma 3.1 we have

$$\|\nabla y_m(t)\|^2 + \int_0^t \|\triangle y_m(s)\|^2 ds \leq C(1 + \|\triangle\phi\|^2_{L^2(Q-r)}).$$

Since $\phi \in L^2(-\gamma, 0; H_0^1(\Omega) \cap H^2(\Omega))$, we derive that

$$\|\nabla y_m(t)\|^2 + \int_0^t \|\triangle y_m(s)\|^2 ds \le C. \tag{3.10}$$

From (Hyp.f) and (3.7)

$$\|f(s, y_m(s))\|_{L^\infty(0,T;L^2(\Omega))} \le C. \tag{3.11}$$

Thus from (3.7), (3.10) and (3.11), we obtain

$$\|y_m'\|_{L^2(0,T;L^2(\Omega))} \le C. \tag{3.12}$$

Step 2. (Passage to the limit).

From the priori estimates (3.5), (3.7), (3.10) and (3.12) for a subsequence we deduce that

$$\begin{aligned} y_m \to y \quad &\text{weakly star in} \quad L^\infty(0,T;H_0^1(\Omega)),\\ y_m \to y \quad &\text{strongly in} \quad L^2(0,T;L^2(\Omega)),\\ y_m' \to y' \quad &\text{weakly in} \quad L^2(0,T;L^2(\Omega)),\\ \triangle y_m \to \triangle y \quad &\text{weakly in} \quad L^2(0,T;L^2(\Omega)),\\ b^m(v, y_m) \to \Xi \quad &\text{weakly in} \quad L^2(0,T;L^2(\Omega)). \end{aligned} \tag{3.13}$$

Since G is a bounded linear operator and $f(t,\cdot) \in C^1(\mathbb{R})$, we can take limit $m \to \infty$ in (3.1). Hence we have

$$\begin{aligned} &(y'(t), w) + (\nabla y(t), \nabla w) - (G(\triangle y(t)), \triangle w) + (\Xi, w)\\ &= (f(t, y(t)), w) + (Bu(t), w), \quad \forall w \in H_0^1(\Omega). \end{aligned} \tag{3.14}$$

Step 3. (y is a solution of (1.1)).

We will show that $\Xi(t,x) \in \varphi(t, x, v(t,x), y(t,x))$ a.e. $(t,x) \in Q$. From (3.13) we infer that

$$y_m(t,x) \to y(t,x) \quad \text{a.e.} \quad (t,x) \in Q.$$

Let $\eta > 0$. Using the theorems of Lusin and Egoroff, we can choose a subset $W \subset Q$ such that meas$(W) < \delta$, $y \in L^\infty(Q - W)$ and $y_m \to y$ uniformly on $Q - W$. Thus, for each $\varepsilon > 0$, there is an $N > \frac{2}{\varepsilon}$ such that $|y_m(t,x) - y(t,x)| < \frac{\varepsilon}{2}$ for all $(t,x) \in Q - W$ and $m > N$. Then, if $|y_m(t,x) - s| < \frac{1}{m}$, we have $|y(t,x) - s| < \varepsilon$ for all $m > N$ and $(t,x) \in Q$. Therefore we have

$$\begin{aligned}\underline{b}_\varepsilon(t,x,v(t,x),y(t,x)) &\le b^m(t,x,v(t,x),y(t,x))\\ &\le \bar{b}_\varepsilon(t,x,v(t,x),y(t,x))\end{aligned}$$

for all $m > N$ and $(t,x) \in Q - W$. Let $r \in L^2(Q)$ and $r \ge 0$. Then

$$\begin{aligned}&\int_{Q-W} \underline{b}_\varepsilon(t,x,v(t,x),y(t,x))r(t,x)dxdt\\ &\le \int_{Q-W} b^m(t,x,v(t,x),y_m(t,x))r(t,x)dxdt\\ &\le \int_{Q-W} \bar{b}_\varepsilon(t,x,v(t,x),y(t,x))r(t,x)dxdt.\end{aligned} \tag{3.15}$$

Letting $m \to \infty$ in (3.15) and using (3.13) we obtain

$$\begin{aligned}&\int_{Q-W} \underline{b}_\varepsilon(t,x,v(t,x),y(t,x))r(t,x)dxdt\\ &\le \int_{Q-W} \Xi(t,x)r(t,x)dxdt\\ &\le \int_{Q-W} \bar{b}_\varepsilon(t,x,v(t,x),y(t,x))r(t,x)dxdt.\end{aligned} \tag{3.16}$$

Letting $\varepsilon \to 0^+$ in (3.16), we infer that $\Xi(t,x) \in \varphi(t,x,v(t,x),y(t,x))$ a.e. in $Q - W$. Letting $\delta \to 0^+$, then we have $\Xi(t,x) \in \varphi(t,x,v(t,x),y(t,x))$ a.e. in Q. Therefore the proof of Theorem 3.1 is complete.

4. Existence of the Solutions of the Optimal Control Problem

We denote by $S(u,v)$ the set of all solutions of the problem (1.1) for a given $(u,v) \in U_{\text{ad}} \times W_{\text{ad}}$. Theorem 3.1 implies that $S(u,v) \ne \phi$ for all $(u,v) \in U_{\text{ad}} \times W_{\text{ad}}$. Let us consider the following optimal control problem (P):

$$\text{Minimize}\{J(y,u,v) : (u,v) \times U_{\text{ad}} \times W_{\text{ad}}, \quad y \in S(u,v)\}.$$

Theorem 4.1. *For a given $(u,v) \in U_{ad} \times W_{ad}$, the following estimate holds:*

$$\begin{aligned}&\sup_{y\in S(u,v)}\{\|y\|^2_{L^\infty(0,T;L^2(\Omega))} + \|y\|^2_{L^\infty(0,T;H^1_0(\Omega))}\\ &\quad + \|y\|^2_{L^2(0,T;H^2(\Omega))} + \|y'\|^2_{L^2(0,T;L^2(\Omega))}\}\\ &\le c_1(1 + \|\nabla\phi_0\|^2_{L^2(\Omega)} + \|\nabla\phi\|_{L^2(-r,0;L^2(\Omega))})\\ &\quad + c_2(\|u\|^2_{L^2(0,T;U)} + \|v\|^2_{L^2(0,T;L^2(\Omega))})\end{aligned}$$

where c_1 and c_2 are positive constants.

Proof. Let $y \in S(u,v)$. Then y is satisfies (2.1)-(2.3). Replacing w by $y(t)$ in (2.1) and then using (Hyp.f) and Young's inequality, we obtain

$$\begin{aligned}&\|y(t)\|^2 + \int_0^t \|\nabla y(s)\|^2 ds \\ &\le c(1 + \|\phi_0\|^2 + \int_0^t \|y(s)\|^2 ds) + \int_0^t \|Bu(s)\|^2 ds + \int_0^t \|\Xi(s)\|^2 ds \\ &\quad + |\int_0^t (G(\nabla y(s)), \nabla y(s)) ds|).\end{aligned} \tag{4.1}$$

By the assumption on b (see (Hyp.b)(iv)) and Remark 2.2, we can easily show that

$$\int_0^t \|\Xi(s)\|^2 ds \le c_1 + c_2\{\int_0^t (\|y(s)\|^2 + \|v(s)\|^2) ds\}. \tag{4.2}$$

From (4.1), (4.2) and Lemma 3.1, we deduce that

$$\begin{aligned}&\|y(t)\|^2 + \int_0^t \|\nabla y(s)\|^2 ds \le c(1+\|\phi_0\|^2 \\ &+\|\nabla\phi\|^2_{L^2(-r,0;L^2(\Omega))}) + \|Bu(s)\|^2_{L^2(0,T;U)} + \|v\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{4.3}$$

Similarly, replacing w by $\triangle y(t)$ in (1.1), we obtain

$$\begin{aligned}&\|\nabla y(t)\|^2 + \int_0^t \|\triangle y(s)\|^2 ds \\ &\le c(1 + \|\nabla\phi_0\|^2 + \|\triangle\phi\|^2_{L^2(-r,0;L^2(\Omega))} + \|Bu(t)\|^2_{L^2(0,T;U)} \\ &\quad + \|v\|^2_{L^2(0,T;L^2(\Omega))}).\end{aligned} \tag{4.4}$$

Moreover, from (1.1), (4.1)-(4.4) we have that

$$\begin{aligned}\|y'\|^2_{L^2(0,T;L^2(\Omega))} &\le c(1 + \|\nabla\phi_0\|^2 + \|\triangle\phi\|^2_{L^2(-r,0;L^2(\Omega))} \\ &\quad +\|Bu\|^2_{L^2(0,T;U)} + \|v\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{4.5}$$

Since B is a bounded linear operator, (4.3), (4.4) and (4.5) complete the proof of Theorem 4.1.

Theorem 4.2. *Assume that the conditions of Theorem 3.1, $($Hyp,U,$W)$, $($Hyp.$g)$ and $($Hyp.$h)$ hold. Then the optimal control problem (P) has at least one solution.*

Proof. Let $d = \inf\{J(y,u,v)|(u,v) \in U_{ad} \times W_{ad}, \quad y \in S(u,v)\}$. By the assumptions g and h, it is clear that $d > -\infty$. Let $(y_n, u_n, v_n) \in S(u_n, v_n) \times U_{\text{ad}} \times W_{\text{ad}}$ be a minimizing sequence, that is,

$$\int_0^t (y_n'(s), w)ds + \int_0^t (\nabla y_n(s), \nabla w)ds - \int_0^t (G(\nabla y_n(s)), \nabla w)ds + \int_0^t (\Xi_n(s), w)ds$$

$$= \int_0^t (Bu_n(s), w)ds + \int_0^t (f(s, y_n(s)), w)ds, \ \forall t \in (0,T), \forall w \in H_0^1(\Omega), \tag{4.7}$$

$$\Xi_n(t,x) \in \varphi(t,x,v_n(t,x),y_n(t,x)) \quad \text{a.e.} \quad (t,x) \in Q,$$

$$y_n(0,x) = \phi_0(x), \tag{4.8}$$

$$y_n(t,x) = \phi(t,x) \quad \text{a.e.} \quad (t,x) \in Q_{-r}$$

and

$$d \le J(y_n, u_n, v_n) \le d + \frac{1}{n}, \quad n = 1, 2, \cdots. \tag{4.9}$$

From (Hyp.h), (u_n, v_n) is bounded in $U_{\text{ad}} \times W_{\text{ad}} \subset L^2(0,T;U) \times L^2(Q)$. Thus a subsequence can be determined such that

$$u_n \to u^* \quad \text{weakly in} \quad L^2(0,T;U). \tag{4.10}$$

By (Hyp.U), U_{ad} is weakly closed, and hence $u^* \in U_{\text{ad}}$. Also, since W_{ad} is compact in $L^2(Q)$ and (v_n) is bounded in W_{ad}, we have

$$v_n \to v^* \quad \text{strongly in} \quad L^2(Q) \tag{4.11}$$

and $v^* \in W_{\text{ad}}$. Therefore, by Theorem 4.1, we see that

$$(y_n) \quad \text{is bounded in} \quad L^\infty(0,T; H_0^1(\Omega) \cap L^2(\Omega)),$$

$$(y_n) \quad \text{is bounded in} \quad L^2(0,T; H^2(\Omega)), \tag{4.12}$$

$$(y_n') \quad \text{is bounded in} \quad L^2(0,T; L^2(\Omega)).$$

This together with the fact that

$$\int_0^t \|\Xi_n(s)\|^2 ds \le c + c\int_0^t (\|y_m(s)\|^2 + \|v_n(s)\|^2) ds$$

implies that $\{\Xi_n\}$ is bounded in $L^2(0,T;L^2(\Omega))$. Therefore, we have

$$
(4.13)\qquad
\begin{aligned}
y_n &\to y^* \quad \text{weakly star in} \quad L^\infty(0,T;H_0^1(\Omega)),\\
y_n &\to y^* \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)),\\
y_n &\to y^* \quad \text{weakly in} \quad L^2(0,T;H^2(\Omega)),\\
y_n' &\to y^{*'} \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)),\\
\Xi_n &\to \Xi^* \quad \text{weakly in} \quad L^2(o,T;L^2(\Omega)).
\end{aligned}
$$

Since $f(t,\cdot)$ belong to $C(\mathbb{R})$, using (4.10), (4.11), (4.13) and letting $n \to \infty$ in (4.7), we conclude that

$$
(4.14)\qquad
\begin{aligned}
&\int_0^t (y^{*'}(s),w)ds + \int_0^t (\nabla y^*(s),\nabla w)ds - \int_0^t (G(\nabla y^*(s)),\nabla w)ds\\
&+ \int_0^t (\Xi^*(s),w)ds\\
&= \int_0^t (Bu^*(s),w)ds + \int_0^t (f(s,y^*(s)),w)ds, \ \ \forall t\in(0,T), \forall w \in H_0^1(\Omega).
\end{aligned}
$$

To show that $y^* \in S(u^*,v^*)$, it is sufficient to show that

$$
(4.15)\qquad \Xi^*(t,x) \in \varphi(t,x,v^*(t,x),y^*(t,x)) \quad \text{a.e.} \quad (t,x)\in Q.
$$

Indeed, by (4.13) and the Aubin-Lions compactness lemma, we get $y_n \to y^*$ strongly in $L^2(0,T;L^2(\Omega))$ and hence $y_n(t,x) \to y^*(t,x)$ a.e. $(t,x)\in Q$. By the theorems of Lusin and Egoroff, for a given $\eta > 0$, we can choose a subset $W \subset Q$ such that meas$(W) < \eta$ and $y_n \to y^*$ uniformly in $Q - W$. Thus for each $\varepsilon > 0$, there is a positive integer N such that

$$
|y_n(t,x) - y^*(t,x)| < \frac{\varepsilon}{2}, \quad \forall (t,x) \in Q - W, \forall n > N.
$$

On the other hand, (4.8) implies that

$$
(4.16)\qquad
\begin{aligned}
&\int_{Q-W} \underline{b}_{\frac{\varepsilon}{2}}(t,x,v_n(t,x),y_n(t,x))\phi(t,x)dxdt\\
&\le \int_{Q-W} \Xi_n(t,x)\phi(t,x)dxdt\\
&\le \int_{Q-W} \bar{b}_{\frac{\varepsilon}{2}}(t,x,v_n(t,x),y_n(t,x))\phi(t,x)dxdt
\end{aligned}
$$

for any $\phi \in L^2(Q)$ with $\phi \geq 0$. Note that for $n > N$,

$$\begin{aligned}\underline{b}_{\frac{\varepsilon}{2}}(t,x,v_n(t,x),y_n(t,x)) &= \text{ess inf}_{|s-y^n|\leq\frac{\varepsilon}{2}} \quad b(t,x,v_n(t,x),s)\\ &\geq \text{ess inf}_{|s-y^*|\leq\varepsilon} \quad b(t,x,v_n(t,x),s)\\ &= \underline{b}_{\varepsilon}(t,x,v_n(t,x),y^*(t,x))\end{aligned}$$

and

$$\begin{aligned}\bar{b}_{\frac{\varepsilon}{2}}(t,x,v_n(t,x),y_n(t,x)) &= \text{ess sup}_{|s-y_n|<\frac{\varepsilon}{2}} \quad b(t,x,v_n(t,x),s)\\ &\leq \text{ess sup}_{|s-y^*|\leq\varepsilon} \quad b(t,x,v_n(t,x),s)\\ &= \bar{b}_{\varepsilon}(t,x,v_n(t,x),y^*(t,x)).\end{aligned}$$

From (4.16) we obtain

$$\begin{aligned}&\int_{Q-W} \underline{b}_{\varepsilon}(t,x,v_n(t,x),y^*(t,x))\phi(t,x)dxdt\\ &\leq \int_{Q-W} \Xi_n(t,x)\phi(t,x)dxdt\\ &\leq \int_{Q} \bar{b}_{\varepsilon}(t,x,v_n(t,x),y^*(t,x))\phi(t,x)dxdt.\end{aligned} \tag{4.17}$$

Letting $n \to \infty$ in (4.17) and using (4.11) and (Hyp.b), we conclude that

$$\begin{aligned}&\int_{Q-W} \underline{b}_{\varepsilon}(t,x,v^*(t,x),y^*(t,x))\phi(t,x)dxdt\\ &\leq \int_{Q-W} \Xi^*(t,x)\phi(t,x)dxdt\\ &\leq \int_{Q-W} \bar{b}_{\varepsilon}(t,x,v^*(t,x),y^*(t,x))\phi(t,x)dxdt.\end{aligned} \tag{4.18}$$

Letting $\varepsilon \to 0^+$ in (4.18), we infer that $\Xi^*(t,x) \in \varphi(t,x,v^*(t,x),y^*(t,x))$ a.e. in $Q-W$ and letting $\eta \to 0^+$ we get

$$\Xi^*(t,x) \in \varphi(t,x,v^*(t,x),y^*(t,x)) \quad \text{a.e. in} \quad Q.$$

Hence $(y^*,u^*,v^*) \in S(u^*,v^*) \times U_{\text{ad}} \times W_{\text{ad}}$ is admissible pair for problem (P). Taking the limit $n \to \infty$ in (4.9) and using the lower semicontinuity of J, we conclude that

$$d \leq J(y^*,u^*,v^*) \leq \lim_{n\to\infty} J(y_n,u_n,v_n) \leq d.$$

Thus (y^*,u^*,v^*) is a solution of the optimal control problem (P).

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