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CRITERIA OF AN ENTIRE SERIES WITH FINITE LOGARITHMIC ORDER

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Abstract. We establish criteria on entire series with finite logarithmic order in terms of several growth scales.

1. INTRODUCTION

Let$f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be a transcendental entire series in the complex plane $\mathbb{C}$ with maximum term $\mu(r, f)(= Max_{1\leq n<+\infty}|a_n|r^n)$ and the central index $\nu(r, f)(= Max\{n : |a_n|r^n = \mu(r, f)\})$. The discontinuous points (are also known as jump points) of $\nu(r, f)$ forms an unbounded strictly increasing sequence, say, $r_1, r_2, \ldots$ with $\mu(r, f) = |a_j|r^{n_j}$ for $r_j \leq r < r_{j+1}$ and $\nu(r, f) = n_j$ for $r_j \leq r < r_{j+1}$. A nonnegative increasing function $S(r)$, defined for $r > 0$, is said to be of *logarithmic order* λ [1] if $\limsup_{r\to+\infty} \log S(r)/ \log\log r = \lambda$. We define *the logarithmic order* of f to be the logarithmic order of $\log^+ M(r, f)$, where $M(r, f)$ is the maximum modulus function of f over $|z| \leq r$.

In Section 2, we establish growth property among the jump points, the central indexes and the logarithmic maximum terms of a transcendental entire series with finite logarithmic order.

In Section 3, we establish criteria on entire series with finite logarithmic order in terms of the growth scale either of $T(r, f)$ where $T(r, f)$ denotes the Nevanlinna characteristic of f, or of $\log^+ M(r, f)$, or of $\log^+ \mu(r, f)$, or of $\nu(r, f)$, or of $N(r, f = a)$ for some $a \in \mathbb{C}$ where $N(r, f = a)$ denotes the integral counting function of a-points of f, or of $n(r, f = a)$ for some $a \in \mathbb{C}$ where $n(r, f = a)$ denotes the non-integral counting function of a-points of f, or of $T_o(r, f)$ where $T_o(r, f)$ denotes the Ahlfors-Shimizu characteristic of f, or of $A(r, f)$ where $A(r, f)$ is the spherical area of the image under f over $|z| \leq r$. Examples of entire series of finite logarithmic order are given in Section 4.

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2. Growth Property Among the Jump Points, The Central Indexes and the Logarithmic Maximum Terms of an Entire Series

First we prove a technical Lemma:

Lemma 2.1. *If $f(z)$ is an entire series in $\mathbb{C}$. Let $r_j (j = 1, 2, \dots)$ be the jump points of $\nu(r, f)$ of $f(z)$, with $r_j < r_{j+1}$. If $\mu > 0$, then the series*

$$\sum_j \frac{\nu(r_j, f) - \nu(r_{j-1}, f)}{(\log r_j)^\mu} \tag{2.1}$$

and the integrals

$$\int^\infty \frac{\nu(t, f)}{t(\log t)^{\mu+1}}\, dt \tag{2.2}$$

and

$$\int^\infty \frac{\log \mu(t, f)}{t(\log t)^{\mu+2}}\, dt \tag{2.3}$$

are either simultaneously convergent or simultaneously divergent.

Proof of Lemma 2.1. We first rewrite (2.1) the series of positive terms as a Stieltjes integral as follows:

$$\begin{aligned}\sum_{j=j_0}^{J} \frac{\nu(r_j, f) - \nu(r_{j-1}, f)}{(\log r_j)^\mu} &= \int_{r_0}^{R} \frac{1}{|\log t|^\mu}\, d\nu(t, f) \\ &= \left[\frac{\nu(t, f)}{(\log t)^\mu}\right]_{t=r_0}^{t=R} + \mu \int_{r_0}^{R} \frac{\nu(t, f)}{t(\log t)^{\mu+1}} dt.\end{aligned} \tag{2.4}$$

From the identity (2.4), we deduce that the series (2.1) and the integral (2.2) are either simultaneously convergent or simultaneously divergent.

Next, by using the identity

$$\int_3^R \frac{\nu(t, f)}{t(\log t)^{\mu+1}} dt = [(\log \mu(t, f))(\log t)^{-\mu-1}]_{t=3}^{t=R} + (\mu+1) \int_3^R \frac{\log \mu(t, f)}{t(\log t)^{\mu+2}} dt, \tag{2.5}$$

we see that the integral (2.2) and the integral (2.3) are either simultaneously convergent or simultaneously divergent. This completes the proof of Lemma 2.1. ∎

For entire series f of finite logarithmic order we have:

Theorem 2.1. *If $\nu(r, f)$ is of finite positive logarithmic order, then we have*

$$\sum_j \frac{\nu(r_j, f) - \nu(r_{j-1}, f)}{(\log r_j)^\mu} \begin{cases} < +\infty, & \text{if } \mu > \lambda_{\log}(f) \\ = +\infty, & \text{if } \mu < \lambda_{\log}(f), \end{cases} \tag{2.6}$$

(2.7)
$$\int^{\infty} \frac{\nu(t,f)}{t(\log t)^{\mu+1}}\, dt \quad \begin{matrix} < +\infty, & \text{if } \mu > \lambda_{\log}(f) \\ = +\infty, & \text{if } \mu < \lambda_{\log}(f), \end{matrix}$$

(2.8)
$$\int^{\infty} \frac{\log \mu(t,f)}{t(\log t)^{\mu+2}}\, dt \quad \begin{matrix} < +\infty, & \text{if } \mu > \lambda_{\log}(f) \\ = +\infty, & \text{if } \mu < \lambda_{\log}(f), \end{matrix}$$

where $\lambda_{\log}(f)$ denotes the logarithmic order of $\nu(r,f)$.

Proof of Theorem 2.1. Since $\nu(r,f)$ is a non-negative increasing function, and $\lambda_{\log}(f)$ is the logarithmic order of $\nu(r,f)$, applying an integral criterion for $\nu(r,f)$ (see [1, Theorem 1.1]), we have the expression (2.7).

By Lemma 1.1, the series (2.1) and the integral (2.2) are either simultaneously convergent or simultaneously divergent, so the expressions (2.7) implies the expressions (2.6).

By Lemma 2.1, the integral (2.2) and the integral (2.3) are either simultaneously convergent or simultaneously divergent, so the expressions (2.7) implies the expressions (2.8). This completes the proof of Theorem 2.1. ■

Although for any given entire series $f(z)$ with finite positive order $\log \mu(r,f)$ and $\nu(r,f)$ both have the same order, the situation is different for functions of finite logarithmic order. Indeed, we have the following.

Theorem 2.2. *Let $f(z)$ be an entire series with finite logarithmic order in $\mathbb{C}$. Then $\log^+ \mu(r,f)$ is of logarithmic order $\lambda_{\log}(f)+1$ where $\lambda_{\log}(f)$ is the logarithmic order of $\nu(r,f)$.*

Proof of Theorem 2.2. The expression (2.8) and the Theorem 1.1 of [1] imply that $\log^+ \mu(r,f)$ is of logarithmic order $\lambda_{\log}(f)+1$, since $\log^+ \mu(r,f)$ is a non-negative increasing function for $r > 0$. ■

3. Criteria

In this section, we establish criteria on entire series with finite logarithmic order in terms of several growth scales.

Theorem 3.1. If $f(z)$ is a transcendental entire series with finite logarithmic order, then the following statements are equivalent:

(1) T(r,f) has logarithmic order λ.

(2) $\log^+ M(r,f)$ has logarithmic order λ.

(3) $\log^+ \mu(r,f)$ has logarithmic order λ.

(4) $\nu(r,f)$ has logarithmic order $\lambda - 1$.

(5) $N(r, f = a)$ has logarithmic order λ for some $a \in \mathbb{C}$.

(6) $n(r, f = a)$ has logarithmic order $\lambda - 1$ for some $a \in \mathbb{C}$.

(7) The logarithmic exponent of convergence of a points of $f(z)$ is $\lambda - 1$ for some $a \in \mathbb{C}$.

(8) $T_0(r, f)$ the Ahlfors-Shimizu characteristic of $f(z)$ has logarithmic order λ.

(9) $A(r, f)$ has logarithmic order $\lambda - 1$.

Proof of Theorem 3.1. (1) and (2) are equivalent is stated in [1, section 2].

Since $\log^+ M(r, f)$ and $\log^+ \mu(r, f)$ are asymptotic if f has finite order, see [4, p. 17, Theorem 1.9] so they have the same logarithmic order, thus (2) and (3) are equivalent.

(3) and (4) are equivalent follows from Theorem 2.2 of this article. By [1, Theorem 7.1], for any two distinct extended complex values a, b,we have

$$T(r, f) \leq N(r, f = a) + N(r, f = b) + o(U(r, f)), \tag{3.1}$$

where $U(r, f) = (\log r)^{\lambda(r)}$ is a logarithmic type function of $T(r, f)$.
Now we prove that (1) and (5) are equivalent. Assume (1), since f is entire, $N(r, f = \infty) = 0$ by the inequality (3.1) and the first fundamental Theorem of R. Nevanlinna, $N(r, f = a)$ is of logarithmic order λ for any $a \in \mathbb{C}$. Conversely if we assume (5) that $N(r, f = a)$ is of logarithmic order λ for some $a \in \mathbb{C}$, since f is entire, $N(r, f = \infty) = 0$ by the inequality (3.1), $T(r, f)$ is of logarithmic order λ.

(5) and (6) are equivalent follows from [1, Theorem 4.1]. (6) and (7) are equivalent follows from [1, Theorem 3.1]. (1) and (8) are equivalent since

$$\mid T(r, f) - T_0(r, f) - \log^+ \mid f(0) \mid\mid \leq \frac{1}{2} \log 2, \tag{3.2}$$

see [3, p.13].

Following the argument in the proof of Theorem 4.1 of [1], we can prove that (8) and (9) are equivalent since the relationship between $A(r, f)$ and $T_0(r, f)$ likes that between $n(r, f = a)$ and $N(r, f = a)$. We omit its details here. ■

Remark. It is obvious that if we replace some $a \in \mathbb{C}$ by every $a \in \mathbb{C}$ in above statement (5), (6) and (7); they are still equivalent to other's statements.

4. Examples of Entire Series with Finite Logarithmic Order

Theorem 4.1. There exist infinitely many transcendental entire series f such that $\log^+ \mu(r, f)$ is of logarithmic order one.

Proof of Theorem 4.1. For each positive number $c > 1$, put $r_n = e^{c^n}$. Let

$$f(z) = \sum_{n=1}^{+\infty} \frac{1}{r_1 \cdots r_n} z^n = \sum_{n=1}^{+\infty} a_n z^n. \tag{4.1}$$

Since $\lim\limits_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to+\infty} \frac{1}{r_{n+1}} = 0$, so the convergence of the radius for the function $f(z)$ is $+\infty$, hence $f(z)$ is an entire series, and then by a result of [5, page 6], $\{r_j\}_{j=1}^{+\infty}$ is the set of jump points of f, further $\inf\{\mu|\mu > 0, \sum_{j=1}^{j=+\infty} \frac{\nu(r_j,f)-\nu(r_{j-1},f)}{(\log r_j)^\mu} < +\infty\} = \inf\{\mu|\mu > 0, \sum_{n=1}^{+\infty} \frac{1}{(c^n)^\mu} < +\infty\} = 0$, it follow from Lemma 2.1, the integral (2.2) is convergent for any $\mu > 0$, then by from Theorem 1.1 of [1], the logarithmic order of $\nu(r, f)$ equals zero, and hence by Theorem 2.2 of this article, $\log^+ \mu(r, f)$ is of logarithmic order one. Since $c > 1$ is arbitrary, the proof of Theorem 4.1 is complete. ■

Theorem 4.2. For each positive number $\lambda(> 1)$, there exists an entire series f such that $\log^+ \mu(r, f)$ is of logarithmic order λ.

Proof of Theorem 4.2. Put $k = \lambda - 1(> 0)$ and $r_n = e^{n^{1/k}}$. Let

$$f(z) = \sum_{n=1}^{+\infty} \frac{1}{r_1 r_2 \cdots r_n} z^n \tag{4.2}$$

then $\{r_j\}$ is the set of jump points of $\nu(r, f)$. $\inf\{\mu|\mu > 0, \sum_{j=1}^{j=+\infty} \frac{\nu(r_j,f)-\nu(r_{j-1},f)}{(\log r_j)^\mu} < +\infty\} = k$, it follows from Lemma 2.1 that the integral (2.2) is convergent for $\mu > k$ and divergent for $\mu < k$. It follows from Theorem 1.1 of [1] that $\nu(r, f)$ has logarithmic order k too. Then by Theorem 2.2 of this article, $\log^+ \mu(r, f)$ is of logarithmic order $\lambda(= k + 1)$. This completes the proof of Theorem 4.2. ■

Remark. Most results of this article were announced in the 15th International Conference on finite or infinite dimensional complex analysis and applications in the Osaka Conference at July 30th of 2007 and has been appeared in [2] but without proof in the version of [2].

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