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CANTOR'S THEOREM IN 2-METRIC SPACES AND ITS APPLICATIONS TO FIXED POINT PROBLEMS

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Abstract. 2-metric space is an interesting nonlinear generalization of metric space which was conceived and studied in details by Gahler. In this paper, for the first time, we establish Cantor's intersection theorem and Baire category theorem in 2-metric spaces. As a departure from normal practice we then apply Cantor's theorem to establish some fixed point theorems in such spaces.

1. Introduction

The concept of a 2-metric space has been initiated by Gähler in a series of papers [3-5]. This space was shown to have a unique nonlinear structure, quite different from a metric space and has subsequently been studied by various workers. Gähler himself and White [20] extended the concept to 2-Banach spaces, where White established Hahn-Banach theorem in a 2-Banach space. Banach-Steinhaus theorem is also available [12] now in 2-Banach spaces. As in other spaces, the fixed point theory of operators has been developed in such spaces also. Perhaps Iseki [7-9] obtained for the first time basic results on fixed point of operators in 2-metric spaces and in 2-Banach spaces. After the works of Iseki, several authors extended and generalized fixed point theorems in 2-metric and 2-Banach spaces for different types of operators (see [6, 8, 13-18] where many more references can be found) including operators of contractive type (for detailed information, one is referred to Iseki [10]).

The concept of subbasis that forms a topological space has also been considered by Gähler [3]. He showed that by suitably defining the members of the subbasis, a topology can be considered in a 2-metric space. In this paper we use this information

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and after introducing a new concept $\delta_c(A) = \sup\{\sigma(a,b,c);\ a,b \in A\}$ where $c \in X$ and $A \subset X$ (the quantity $\delta_c(A)$ is not really the diameter of A, unlike metric spaces), we prove analogues of Cantor's intersection theorem and Baire category theorem in 2-metric spaces. As far as our knowledge is concerned, these two important results were not established before. Further we observe that the intersection theorem along with the idea of a set S_a (defined below) may be conveniently used to prove Banach's fixed point theorem in 2-metric spaces. Later on, some other fixed point theorems have also been obtained. This approach is entirely different from the usual sequencial approach and follows the line of [1] and [11].

2. Definitions and Lemmas

We first recall the following definition of 2-metric spaces from [3].

Definition 1. Let X be a non-empty set and let $\sigma(.,.,.)$ be a mapping from $X \times X \times X$ to R i.e. $\sigma : X^3 \to R$ satisfying the following conditions:

(i) for every pair of distinct points a, b there exists a point $c \in X$ such that $\sigma(a,b,c) \neq 0$.

(ii) $\sigma(a,b,c) = 0$ only if at least two of the three points are same.

(iii) $\sigma(a,b,c) = \sigma(a,c,b) = \sigma(b,c,a)$ for all $a,b,c \in X$.

(iv) $\sigma(a,b,c) \leq \sigma(a,b,d) + \sigma(a,d,c) + \sigma(d,b,c)$ for all a,b,c and $d \in X$.

Then σ is called a 2-metric on X and (X,σ) is called a 2-metric space which will sometimes be denoted simply by X, when there is no confusion.

It can be easily seen that σ is a non-negative function. We shall assume throughout that X is an infinite set.

Definition 2. (cf. [3]). Let (X,σ) be a 2-metric space. Let $a,b \in X$ and $r > 0$. The subset

$$B_r(a,b) = \{c \in X; \sigma(a,b,c) < r\}$$

of X will be called a 2-ball centered at a and b with radius r.

From the definition of a 2-metric, it is clear that $B_r(a,b)$ is the same as $B_r(b,a)$. Gähler [3] observed that a topology can be generated in X by taking the collection of all 2-balls as a subbasis, which we call here the 2-metric topology, to be denoted by τ. Thus (X,τ) is a $2-$metric topological space. Members of τ are called 2-open sets and their complements, 2-closed sets.

Lemma 1. *A subset U of (X,τ) is 2-open if and only if for any $x \in U$ there are finite number of points $a_1, a_2, ..., a_n \in X, r_1, r_2, ..., r_n > 0$ such that $x \in B_{r_1}(x,a_1) \bigcap .... \bigcap B_{r_n}(x,a_n) \subset U$.*

Proof. Since each set of the form $B_{r_1}(x, a_1) \bigcap .... \bigcap B_{r_n}(x, a_n)$ is 2-open by definition, the sufficiency of the condition follows immediately. Conversely let U be 2-open and $x \in U$. Then there exists a finite number of 2-balls $B_{r_i}(a_i, b_i), i = 1, 2, 3...., m$ (say) such that

$$x \in \bigcap_{i=1}^{m} B_{r_i}(a_i, b_i) \subset U.$$

Since $x \in B_{r_i}(a_i, b_i)$, so $\sigma(x, a_i, b_i) = s_i < r_i$. Choose $t_i < \frac{r_i - s_i}{2}$. Then $B_{t_i}(x, a_i) \bigcap B_{t_i}(x, b_i) \subset B_{r_i}(a_i, b_i)$ and this is true for $i = 1, 2, ..., m$. So

$$\begin{aligned} x \in B_{t_1}(x, a_1) \cap B_{t_1}(x, b_1) \cap B_{t_2}(x, a_2) \cap B_{t_2}(x, b_2) \cap .... \\ \cap B_{t_m}(x, a_m) \cap B_{t_m}(x, b_m) \\ \subset \bigcap_{i=1}^{m} B_{r_i}(a_i, b_i) \subset U. \end{aligned}$$

This proves the lemma.

Definition 3. For $A \subset (X, \tau)$, the 2-closure of A, denoted by $\bar{A}$ is defined to be the intersection of all 2-closed sets containing A.

Definition 4. $x \in (X, \tau)$ is called a 2-limit point of $A \subset X$ if for any 2-open set U containing $x,\ A \cap (U - \{x\}) \neq \phi$.

As in a topological space, $\bar{A}$ can also be defined by $\bar{A} = A \cup \partial A$ where ∂A is the derived set of A that consists of all 2-limit points of A. For any $A \subset X,\ \bar{A}$ is clearly a 2-closed set.

Lemma 2. $A \subset (X, \tau)$ *is 2-closed if and only if* $\bar{A} = A$.

The proof is omitted.

Lemma 3. (X, τ) *is* T_1.

Proof. Let $a, b \in X, a \neq b$. *Then there is a point* $c \in X$ *such that* $\sigma(a, b, c) = r\ (say)\ > 0$. *If* $s = \frac{r}{2}$ *then* $B_s(a, c)$ *and* $B_s(b, c)$ *are two 2-open sets with* $a \in B_s(a, c),\ b \in B_s(b, c)$ *but* $a \notin B_s(b, c), b \notin B_s(a, c)$. *Hence the lemma.*

The definition of convergence of a sequence in (X, σ) is known in the following form.

Definition 5. (cf. [7]). A sequence $\{x_n\}$ in (X, σ) is said to converge to $x \in X$ if for any $a \in X,\ \sigma(x_n, x, a) \to 0$ as $n \to \infty$.

This fact is written as $x_n \to x$ as $n \to \infty$ or $\lim_{n \to \infty} x_n = x$.

Lemma 4. *A sequence* $\{x_n\}$ *is convergent to* x *in* (X, σ) *if and only if for any 2-open set* U *containing* x *there exists a positive integer* m *such that* $x_n \in U\ \forall\ n \geq m$.

Proof. Assume first the given condition. Let $a \in X$ and $\epsilon > 0$. Since $B_\epsilon(x,a)$ is a 2-open set containing x, there exists $m \in N$ such that $x_n \in B_\epsilon(x,a)\ \forall\ n \geq m$ i.e. $\sigma(x_n,x,a) < \epsilon\ \forall\ n \geq m$ which shows that $\sigma(x_n,x,a) \to 0$ as $n \to \infty$. Thus $\{x_n\}$ converges to x in (X,σ).

Conversely let $\{x_n\}$ be convergent to x in (X,σ). Let U be a 2-open set with $x \in U$. From lemma 1, we have $x \in B_{r_1}(x,a_1) \cap ..... \cap B_{r_k}(x,a_k) \subset U$ for some $a_1, ..., a_k \in X$ and $r_1, r_2, ..., r_k > 0$. Since $\sigma(x_n,x,a_i) \to 0$ as $n \to \infty$, there exists $m_i \in N$ such that $\sigma(x_n,x,a_i) < r_i$ for all $n \geq m_i$ i.e. $x_n \in B_{r_i}(x,a_i)\ \forall\ n \geq m_i$ and this is true for each $i = 1, 2, ..., k$. Taking $m = max\{m_1, ..., m_k\}$ we obtain $x_n \in B_{r_1}(x,a_1) \cap ..... \cap B_{r_k}(x,a_k) \subset U,\ \forall\ n \geq m$
and the lemma is proved.

Note 1. It is known that in a metric space, a set A is closed if and only if every convergent sequence of points of A converges to a point of A. But the situation appears to be different in a 2-metric space, because of the non-availability of the first axiom of countability. If however the 2-metric topology τ is first countable (for example if X is countable) then an analogous conclusion is true in (X,σ).

Definition 6. (cf. [7]). A sequence $\{x_n\}$ in (X,σ) is said to be a <u>Cauchy sequence</u> if for any $a \in X, \sigma(x_m,x_n,a) \to 0$ as $m,n \to \infty$.

Definition 7. ([7, 8]). (X,σ) is said to be <u>complete</u> if every Cauchy sequence in X converges to a point of X.

Note 2. In a complete 2-metric space, a convergent sequence need not be a Cauchy sequence {see [15], Example 0.1}.

Definition 8. (X,σ) is said to be <u>compact</u> if every sequence in X has a convergent subsequence.

Definition 9. $A \subset X$ is said to be <u>dense</u> in X if $\bar{A} = X$.

Definition 10. $A \subset X$ is said to be <u>no-where dense</u> if $int(\bar{A}) = \phi$ where interior of a set B is defined to be the union of all 2-open sets contained in B.

Definition 11. A mapping $T : (X,\sigma) \to (Y,\sigma_1)$ where (Y,σ_1) is another 2-metric space, is called <u>continuous</u> at $x \in X$ if for any 2-open set V containing $f(x)$ in Y, there is a 2-open set U containing x in X such that $T(U) \subset V$.

Lemma 5. *If $T : (X,\sigma) \to (Y,\sigma_1)$ is continuous at $x \in X$, then $x_n \to x$ in (X,σ) implies $T(x_n) \to T(x)$ in (Y,σ_1).*

The proof is straight forward and so omitted.

If T is continuous at each point x of (X,σ) then T is said to be a continuous function.

3. Cantor's and Baire's Theorem in 2-Metric Spaces

In this section we prove an analogue of Cantor's intersection theorem for complete 2-metric spaces and use it to show that such a space cannot be expressed as a countable union of no-where dense sets under some general situations.

For $A \subset X$, we define

$$\delta_c(A) = sup\{\sigma(a,b,c);\ a,b \in A\}$$

where $c \in X$.
The quantity $\delta_c(A)$ need not be considered as the diameter of A. However if (X,σ) is bounded in the sense of Iseki [7] (i.e. $sup\{\sigma(a,b,c\};\ a,b,c \in X\} < \infty$) then for every $A \subset X,\ \delta_c(A)$ is finite $\forall\ c \in X$.

The idea of $\delta_c(A)$ is helpful to prove the following theorems.

Theorem 1. *Suppose that (X,σ) is a complete 2-metric space. If $\{F_n\}$ is any decreasing sequence (i.e. $F_{n+1} \subset F_n\ \forall\ n \in N$) of 2-closed sets with $\delta_a(F_n) \to 0$ as $n \to \infty\ \forall\ a \in X$ then $\bigcap\limits_{n=1}^{\infty} F_n$ is non-empty and contains at most one point.*

Proof. For each positive integer n, let x_n be a point of F_n. We show that $\{x_n\}$ is a Cauchy sequence in X. Since $\{F_n\}$ is decreasing, $x_m \in F_n\ \forall\ m \geq n$. Now for any $a \in X,\ m \geq n$,

$$\sigma(x_m, x_n, a) \leq \delta_a(F_n) \to 0 \text{ as } n \to \infty.$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $x_n \to x$ (say) in X . We claim that $x \in \bigcap F_n$. We may assume that $x_k \neq x$ from some k onwards, otherwise there is nothing to prove. Let $n \in N$ be fixed. Let U be any 2-open set containing x. By Lemma 4, there is $n_1 \in N$ such that $x_k \in U\ \forall\ k \geq n_1$. Then $x_k \in [U - \{x\}] \bigcap F_n\ \forall\ k \geq max\{n, n_1\}$. This shows that $x \in \bar{F_n} = F_n$, since F_n is 2-closed. As this is true for all $n \in N,\ x \in \bigcap\limits_{n=1}^{\infty} F_n$.

Finally we prove that $\bigcap\limits_{n=1}^{\infty} F_n$ contains at most one point. If possible let us suppose that it contains two distinct points x and y. Choose $z \in X,\ z \neq x, y$. From the definition of $\delta_z(F_n)$,

$$\sigma(x,y,z) \leq \delta_z(F_n)\ \forall\ n \in N.$$

Since $\delta_z(F_n) \to 0$ as $n \to \infty,\ \sigma(x,y,z) = 0$ which is a contradiction. This proves the theorem.

To prove the converse of theorem 1, the following lemma is needed.

Lemma 6. *For any $A \subset X$ and $a \in X$*

$$\delta_a(A) = \delta_a(\bar{A}).$$

Proof. Since $A \subset \bar{A}$, it follows that $\delta_a(A) \leq \delta_a(\bar{A})$. To prove the converse inclusion, let $x, y \in \bar{A}$. If both x, y belong to A, then clearly $\sigma(x, y, a) \leq \delta_a(A)$. So suppose first that one of them, say, $x \notin A$ but $y \in A$. Let $\epsilon > 0$ be arbitrary. Since $x \in \bar{A}$ and $B_\epsilon(x, y) \cap B_\epsilon(x, a)$ is a 2-open set containing x, there exists $z \in A \cap [B_\epsilon(x, y) \cap B_\epsilon(x, a)]$. Then

$$\begin{aligned} \sigma(x, y, a) &\leq \sigma(x, z, a) + \sigma(y, z, a) + \sigma(x, y, z) \\ &\leq \delta_a(A) + 2\epsilon. \end{aligned}$$

Since this is true for every $\epsilon > 0$, we conclude that

$$\sigma(x, y, a) \leq \delta_a(A) \text{ for } y \in A \text{ and } x \in \bar{A}.$$

Finally, if $x, y \in \bar{A} - A$ then repeating the same argument we can show that in this case also $\sigma(x, y, a) \leq \delta_a(A)$. Hence

$$\delta_a(\bar{A}) = sup\{\sigma(x, y, a);\ x, y \in \bar{A}\} \leq \delta_a(A).$$

and so $\delta_a(A) = \delta_a(\bar{A})$.

This proves the lemma.

The converse of Theorem 1 is contained in the following theorem.

Theorem 2. *If in a 2-metric space (X, σ), for any decreasing sequence of 2-closed sets $\{F_n\}$ with $\delta_a(F_n) \to 0$ as $n \to \infty\ \forall\ a \in X$, $\bigcap_{n=1}^{\infty} F_n$ consists of a single point then (X, σ) is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Let $F_n = \{x_n, x_{n+1}, ...\}$ for any $n \in N$. Then $F_n \supset F_{n+1}$ and so $\bar{F_n} \supset \bar{F}_{n+1}\ \forall\ n \in N$. So $\{\bar{F_n}\}$ is a decreasing sequence of 2-closed sets. For $a \in X$ and $\epsilon > 0$ arbitrary, there is $n_1 \in N$ such that

$$\sigma(x_m, x_n, a) \quad < \quad \epsilon\ \forall\ m, n \geq n_1.$$

This shows that $\delta_a(F_{n_1}) \leq \epsilon$ and so by Lemma 6 $\delta_a(\bar{F}_{n_1}) \leq \epsilon$. Since $\{\bar{F}_n\}$ is decreasing, for $n \geq n_1$, $\delta_a(\bar{F}_n) \leq \delta_a(\bar{F}_{n_1}) \leq \epsilon$. Therefore $\delta_a(\bar{F}_{n)} \to 0$ as $n \to \infty$. Hence by the given condition, $\bigcap_{n=1}^{\infty} \bar{F}_n = \{x_0\}$, say. This gives that for any $a \in X$,

$$\sigma(x_n, x_0, a) \leq \delta_a(\bar{F_n}) \to 0 \text{ as } n \to \infty$$

which implies $x_n \to x_0$ in X and this proves the theorem.

Combining Theorems 1 and 2 we obtain the analogue of Cantor's intersection theorem in 2-metric spaces.

Theorem 3. *A 2-metric space (X, σ) is complete if and only if for any decreasing sequence of 2-closed sets $\{F_n\}$ with $\delta_a(F_n) \to 0$ as $n \to \infty\ \forall\ a \in X$, $\bigcap_{n=1}^{\infty} F_n$ consists of a single point.*

The following lemma will be required for the next theorem.

Lemma 7. *For any $a, b \in X$ and $r > 0$,*

$$C_r(a, b) = \{c \in X;\ \sigma(a, b, c) \leq r\}$$

is a 2-closed set.

Proof. We will show that no point outside $C_r(a, b)$ is a 2-limit point of $C_r(a, b)$. Let $d \notin C_r(a, b)$. Then $\sigma(a, b, d) > r$. If possible, let d be a 2-limit point of $C_r(a, b)$. Let $\epsilon > 0$ be given. Since $B_\epsilon(a, d) \cap B_\epsilon(b, d)$ is a 2-open set containing d, there exists $e \in C_r(a, b) \cap [[B_\epsilon(a, d) \cap B_\epsilon(b, d)] - \{d\}]$. Then

$$\begin{aligned} \sigma(a, b, d) &\leq \sigma(a, b, e) + \sigma(b, e, d) + \sigma(a, e, d) \\ &< r + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $\sigma(a, b, d) \leq r$ which is a contradiction. Thus d cannot be a 2-limit point of $C_r(a, b)$. Hence $C_r(a, b)$ contains all its 2-limit points and so $C_r(a, b)$ is 2-closed. This proves the lemma.

In the next theorem, we prove an analogue of Baire Category theorem for 2-metric spaces.

Theorem 4. *A complete 2-metric space (X, σ) satisfying the condition*

(A) for every pair of points $x, y \in X$, there exists a sequence of 2-closed balls $\{B_n\}$ with centre at x and y with $\delta_a(B_n) \to 0$ as $n \to \infty\ \forall\ a \in X$
cannot be written as a countable union of no-where dense sets.

Proof. If possible, assume that

$$X = \bigcup_{n \in N} X_n = \bigcup_{n \in N} \bar{X_n}$$

where each X_n is no-where dense i.e. $\bar{X_n}$ does not contain any non-empty 2-open set. Let U be any 2-open set. Since X_1 is no-where dense, $\bar{X_1}$ cannot contain U. So there exists $x_1 \in U$ such that $x_1 \notin \bar{X}_1$. Since $U - \bar{X}_1$ is 2-open and $x_1 \in U - \bar{X}_1$, by Lemma 1, there exist $y_1, y_2, ..., y_n$ and $r_1, r_2, ..., r_n$ all positive such that

$$x_1 \in B_{r_1}(x_1, y_1) \cap .... \cap B_{r_n}(x_1, y_n) = V_1 \text{ (say) } \subset U - \bar{X}_1.$$

Without any loss of generality, because of the condition (A), we can choose $B_{r_1}(x_1, y_1)$ such that $\delta_a(B_{r_1}(x_1, y_1)) < 1 \ \forall \ a \in X$. Then $\delta_a(V_1) < 1 \ \forall \ a \in X$. Choose

$$U_1 = B_{r_1/2}(x_1, y_1) \cap .... \cap B_{r_n/2}(x_1, y_n).$$

Then by Lemma 7

$$\begin{aligned}\bar{U}_1 \subset C_{r_1/2}(x_1, y_1) \cap .... \cap C_{r_n/2}(x_1, y_n) &\subset V_1 \\ &\subset U - \bar{X}_1\end{aligned}$$

and $\delta_a(\bar{U}_1) \leq \delta_a(V_1) < 1 \ \forall \ a \in X$.

Again since U_1 is 2-open and X_2 is no-where dense, $U_1 - \bar{X}_2 \neq \phi$. So there exists $x_2 \in U_1 - \bar{X}_2$. Proceeding as above we can find a 2-open set U_2 such that

$$x_2 \in U_2 \subset \bar{U}_2 \subset U_1 - \bar{X}_2$$

and $\delta_a(\bar{U}_2) < 1/2 \ \forall \ a \in X$.

Continuing in this way we obtain a sequence of 2-closed sets $\{\bar{U}_n\}$ such that $\bar{U}_{n+1} \subset \bar{U}_n \ \forall \ n \in N, \ \delta_a(\bar{U}_n) < 1/n \ \forall \ a \in X$ i.e. $\delta_a(\bar{U}_n) \to 0$ as $n \to \infty, \ \forall \ a \in X$. By Theorem 1, $\bigcap_{n=1}^{\infty} \bar{U}_n$ is non-empty and contains at most one point. Let $\bigcap_{n=1}^{\infty} \bar{U}_n = \{x_0\}$. Since $\bar{U}_n \cap \bar{X}_n = \phi \ \forall \ n \in N, \ x_0 \notin \bigcap_{n=1}^{\infty} \bar{X}_n$ which is a contradiction.

This proves the theorem.

4. Application to Fixed Point Problems

In this section, we introduce a set S_t, which behaves reasonably well in dealing with some fixed point problems in (X, σ). In metric spaces, such a set contributed widely in determining the fixed point of operators (see [1, 11]).

Throughout the section we assume that (X, σ) is first countable. Let $T : X \to X$ be a mapping. For $t > 0$ we define

$$S_t = \{x \in X; \ \sigma(x, Tx, y) \leq t \ \forall \ y \in X\}.$$

If T has a fixed point x, then S_t is not void and contains x. If X is bounded, then also S_t is not void for suitable t's.

The idea of contractive mapping is available in [2] for metric spaces. If (X, ρ) is a metric space then $T : X \to X$ is said to be contractive if

$$\rho(Tx, Ty) < \rho(x, y)$$

for all $x, y \in X,\ x \neq y$. In the 2-metric setting we give the following definition of contractive mapping.

Definition 12. A mapping $T : (X, \sigma) \to (X, \sigma)$ is said to be contractive if $\sigma(Tx, Ty, a) < \sigma(x, y, a)\ \forall\ x, y, a \in X$ where $x \neq y \neq a$, and $\sigma(Tx, Ty, a) = 0$ if any two of x, y and a are equal.

The following basic property of the set S_t will be used.

Theorem 5. *Let $T : X \to X$ be contractive. Then S_t is a 2-closed set,* $\forall\ t > 0$.

Proof. Since (X, σ) is first countable T_1 (by Lemma 2), to prove that S_t is 2-closed, it is sufficient to show that any convergent sequence $\{x_n\}$ in S_t converges to a point $x \in S_t$. Let $\{x_n\}$ be a convergent sequence in S_t such that $x_n \to x$ in X. Let $\epsilon > 0$ be given. Let $y \in X$. Since $B_\epsilon(x, Tx) \cap B_\epsilon(x, y)$ is a 2-open set containing x, there is a $n_0 \in N$ such that $x_n \in B_\epsilon(x, Tx) \cap B_\epsilon(x, y)\ \forall\ n \geq n_0$ (by Lemma 4). Using the fact that T is contractive and $x_n \in S_t$ for $n \in N$, we obtain for $n \geq n_0$

$$\begin{aligned}
\sigma(x, Tx, y) &\leq \sigma(x, x_n, y) + \sigma(Tx, x_n, x) + \sigma(x_n, Tx, y) \\
&< 2\epsilon + \sigma(x_n, Tx_n, y) + \sigma(Tx_n, Tx, y) + \sigma(Tx_n, Tx, x_n) \\
&< 2\epsilon + \sigma(x_n, Tx_n, y) + \sigma(x_n, x, y) \\
&\leq t + 3\epsilon.
\end{aligned}$$

Since this is true for arbitrary $\epsilon > 0$, we must have $\sigma(x, Tx, y) \leq t$. This is true for all $y \in X$ and so $x \in S_t$ which proves the lemma.

Let $\{\alpha_n\}$ be a decreasing sequence of positive numbers tending to zero and let T be a self mapping on X. Let

$$S_n = S_{\alpha_n} = \{x \in X;\ \sigma(x, Tx, y) \leq \alpha_n\ \forall\ y \in X\}.$$

Then $S_{n+1} \subset S_n$ for each $n \in N$.

Theorem 6. *Let $T : X \to X$ be continuous. Then in any compact set S in (X, σ), the sets S_n are empty for all sufficiently large values of n, provided S does not contain any fixed point of T.*

Proof. If possible, suppose the converse. Then there exist positive integers $n_1, n_2, n_3, ...$ tending to infinity such that none of the sets $S_{n_1}, S_{n_2}, ...$ are empty in S.

Let $x_{n_r} \in S_{n_r} \cap S \ \forall \ r \in N$. Since S is compact, there exists a subsequence $x_{n_{t_1}}, x_{n_{t_2}}, ...$ of $\{x_{n_r}\}$ converging to a point x of S. Clearly

$$\sigma(x_{n_{t_r}}, Tx_{n_{t_r}}, y) \leq \alpha_{n_{t_r}} \ \forall \ y \in X.$$

Consider now the 2-open set $B_\epsilon(x, y) \cap B_\epsilon(x, Tx)$ containing x where $\epsilon > 0$ is given and $y \in X$. Since $x_{n_{t_r}} \to x$ as $r \to \infty$, by Lemma 4, there exists $r_0 \in N$ such that

$$x_{n_{t_r}} \in B_\epsilon(x, y) \cap B_\epsilon(x, Tx) \ \forall \ r \geq r_0.$$

Again since T is continuous, $Tx_{n_{t_r}} \to Tx$ as $r \to \infty$ and so by Lemma 4 there exists $r_1 \in N$ such that

$$Tx_{n_{t_r}} \in B_\epsilon(Tx, y) \ \forall \ r \geq r_1.$$

Then for all $r > max\{r_0, r_1\}$,

$$\begin{aligned}
\sigma(x, Tx, y) &\leq \sigma(x, x_{n_{t_r}}, y) + \sigma(x, x_{n_{t_r}}, Tx) + \sigma(x_{n_{t_r}}, Tx, y) \\
&< \epsilon + \epsilon + \sigma(x_{n_{t_r}}, Tx, y) \\
&\leq 2\epsilon + \sigma(x_{n_{t_r}}, Tx_{n_{t_r}}, y) + \sigma(x_{n_{t_r}}, Tx, Tx_{n_{t_r}}) + \sigma(Tx_{n_{t_r}}, Tx, y) \\
&< 3\epsilon + 2\alpha_{n_{t_r}}.
\end{aligned}$$

As $\alpha_{n_{t_r}} \to 0$ as $r \to \infty$, we have

$$\sigma(x, Tx, y) \leq 3\epsilon \ \forall \ y \in X.$$

Since $\epsilon > 0$ is arbitrary, we must have $\sigma(x, Tx, y) = 0 \ \forall \ y \in X$ and so $Tx = x$ which contradicts the fact that S does not contain any fixed point of T. This proves the theorem.

The statement of Theorem 7 is a particular case of Theorem 1 [13], but the proof presented here is entirely non-traditional, and may be considered as an alternative proof of an analogue of Banach's fixed point theorem, using Theorem 3 and properties of the set S_n.

Theorem 7. *Let (X, σ) be a complete bounded 2-metric space and $T : X \to X$ be a mapping such that*
$\sigma(Tx, Ty, a) \leq \alpha\sigma(x, y, a), \ 0 < \alpha < 1, \ \forall \ x, y, a \in X$
where in the case of strict inequality, $x \neq y \neq a$ and $\sigma(Tx, Ty, a) = 0$ if any two of x, y, a are equal. Then T has a unique fixed point in X.

Proof. Since X is bounded,

$$sup\{\sigma(a, b, c); \ a, b, c \in X\} = M \ (\text{say}) \ < \infty.$$

Let $x \in X$. Consider the sequence $\{T^n x\}$ of iterates of x by T. Then

$$\begin{aligned}\sigma(T^n x, T^{n+1}x, a) &\leq \alpha\sigma(T^{n-1}x, T^n x, a)\\ &\leq \alpha^2\sigma(T^{n-2}x, T^{n-1}x, a)\\ &\quad\vdots\\ &\leq \alpha^n\sigma(x, Tx, a)\\ &\leq \alpha^n.M \ \forall\ a \in X.\end{aligned}$$

Since $0 < \alpha < 1$, it immediately follows that no set $S_n = S_{t_n} = \{z \in X;\ \sigma(z, Tz, a) \leq t_n\ \forall\ a \in X\}$ is empty where $\{t_n\}$ is a decreasing sequence converging to zero. Evidently T is also contractive and so by Theorem 5, each S_n is 2-closed. Also $S_{n+1} \subset S_n\ \forall\ n \in N$.

Note that for any $x, y \in S_n$ and $a \in X$,

$$\begin{aligned}\sigma(x, y, a) &\leq \sigma(x, Tx, a) + \sigma(Tx, y, a) + \sigma(x, Tx, y)\\ &\leq 2t_n + \sigma(Tx, Ty, a) + \sigma(y, Ty, a) + \sigma(Tx, Ty, y)\\ &\leq 3t_n + \alpha\sigma(x, y, a) + \alpha\sigma(x, y, y)\end{aligned}$$

which implies

$$\sigma(x, y, a) \ \leq \ \frac{3t_n}{1-\alpha}.$$

This shows that $\delta_a(S_n) \leq \frac{3t_n}{1-\alpha}$ which tends to zero as $n \to \infty$. By Theorem 3, $\bigcap_{n=1}^{\infty} S_n$ contains exactly one point. Let $\bigcap_{n=1}^{\infty} S_n = \{x_0\}$. Then

$$\sigma(x_0, Tx_0, a) \leq t_n\ \forall\ n \in N \text{ and } \forall\ a \in X.$$

So $\sigma(x_0, Tx_0, a) = 0\ \forall\ a \in X$ and thus $Tx_0 = x_0$.

If u and v are two distinct fixed points of T then for a point $a \in X,\ a \neq u$ or v,

$$\begin{aligned}\sigma(u, v, a) &= \sigma(Tu, Tv, a)\\ &\leq \alpha\sigma(u, v, a)\\ &< \sigma(u, v, a)\end{aligned}$$

which is a contradiction. Hence T has a unique fixed point.

5. Some Further Fixed Point Theorems

In Theorem 8 we omit the completeness of the space (X,σ) and instead of, assume the convergence of a subsequence of a sequence of iterates. We observe that the proof is conveniently accomplished with the sets S_n.

Theorem 8. *Let* (X,σ) *be bounded and* $T:(X,\sigma)\to(X,\sigma)$ *be a mapping such that*

$$\sigma(Tx,Ty,a)\le\alpha\sigma(x,y,a),\ 0<\alpha<1\ \forall\ x,y,a\in X.$$

Let there be a point $x\in X$ *such that the sequence of iterates* $\{T^{\,n}x\}$ *contains a subsequence* $\{T^{n_r}x\}$ *that converges to* $x_0\in X$*. Then* x_0 *is a unique fixed point of* T*.*

Proof. As in Theorem 7

$$sup\{\sigma(a,b,c);\ a,b,c\in X\}=M\ (\text{say})\ <\infty.$$

Let

$$S_n=S_{t_n}=\{z\in X;\ \sigma(z,Tz,a)\le t_n\ \forall\ a\in X\}$$

where $\{t_n\}$ is a decreasing sequence tending to zero. Now we have

$$\begin{aligned}\sigma(T^{n_r}x,T^{n_r+1}x,a)\ &\le\ \alpha.\sigma(T^{n_r-1}x,T^{n_r}x,a)\\ &\le\ \alpha^2.\sigma(T^{n_r-2}x,T^{n_r-1}x,a)\\ &\ \ \vdots\\ &\le\ \alpha^{n_r}.\sigma(x,Tx,a)\\ &\le\ \alpha^{n_r}.M\ \forall\ a\in X.\end{aligned}$$

Since $0<\alpha<1$, no S_n is empty. In fact S_n contains $T^{n_r}x$ for all sufficiently large values of r.

Let $n\in N$ and $a\in X$. Let $\epsilon>0$ be given. Since $B_\epsilon(x_0,Tx_0)\cap B_\epsilon(x_0,a)$ is a 2-open set containing x_0 and $T^{n_r}x\to x_0$, by Lemma 4 we can find a positive integer r_0 such that

$$T^{n_r}x\in B_\epsilon(x_0,Tx_0)\cap B_\epsilon(x_0,a)\ \forall\ r\ge r_0.$$

Let n be fixed and choose $r \geq r_0$ such that $\alpha^{n_r}.M < t_n$. Then

$$\begin{aligned}\sigma(x_0, Tx_0, a) &\leq \sigma(x_0, T^{n_r}x, a) + \sigma(x_0, T^{n_r}x, Tx_0) + \sigma(T^{n_r}x, Tx_0, a)\\ &< 2\epsilon + \sigma(T^{n_r}x, Tx_0, a)\\ &\leq 2\epsilon + \sigma(T^{n_r}x, T^{n_r+1}x, a) + \sigma(T^{n_r}x, T^{n_r+1}x, Tx)\\ &\quad +\sigma(T^{n_r+1}x, Tx_0, a)\\ &\leq 2\epsilon + \alpha^{n_r}.\sigma(x, Tx, a) + \alpha^{n_r}.\sigma(x, Tx, Tx_0)\\ &\quad +\alpha.\sigma(T^{n_r}x, x_0, a)\\ &\leq 2\epsilon + t_n + \alpha^{n_r}.\alpha.\sigma(x, x, x_0) + \alpha.\epsilon\\ &= t_n + (2+\alpha)\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$\sigma(x_0, Tx_0, a) \leq t_n \ \forall\ a \in X$$

and this is true for all positive integers n. Letting $n \to \infty$, we obtain $\sigma(x_0, Tx_0, a) = 0\ \forall\ a \in X$ which implies $Tx_0 = x_0$, i.e. x_0 is a fixed point of T. The proof of the uniqueness is omitted. This completes the proof.

Edelstein [2] proved a fixed point theorem for contractive type mappings in metric spaces for which a simple proof is available in [19]. We prove an analogue of Edelstein's theorem in 2-metric settings.

Theorem 9. *Suppose $T : X \to X$ is contractive and X is uncountable. If there exists a point $x \in X$ such that the sequence of iterates $\{T^n x\}$ contains a subsequence $\{T^{n_i}x\}$ converging to $x_0 \in X$, then x_0 is the unique fixed point of T.*

Proof. In the sequence $\{T^n x\}$ if $T^r x = T^{r+1}x$ for some r then $x_0 = T^r x$ is a fixed point of T.

So let $T^r x \neq T^{r+1}x$ for all $r \in N$. Also let $Tx_0 \neq x_0$, for otherwise x_0 is a fixed point of T. Choose an element $a \in X$ distinct from x_0, Tx_0 and $T^r x,\ r = 1, 2, \ldots$. Then we have

$$\sigma(Tx_0, T^2x_0, a) < \sigma(x_0, Tx_0, a). \tag{1}$$

Consider the set of non-negative real numbers $\{\sigma(T^n x, T^{n+1}x, a)\}_{n=0}^{\infty}$. We first show that $\sigma(x_0, Tx_0, a)$ is a limit point of this set. For this, let $\epsilon > 0$ be given. Since $B_{\epsilon/3}(x_0, Tx_0) \cap B_{\epsilon/3}(x_0, a)$ is a 2-open set containing x_0 and $T^{n_i}x \to x_0$ as $i \to \infty$, using Lemma 4 we have $k \in N$ such that

$$T^{n_i}x \in B_{\epsilon/3}(x_0, Tx_0) \cap B_{\epsilon/3}(x_0, a)\ \forall\ i \geq k.$$

Choose an $i > k$. Then

$$
\begin{aligned}
\sigma(x_0, Tx_0, a) &\le \sigma(x_0, T^{n_i}x, a) + \sigma(x_0, T^{n_i}x, Tx_0) + \sigma(Tx_0, T^{n_i}x, a) \\
&< \frac{2\epsilon}{3} + \sigma(Tx_0, T^{n_i+1}x, a) + \sigma(T^{n_i+1}x, T^{n_i}x, a) \\
&\quad + \sigma(Tx_0, T^{n_i+1}x, T^{n_i}x) \qquad (2) \\
&\le \frac{2\epsilon}{3} + \sigma(x_0, T^{n_i}x, a) + \sigma(T^{n_i+1}x, T^{n_i}x, a) \\
&< \epsilon + \sigma(T^{n_i}x, T^{n_i+1}x, a).
\end{aligned}
$$

Similarly one can show that for all large i

$$\sigma(T^{n_i}x, T^{n_i+1}x, a) < \sigma(x_0, Tx_0, a) + \epsilon. \tag{3}$$

From (2) and (3) it follows that for all large i

$$|\sigma(x_0, Tx_0, a) - \sigma(T^{n_i}x, T^{n_i+1}x, a)| < \epsilon.$$

This proves our assertion.

We now note that for a fixed n_i we have

$$\sigma(T^n x, T^{n+1}x, a) < \sigma(T^{n_i+1}x, T^{n_i+2}x, a) \ \forall \ n > n_i + 1.$$

Therefore the limit point $\sigma(x_0, Tx_0, a)$ satisfies the condition

$$\sigma(x_0, Tx_0, a) \le \sigma(T^{n_i+1}x, T^{n_i+2}x, a)$$

and this is true for all $i \in N$. We obtain for $i > k$

$$
\begin{aligned}
\sigma(T^{n_i+1}x, T^{n_i+2}x, a) &\le \sigma(T^{n_i+1}x, Tx_0, T^{n_i+2}x) + \sigma(T^{n_i+1}x, Tx_0, a) \\
&\quad + \sigma(Tx_0, T^{n_i+2}x, a) \\
&< \sigma(T^{n_i}x, x_0, a) + \sigma(Tx_0, T^2x_0, a) + \sigma(T^2x_0, T^{n_i+2}x, a) \\
&\quad + \sigma(Tx_0, T^2x_0, T^{n_i+2}x) \\
&< \sigma(Tx_0, T^2x_0, a) + \sigma(T^{n_i}x, x_0, a) + \sigma(T^{n_i}x, x_0, a) \\
&< \sigma(Tx_0, T^2x_0, a) + \epsilon/3 + \epsilon/3 \\
&< \sigma(Tx_0, T^2x_0, a) + \epsilon.
\end{aligned}
$$

Thus from above we have

$$\sigma(x_0, Tx_0, a) < \sigma(Tx_0, T^2x_0, a) + \epsilon$$

and so

$$\sigma(x_0, Tx_0, a) \leq \sigma(Tx_0, T^2x_0, a)$$

which contradicts (1). Hence $Tx_0 = x_0$ or $T^{r+1}x = T^r x$ for some r and the proof is complete.

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