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SOME NEW FAMILIES OF GENERALIZED EULER AND GENOCCHI POLYNOMIALS

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Abstract. The main object of this paper is to introduce and investigate a new generalization of the family of Euler polynomials by means of a suitable generating function. We establish several interesting properties of these general polynomials and derive explicit representations for them in terms of a certain generalized Hurwitz-Lerch Zeta function and in series involving the familiar Gaussian hypergeometric function. Finally, we propose an analogous generalization of the closely-related Genocchi polynomials and show how we can fruifully exploit some potentially useful linear connections of all these three important families of generalized Bernoulli, Euler and Genocchi polynomials with one another.

1. INTRODUCTION AND PRELIMINARIES

It is well known that the Bernoulli numbers B_n and the Bernoulli polynomials $B_n(x)$ are of fundamental importance in several parts of analysis and in the calculus of finite differences and have applications in various other fields such as statistics, numerical analysis, combinatorics, and so on. Another polynomial set which is related to the Bernoulli polynomials and also has interesting properties is the set of the Euler polynomials $E_n(x)$. The Euler polynomials $E_n(x)$ and the Euler numbers E_n play an important rôle in various fields like analysis, number theory, differential geometry and algebraic topology. Some interesting generalizations of the classical Bernoulli polynomials and numbers were investigated by Apostol [2, p. 165], Srivastava [39, pp. 83–84], Luo and Srivastava [33, p. 292], Luo *et al.* [31] and (more recently) by Srivastava *et al.* [41] (see also [23, 25, 26, 34] and [36] *and* the references cited in each of these earlier works).

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The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ of (real or complex) order α, are usually defined by means of the following generating functions (see, for details, [35, 38] and [40, p. 61]; see also [43, 44, 46] and [48]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\, \frac{t^n}{n!} \qquad \left(|t| < 2\pi;\ 1^{\alpha} := 1\right) \tag{1}$$

and

$$\left(\frac{2}{e^t+1}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\, \frac{t^n}{n!} \qquad \left(|t| < \pi;\ 1^{\alpha} := 1\right), \tag{2}$$

so that, obviously,

$$B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x) \qquad (n \in \mathbb{N}_0). \tag{3}$$

Throughout our present investigation, we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \cdots\}, \quad \mathbb{N}_0 := \{0, 1, 2, 3, \cdots\} = \mathbb{N} \cup \{0\} \quad \text{and}$$

$$\mathbb{Z}^- := \{-1, -2, -3, \cdots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

Also, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of *positive* real numbers and $\mathbb{C}$ denotes the set of complex numbers. Moreover, throughout this investigation, it is tacitly assumed that $\ln z$ denotes the *principal* branch of the many-valued function $\ln z$ *with* the imaginary part $\Im(\ln z)$ constrained by $-\pi < \Im(\ln z) \leqq \pi$.

The Bernoulli numbers B_n and the Euler numbers E_n are defined here by

$$B_n := B_n(0) = B_n^{(1)}(0) \quad \text{and} \quad E_n := E_n(0) = E_n^{(1)}(0) \qquad (n \in \mathbb{N}_0), \tag{4}$$

respectively.

The Apostol-Bernoulli polynomials $\mathcal{B}_n(x; \lambda)$ and the Apostol-Euler polynomials $\mathcal{E}_n(x; \lambda)$ are defined by means of the following generating functions (see [2] and [39]):

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda)\, \frac{t^n}{n!} \qquad (|t + \ln \lambda| < 2\pi) \tag{5}$$

and (see, for example, [33])

$$\frac{2e^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x; \lambda)\, \frac{t^n}{n!} \qquad (|t + \ln \lambda| < \pi), \tag{6}$$

respectively. Here, of course, we write

$$B_n(x) = \mathcal{B}_n(x;1) \quad \text{and} \quad \mathcal{B}_n(\lambda) := \mathcal{B}_n(0;\lambda), \tag{7}$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers. Furthermore, we have

$$E_n(x) = \mathcal{E}_n(x;1) \quad \text{and} \quad \mathcal{E}_n(\lambda) := \mathcal{E}_n(0;\lambda), \tag{8}$$

where $\mathcal{E}_n(\lambda)$ denotes the so-called Apostol-Euler numbers.

Motivated by the generalizations in (1) and (2) of the classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ involving a real or complex parameter α, Luo and Srivastava [33] introduced and investigated the so-called Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α and the Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α, which are defined by (*cf.* Luo and Srivastava [33, p. 292])

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda)\, \frac{t^n}{n!} \qquad \left(|t + \ln\lambda| < 2\pi;\ 1^{\alpha} := 1\right) \tag{9}$$

and

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\, \frac{t^n}{n!} \qquad \left(|t + \ln\lambda| < \pi;\ 1^{\alpha} := 1\right), \tag{10}$$

respectively. In particular, we write

$$B_n^{(\alpha)}(x) = \mathcal{B}_n^{(\alpha)}(x;1), \ \mathcal{B}_n^{(\alpha)}(\lambda) := \mathcal{B}_n^{(\alpha)}(0;\lambda) \ \text{and} \ \mathcal{B}_n(x;\lambda) := \mathcal{B}_n^{(1)}(x;\lambda), \tag{11}$$

where $\mathcal{B}_n^{(\alpha)}(\lambda)$ denotes the so-called *Apostol-Bernoulli numbers of order* α. Moreover, we have

$$E_n^{(\alpha)}(x) = \mathcal{E}_n^{(\alpha)}(x;1), \quad \mathcal{E}_n^{(\alpha)}(\lambda) := \mathcal{E}_n^{(\alpha)}(0;\lambda) \ \text{and} \ \mathcal{E}_n(x;\lambda) := \mathcal{E}_n^{(1)}(x;\lambda), \tag{12}$$

where $\mathcal{E}_n^{(\alpha)}(\lambda)$ denotes the so-called *Apostol-Euler numbers of order* α.

The Apostol-Bernoulli polynomials and the Apostol-Euler polynomials and the corresponding numbers have been studied by several authors in addition to those referred to above (*cf., e.g.,* [6, 24, 29, 33, 37] and [45]; see also [21]). Various q-extensions of the Apostol-Bernoulli polynomials and the Apostol-Euler polynomials and the corresponding numbers were also studied recently by many authors with great interest. For example, Choi *et al.* [3] (see also [4]) studied some q-extensions of the Apostol-Bernoulli polynomials and the Apostol-Euler polynomials of order ℓ $(\ell \in \mathbb{N})$. Hwang *et al.* [12] and Kim *et al.* [16] studied the Apostol type q-Euler numbers and q-Euler polynomials.

Next, in the usual notation, let $\Phi(z, s, a)$ denote the Hurwitz-Lerch Zeta function defined by

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{13}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1\right).$$

Recently, Lin and Srivastava [20] introduced and investigated the following generalization of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$:

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s} \tag{14}$$

$$\begin{gathered}\big(\mu \in \mathbb{C};\ a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \rho, \sigma \in \mathbb{R}^+;\ \rho < \sigma \ \text{ when } \ s, z \in \mathbb{C}; \\ \rho = \sigma \ \text{ and } \ s \in \mathbb{C} \ \text{ when } \ |z| < \delta := \rho^{-\rho}\, \sigma^{\sigma};\ \rho = \sigma \ \text{ and} \\ \Re(s - \mu + \nu) > 1 \ \text{ when } \ |z| = \delta\big),\end{gathered}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined, in terms of the familiar Gamma function, by

$$\begin{aligned}(\lambda)_\nu :&= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \\ &= \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}\end{aligned} \tag{15}$$

it being understood *conventionally* that $(0)_0 := 1$. Clearly, we find from the definitions (13) and (14) that

$$\Phi_{\nu,\nu}^{(\sigma,\sigma)}(z, s, a) = \Phi_{\mu,\nu}^{(0,0)}(z, s, a) = \Phi(z, s, a) \tag{16}$$

and

$$\Phi_{\mu,1}^{(1,1)}(z, s, a) = \Phi_\mu^*(z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s}, \tag{17}$$

where, as already observed by Lin and Srivastava [20, p. 730], the function $\Phi_\mu^*(z, s, a)$ considered by Goyal and Laddha [7, p. 100, Equation (1.5)] is essentially a *fractional* derivative of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ of order $\mu - 1$ $\left(\Re(\mu) > 0\right)$. For further results involving these classes of generalized Hurwitz-Lerch Zeta functions, see the aforementioned works by Garg *et al.* [6] and Lin *et al.* [21].

In the present sequel to our earlier work [41] on novel generalizations of the family of Bernoulli polynomials, we aim here at introducing and investigating various interesting properties of a new class of generalized Euler polynomials. We develop an explicit series representation for these generalized Euler polynomials in terms of the generalized Hurwitz-Lerch Zeta function $\Phi_{\mu}^{*}(z,s,a)$ defined by (17). We present two other series representations for the generalized Euler polynomials involving the familiar Gaussian hypergeometric function. Relevant connections of the results presented here with those that were obtained in earlier works are also indicated precisely. Finally, in Section 4, we briefly consider an analogous set of problems associated with the closely-related Genocchi polynomials $G_n(x)$, the Genocchi polynomials $G_n^{(\alpha)}(x)$ of a (real or complex) order α and the Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$, which are defined by (see, for example, [9-11, 13-15, 17-19, 22, 27, 28, 30] and [47]; see also the references cited in each of these earlier works)

$$\left(\frac{2t}{e^t+1}\right)^{\alpha}\cdot e^{xt}=\sum_{n=0}^{\infty}G_n^{(\alpha)}(x)\;\frac{t^n}{n!}\qquad\left(|t|<\pi;\;1^{\alpha}:=1\right), \tag{18}$$

so that, obviously,

$$G_n(x):=G_n^{(1)}(x)\qquad(n\in\mathbb{N}_0),$$

and

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}\cdot e^{xt}=\sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(x;\lambda)\;\frac{t^n}{n!}\qquad\left(|t+\ln\lambda|<\pi;\;1^{\alpha}:=1\right) \tag{19}$$

with, of course,

$$G_n^{(\alpha)}(x)=\mathcal{G}_n^{(\alpha)}(x;1),\qquad\mathcal{G}_n^{(\alpha)}(\lambda):=\mathcal{G}_n^{(\alpha)}(0;\lambda) \tag{20}$$

and

$$\mathcal{G}_n(x;\lambda):=\mathcal{G}_n^{(1)}(x;\lambda)\qquad\text{and}\qquad\mathcal{G}_n(\lambda):=\mathcal{G}_n^{(1)}(\lambda), \tag{21}$$

where $\mathcal{G}_n(\lambda)$, $\mathcal{G}_n^{(\alpha)}(\lambda)$ and $\mathcal{G}_n(x;\lambda)$ denote the so-called Apostol-Genocchi numbers, the Apostol-Genocchi numbers of order α and the Apostol-Genocchi polynomials, respectively.

2. A Family of Generalized Euler Polynomials

The following natural generalization and unification of the Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α as well as the generalized Bernoulli numbers $B_n(a,b)$ studied by Guo and Qi [8] and the generalized Bernoulli polynomials

$B_n(x;a,b)$ studied by Luo *et al.* [31] was introduced and investigated recently by Srivastava *et al.* [41].

Definition 1. (see, for details, [41, p. 254, Equation (20)]). Let $a,b,c \in \mathbb{R}^+$ $(a \neq b)$ and $n \in \mathbb{N}_0$. Then the *generalized Bernoulli polynomials* $\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)$ *of order* $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$\left(\frac{t}{\lambda b^t - a^t}\right)^{\alpha} \cdot c^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)\, \frac{t^n}{n!} \qquad \left(\left|t \ln\left(\frac{b}{a}\right) + \ln\lambda\right| < 2\pi;\ 1^{\alpha} := 1;\ x \in \mathbb{R}\right). \tag{22}$$

Here, in this sequel to the work by Srivastava *et al.* [41], we introduce and investigate a similar generalization of the family of Euler polynomials defined as follows.

Definition 2. Let $a,b,c \in \mathbb{R}^+$ $(a \neq b)$ and $n \in \mathbb{N}_0$. Then the *generalized Euler polynomials* $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ *of order* $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$\left(\tfrac{2}{\lambda b^t + a^t}\right)^{\alpha} \cdot c^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\, \frac{t^n}{n!} \qquad \left(\left|t \ln\left(\frac{b}{a}\right) + \ln\lambda\right| < \pi;\ 1^{\alpha} := 1;\ x \in \mathbb{R}\right). \tag{23}$$

Since the parameter $\lambda \in \mathbb{C}$, by comparing Definitions 1 and 2, we can easily deduce the following potentially useful lemma.

Lemma 1. *The families of the generalized Bernoulli polynomials* $\mathfrak{B}_n^{(l)}(x;\lambda;a,b,c)$ $(l \in \mathbb{N}_0)$ *and the generalized Euler polynomials* $\mathfrak{E}_n^{(l)}(x;\lambda;a,b,c)$ $(l \in \mathbb{N}_0)$ *are related by*

$$\mathfrak{B}_n^{(l)}(x;\lambda;a,b,c) = \left(-\frac{1}{2}\right)^{l} \frac{n!}{(n-l)!}\, \mathfrak{E}_{n-l}^{(l)}(x;-\lambda;a,b,c) \qquad (n,l \in \mathbb{N}_0;\ n \geqq l) \tag{24}$$

or, *equivalently*, *by*

$$\mathfrak{E}_n^{(l)}(x;\lambda;a,b,c) = (-2)^{l} \frac{n!}{(n+l)!}\, \mathfrak{B}_{n+l}^{(l)}(x;-\lambda;a,b,c) \qquad (n,l \in \mathbb{N}_0). \tag{25}$$

Remark 1. The connection formulas (24) and (25) can be applied to deduce various properties of the generalized Bernoulli polynomials $\mathfrak{B}_n^{(l)}(x;\lambda;a,b,c)$

$(l \in \mathbb{N}_0)$ from the corresponding properties of the generalized Euler polynomials $\mathfrak{E}_n^{(l)}(x;\lambda;a,b,c)$ $(l \in \mathbb{N}_0)$ and *vice versa*.

Remark 2. If we set $\alpha = 1$ in (23), we arrive at a new special case of the generalized Euler polynomials given by

$$\frac{2c^{xt}}{\lambda b^t + a^t} = \sum_{n=0}^{\infty} \mathfrak{E}_n(x;\lambda;a,b,c)\,\frac{t^n}{n!}. \tag{26}$$

On the other hand, by setting $\lambda = 1$ in (23), we obtain another new generalized Euler polynomials given by

$$\left(\frac{2}{b^t + a^t}\right)^{\alpha} \cdot c^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;a,b,c)\,\frac{t^n}{n!}, \tag{27}$$

which, in the *further* special case when $\alpha = 1$, yields the Euler polynomials studied by Luo *et al.* [32]. Moreover, in its special case when

$$a = 1 \qquad \text{and} \qquad b = c = e,$$

the generalized Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ defined by (23) would lead us at once to the Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ defined by (10).

The generalized Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ defined by (23) possess a number of interesting properties which we state here as Theorems 1 to 5 below.

Theorem 1. *Let* $a,b,c \in \mathbb{R}^+$ $(a \neq b)$ *and* $x \in \mathbb{R}$. *Then*

$$\mathfrak{E}_n^{(\alpha)}(x+1;\lambda;a,b,c) = \sum_{k=0}^{n} \binom{n}{k} (\ln c)^{n-k}\ \mathfrak{E}_k^{(\alpha)}(x;\lambda;a,b,c), \tag{28}$$

$$\mathfrak{E}_n^{(\alpha)}(x+\alpha;\lambda;a,b,c) = \mathfrak{E}_n^{(\alpha)}\left(x;\lambda;\frac{a}{c},\frac{b}{c},c\right), \tag{29}$$

$$\mathfrak{E}_n^{(\alpha)}(\alpha-x;\lambda;a,b,c) = \mathfrak{E}_n^{(\alpha)}\left(-x;\lambda;\frac{a}{c},\frac{b}{c},c\right) \tag{30}$$

and

$$\mathfrak{E}_n^{(\alpha)}(\alpha-x;\lambda;a,b,c) = (-1)^n\ \mathfrak{E}_n^{(\alpha)}\left(x;\lambda;\frac{c}{a},\frac{c}{b},c\right). \tag{31}$$

Proof. First of all, it follows from (23) that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x+1;\lambda;a,b,c)\ \frac{t^n}{n!} \\
&\quad = \left(\frac{2}{\lambda b^t + a^t}\right)^{\alpha} \cdot c^{(x+1)t} \\
&\quad = \left(\frac{2}{\lambda b^t + a^t}\right)^{\alpha} \cdot c^{xt} \cdot c^t \\
&\quad = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathfrak{E}_k^{(\alpha)}(x;\lambda;a,b,c)\ (\ln c)^n\ \frac{t^{n+k}}{n!\ k!} \\
&\quad = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathfrak{E}_k^{(\alpha)}(x;\lambda;a,b,c)\ (\ln c)^{n-k}\ \frac{t^n}{(n-k)!\ k!},
\end{aligned} \tag{32}$$

where we have made use of the generating function (23) once again as well as the following elementary series identity [42, p. 100, Equation 2.1 (1)]:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,\ n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,\ n-k)\,. \tag{33}$$

By comparing the coefficients of t^n in the first and the last members of (32), we arrive at the result (28) asserted by Theorem 1. Similarly, by applying the generating function (23), we have

$$\sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x+\alpha;\lambda;a,b,c)\ \frac{t^n}{n!} = \left(\frac{2}{\lambda b^t + a^t}\right)^{\alpha} \cdot c^{(x+\alpha)t},$$

which, by simple manipulations and by equating the coefficients of t^n on both sides, leads us to the result (29) asserted by Theorem 1.

The third assertion (30) of Theorem 1 follows easily from the second assertion (29) upon replacing x by $-x$.

Lastly, from the definition (23) and by easy computation, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(\alpha-x;\lambda;a,b,c)\ \frac{t^n}{n!} &= \left(\frac{2}{\lambda b^t + a^t}\right)^{\alpha} \cdot c^{(\alpha-x)t} \\
&= \left(\frac{2}{\lambda (c/b)^{-t} + (c/a)^{-t}}\right)^{\alpha} \cdot c^{-xt} \\
&= \sum_{n=0}^{\infty} \mathfrak{E}_k^{(\alpha)}\left(x;\lambda;\frac{c}{a},\frac{c}{b},c\right) \frac{(-t)^n}{n!},
\end{aligned}$$

which, upon equating coefficients of t^n in the first and the last members, leads us to the assertion (31) of Theorem 1. ■

Remark 3. It is easily seen by comparing the assertions (30) and (31) that

$$\mathfrak{E}_n^{(\alpha)}\left(-x;\lambda;\frac{a}{c},\frac{b}{c},c\right)=(-1)^n\,\mathfrak{E}_n^{(\alpha)}\left(x;\lambda;\frac{c}{a},\frac{c}{b},c\right). \tag{34}$$

Remark 4. By setting $\alpha=\lambda=1$ in Theorem 1, we obtain the corresponding known results due to Luo *et al.* [32].

Theorem 2. *Let* $a,b,c\in\mathbb{R}^+$ $(a\neq b)$ *and* $x\in\mathbb{R}$*. Then each of the following recurrence relations holds true for the generalized Euler polynomials* $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$:

$$\begin{aligned}&\frac{\alpha}{2}\ln\left(\frac{b}{a}\right)\sum_{k=0}^{n}\binom{n}{k}(\ln a)^k\,\mathfrak{E}_{n-k}^{(\alpha+1)}(x;\lambda;a,b,c)\\&\quad=\mathfrak{E}_{n+1}^{(\alpha)}(x;\lambda;a,b,c)-(x\ln c-\alpha\ln b)\,\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c),\end{aligned} \tag{35}$$

$$\begin{aligned}&\frac{\alpha\lambda}{2}\ln\left(\frac{b}{a}\right)\sum_{k=0}^{n}\binom{n}{k}\left(\ln\frac{b}{c}\right)^k\,\mathfrak{E}_{n-k}^{(\alpha+1)}(x+1;\lambda;a,b,c)\\&\quad=\mathfrak{E}_{n+1}^{(\alpha)}(x;\lambda;a,b,c)+(x\ln c-\alpha\ln a)\,\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\end{aligned} \tag{36}$$

and

$$\begin{aligned}&\frac{\alpha\lambda}{2}\ln\left(\frac{b}{a}\right)\sum_{k=0}^{n}\binom{n}{k}(\ln b)^k\,\mathfrak{E}_{n-k}^{(\alpha+1)}(x;\lambda;a,b,c)\\&\quad=\mathfrak{E}_{n+1}^{(\alpha)}(x;\lambda;a,b,c)-(x\ln c-\alpha\ln a)\,\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c).\end{aligned} \tag{37}$$

Proof. From the generating function (23), we have

$$\sum_{n=0}^{\infty}\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\,\frac{t^n}{n!}=\frac{2^\alpha}{\left(\lambda e^{t\ln(b/a)}+1\right)^\alpha}\cdot e^{t(x\ln c-\alpha\ln a)}. \tag{38}$$

Upon differentiating both sides of (38) with respect to t and after making some simple manipulations, if we equate the coefficients of like powers of t, we easily arrive at the assertions (35), (36) and (37) of Theorem 2. ∎

Remark 5. By setting

$$a=1\qquad\text{and}\qquad b=c=e$$

in (35) and (36), we obtain the corresponding known results in [24, p. 919] and [45, p. 1324], respectively. Furthermore, on making the same substitutions in (37), we get a (presumably new) result for the Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ given by

$$\mathfrak{E}_{n+1}^{(\alpha)}(x;\lambda) = x\,\mathfrak{E}_n^{(\alpha)}(x;\lambda) + \frac{\alpha\lambda}{2}\,\mathfrak{E}_n^{(\alpha+1)}(x;\lambda). \tag{39}$$

Theorem 3. *Let* $a,b,c \in \mathbb{R}^+$ $(a \neq b)$ *and* $x,y \in \mathbb{R}$. *Then*

$$\mathfrak{E}_n^{(\alpha+\beta)}(x+y;\lambda;a,b,c) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_{n-k}^{(\alpha)}(x;\lambda;a,b,c)\,\mathfrak{E}_k^{(\beta)}(y;\lambda;a,b,c) \tag{40}$$

and

$$\mathfrak{E}_n^{(\alpha)}(x+y;\lambda;a,b,c) = \sum_{k=0}^{n} \binom{n}{k} (y\ln c)^{n-k}\,\mathfrak{E}_k^{(\alpha)}(x;\lambda;a,b,c). \tag{41}$$

Proof. Upon multiplying (38) by itself with α and x replaced by β and y, respectively, if we use the defining generating function (23) on the right-hand side and simplify the resulting equation, we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\,\mathfrak{E}_k^{(\beta)}(y;\lambda;a,b,c)\,\frac{t^{n+k}}{n!\,k!}\\ &\qquad = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha+\beta)}(x+y;\lambda;a,b,c)\,\frac{t^n}{n!}.\end{aligned} \tag{42}$$

Thus, by applying the series identity (33) and comparing the coefficients of t^n on both sides of (42), we get the assertion (40) of Theorem 3.

Next, in order to prove the assertion (41) of Theorem 3, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x+y;\lambda;a,b,c)\,\frac{t^n}{n!} &= \left(\frac{2}{\lambda b^t + a^t}\right)^{\alpha} \cdot c^{(x+y)t}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathfrak{E}_k^{(\alpha)}(x;\lambda;a,b,c)\,(y\ln c)^n\,\frac{t^{n+k}}{n!\,k!},\end{aligned}$$

which, by using series manipulation and comparing like powers of t, leads us easily to the result (41). ∎

Remark 6. In its special case when

$$a = 1 \quad \text{and} \quad b = c = e,$$

the assertion (40) immediately yields the corresponding known result given by Luo [24, p. 919, Equation (9)]. Moreover, if we set $\alpha = \lambda = 1$ in (41), we obtain another known result due to Luo *et al.* [32, p. 3899].

Theorem 4. *Let* $a, b, c \in \mathbb{R}^+$ $(a \neq b)$ *and* $x \in \mathbb{R}$. *Then, for any* $l \in \mathbb{N}_0$ *and* $\xi, \eta \in \mathbb{R}$,

$$\frac{\partial^l}{\partial x^l}\left\{\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\right\} = \frac{n!}{(n-l)!}(\ln c)^l\ \mathfrak{E}_{n-l}^{(\alpha)}(x;\lambda;a,b,c) \tag{43}$$

and

$$\begin{aligned}&\int_\xi^\eta \mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\,dx\\ &\quad = \frac{1}{(n+1)\ln c}\left[\mathfrak{E}_{n+1}^{(\alpha)}(\eta;\lambda;a,b,c) - \mathfrak{E}_{n+1}^{(\alpha)}(\xi;\lambda;a,b,c)\right] \quad (\eta > \xi).\end{aligned} \tag{44}$$

Proof. The assertion (43) of Theorem 4 follows easily from the generating function (23) by its successive differentiation with respect to x and then using the principle of mathematical induction on $l \in \mathbb{N}_0$. Also, by taking $l = 1$ in (43) and integrating both sides of the resulting equation with respect to x over the interval (ξ, η) $(\eta > \xi)$, we obtain the integral formula (44) as asserted by Theorem 4. ■

Remark 7. By setting $\alpha = \lambda = 1$ in (43) and (44), we arrive at the known results due to Luo *et al.* [32, p. 3897]. Also, in its special case when

$$a = 1 \qquad \text{and} \qquad b = c = e,$$

Theorem 4 immediately yields the corresponding results due to Luo [24, p. 919, Equations (7) and (8)].

Theorem 5. *The following relationship holds true*:

$$\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c) = 2^\alpha \left[\ln\left(\frac{b}{a}\right)\right]^n \Phi_\alpha^*\left(-\lambda, -n, \frac{x\ln c - \alpha\ln a}{\ln b - \ln a}\right), \tag{45}$$

$$\left(a, b, c \in \mathbb{R}^+\ (a \neq b);\ \alpha \in \mathbb{C};\ |\lambda| < 1;\ x \in \mathbb{R} \setminus \mathbb{Z}_0^-\right)$$

between the generalized Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ *and the generalized Hurwitz-Lerch Zeta function* $\Phi_\mu^*(z,s,a)$ *defined by* (17).

Proof. We begin by rewriting (38) in the following form:

$$\sum_{n=0}^\infty \mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\,\frac{t^n}{n!} = 2^\alpha\left(1 + \lambda e^{t\ln\left(\frac{b}{a}\right)}\right)^{-\alpha} \cdot e^{t(x\ln c - \alpha\ln a)}. \tag{46}$$

Upon expressing the factor:

$$\left(1+\lambda e^{t\ln\left(\frac{b}{a}\right)}\right)^{-\alpha}$$

in a series form, if we write the exponential series for all of the exponential factors collected together, we find from (46) that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\,\frac{t^n}{n!} \\ = 2^{\alpha}\sum_{n=0}^{\infty}\left[\ln\left(\frac{b}{a}\right)\right]^n\left[\sum_{k=0}^{\infty}(\alpha)_k\,\frac{(-\lambda)^k}{k!}\,\left(k+\frac{x\,\ln c-\alpha\,\ln a}{\ln b-\ln a}\right)^n\right]\,\frac{t^n}{n!}.\end{aligned} \tag{47}$$

Now, if we first use the definition (17) and then compare the coefficients of t^n from both sides of the resulting equation, we arrive at the assertion (45) of Theorem 5. ∎

Remark 8. In its special case when

$$a=1 \qquad \text{and} \qquad b=c=e,$$

Theorem 5 immediately yields a known relationship given by Luo [29, p. 339, Equation (2.8)].

3. Series Representations Involving the Gaussian Hypergeometric Function

In this section, we present several explicit series representations for the generalized Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ of order $\alpha\in\mathbb{C}$, which involve the familiar Gaussian hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;z)$ defined by (see, for example, [42, p. 29, Equation 1.2 (4)]):

$${}_2F_1(\alpha,\beta;\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n}\,\frac{z^n}{n!} \tag{48}$$

$$\Big(\alpha,\beta\in\mathbb{C};\;\gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-;\;|z|<1;\;\Re(\gamma-\alpha-\beta)>0\quad\text{when}\quad|z|=1\;\;(z\neq-1);$$
$$\Re(\gamma-\alpha-\beta)>-1\quad\text{when}\quad z=-1\Big)$$

in terms of the Pochhammer symbol $(\lambda)_\nu$ given by (15).

Theorem 6. *Each of the following explicit series representations holds true*:

$$\begin{aligned}
&\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\\
&\quad = 2^{\alpha}\sum_{k=0}^{n}\binom{n}{k}\binom{\alpha+k-1}{k}\frac{\lambda^k}{(\lambda+1)^{\alpha+k}}\left[\ln\left(\frac{b}{a}\right)\right]^k (x\ln c-\alpha\ln a)^{n-k}\\
&\qquad \cdot\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^k\,{}_2F_1\left(k-n,1;k+1;-\frac{j\ln\left(\frac{b}{a}\right)}{x\ln c-\alpha\ln a}\right)\\
&\qquad \left(a,b,c\in\mathbb{R}^+\ \ (a\neq b);\ \alpha\in\mathbb{C};\ \lambda\in\mathbb{C}\setminus\{-1\}\right)
\end{aligned}\tag{49}$$

and

$$\begin{aligned}
&\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\\
&\quad = 2^{\alpha}\sum_{k=0}^{n}\binom{n}{k}\binom{\alpha+k-1}{k}\frac{\lambda^k}{(\lambda+1)^{\alpha+k}}\left[\ln\left(\frac{b}{a}\right)\right]^k\\
&\qquad \sum_{j=0}^{k}(-1)^j\binom{k}{j}j^k\cdot\left[x\ln c-\alpha\ln a+j\ln\left(\frac{b}{a}\right)\right]^{n-k}\\
&\qquad {}_2F_1\left(k-n,k;k+1;\frac{j\ln\left(\frac{b}{a}\right)}{x\ln c-\alpha\ln a+j\ln\left(\frac{b}{a}\right)}\right)\\
&\qquad \left(a,b,c\in\mathbb{R}^+\ \ (a\neq b);\ \alpha\in\mathbb{C};\ \lambda\in\mathbb{C}\setminus\{-1\}\right)
\end{aligned}\tag{50}$$

where ${}_2F_1(a,b;c;z)$ *denotes the Gaussian hypergeometric function defined by* (48)*.*

Proof. First of all, by making use of the Taylor-Maclaurin series expansion and the Leibniz rule in (38), we get

$$\begin{aligned}
&\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\\
&\quad = \left(\frac{2}{\lambda+1}\right)^{\alpha}\sum_{j=0}^{n}\binom{n}{j}(x\ln c-\alpha\ln a)^{n-j}\\
&\qquad \cdot D_t^j\left\{\left[1+\frac{\lambda}{\lambda+1}\left(e^{t\ln(b/a)}-1\right)\right]^{-\alpha}\right\}\Bigg|_{t=0}\qquad \left(D_t:=\frac{d}{dt}\right).
\end{aligned}\tag{51}$$

Next, by using the binomial expansion and the following known formula [40, p. 58, Equation 1.5 (15)]:

$$(e^z-1)^{\ell}=\ell!\sum_{r=\ell}^{\infty}S(r,\ell)\,\frac{z^r}{r!},\tag{52}$$

where $S(n,k)$ denotes the Stirling numbers of the second kind defined by (see, for details, [5, p. 207, Theorem B] and [40, p. 58 *et seq.*])

$$z^n=\sum_{k=0}^{n}\binom{z}{k}k!\,S(n,k),$$

so that

$$S(n,0)=\delta_{n,0},\quad S(n,1)=S(n,n)=1 \quad \text{and} \quad S(n,n-1)=\binom{n}{2},$$

$\delta_{m,n}$ being the Kronecker symbol, we thus find from (51) that

$$\begin{aligned}
&\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\\
&= 2^{\alpha}\sum_{j=0}^{n}\binom{n}{j}(x\ln c-\alpha\ln a)^{n-j}\\
&\quad\cdot\left[\sum_{r=0}^{j}\binom{\alpha+r-1}{r}r!\,\frac{(-\lambda)^r}{(\lambda+1)^{\alpha+r}}\,S(j,r)\left[\ln\left(\frac{b}{a}\right)\right]^j\right],
\end{aligned}\tag{53}$$

which, upon inverting the order of summation, yields

$$\begin{aligned}
&\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\\
&= 2^{\alpha}\sum_{r=0}^{n}\binom{\alpha+r-1}{r}r!\,\frac{(-\lambda)^r}{(\lambda+1)^{\alpha+r}}\\
&\quad\cdot\left[\sum_{j=r}^{n}\binom{n}{j}(x\ln c-\alpha\ln a)^{n-j}\,S(j,r)\left[\ln\left(\frac{b}{a}\right)\right]^j\right].
\end{aligned}\tag{54}$$

Now, by making use of the following known result [40, p. 58, Equation 1.5 (20)]:

$$S(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n,\tag{55}$$

we readily obtain

$$\begin{aligned}
&\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\\
&= 2^{\alpha}\sum_{r=0}^{n}\binom{\alpha+r-1}{r}\frac{(-\lambda)^r}{(\lambda+1)^{\alpha+r}}\sum_{j=r}^{n}\binom{n}{j}\left[\ln\left(\frac{b}{a}\right)\right]^j\\
&\quad\cdot(x\ln c-\alpha\ln a)^{n-j}\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}k^j,
\end{aligned}\tag{56}$$

which, upon interchanging the order of the j- and k-summations, yields

$$\begin{aligned}
&\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\\
&= 2^{\alpha}\sum_{r=0}^{n}\binom{\alpha+r-1}{r}\frac{(-\lambda)^r}{(\lambda+1)^{\alpha+r}}\sum_{k=0}^{r}(-1)^k\binom{r}{k}\\
&\quad\cdot\sum_{j=r}^{n}\binom{n}{j}\left[\ln\left(\frac{b}{a}\right)\right]^j(x\ln c-\alpha\ln a)^{n-j}\,k^j.
\end{aligned}\tag{57}$$

Finally, by simplifying the innermost summation on the right-hand side of (57) and expressing it in terms of the Gaussian hypergeometric function defined by (48), we easily arrive at the first result (49) asserted by Theorem 6.

If we apply the familiar Pfaff-Kummer transformation (see, for example, [1, p. 559, Equation (15.3.4)]):

$$
{}_2F_1\left(\alpha,\beta;\gamma;z\right) = (1-z)^{-\alpha}\ {}_2F_1\left(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1}\right) \qquad \left(\alpha,\beta\in\mathbb{C};\ \gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ |\arg(1-z)|<\pi\right) \tag{58}
$$

in (49), we are led immediately to the second result (50) asserted by Theorem 6. ■

Remark 9. By setting

$$
a=1 \qquad \text{and} \qquad b=c=e,
$$

Theorem 6 yields the corrersponding known results proven earlier by Luo [24, p. 921, Equation (22); p. 920, Equation (16)].

In our proof of the explicit series representation (64) asserted by Theorem 7 below, we shall make use of the following lemma.

Lemma 2. *The generalized Euler polynomials* $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ *can be represented in series of the Euler polynomials* $E_n^{(\alpha)}(x)$ *of order* $\alpha\in\mathbb{C}$ *as follows*:

$$
\begin{aligned}
&\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c) \\
&= e^{-x\left(\frac{\ln c\,\ln\lambda}{\ln b-\ln a}\right)}\left[\ln\left(\frac{b}{a}\right)\right]^n \sum_{k=0}^{\infty}\sum_{j=0}^{n}(-\alpha)^j\binom{n}{j} \\
&\quad\cdot\left(\frac{\ln a}{\ln b-\ln a}\right)^j E_{n+k-j}^{(\alpha)}\left(\frac{x\ \ln c}{\ln b-\ln a}\right)\frac{(\ln\lambda)^k}{k!}.
\end{aligned} \tag{59}
$$

Proof. By using the definition (23), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\,\frac{t^n}{n!} \\
&= \left(\frac{2}{\lambda b^t+a^t}\right)^{\alpha}\cdot c^{xt} = \left(\frac{2}{e^{t\ln(b/a)+\ln\lambda}+1}\right)^{\alpha}\cdot e^{(x\ln c-\alpha\ln a)t}.
\end{aligned} \tag{60}
$$

Upon setting

$$
T := t\ln\left(\frac{b}{a}\right) \qquad \text{and} \qquad X := \frac{x\ \ln c}{\ln b-\ln a}, \tag{61}
$$

the generating relation (60) can be written as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!} \\ = e^{-X\ln\lambda}\cdot e^{-\alpha\left(\frac{\ln a}{\ln b-\ln a}\right)T}\left(\frac{2}{e^{T+\ln\lambda}+1}\right)^{\alpha}\cdot e^{X(T+\ln\lambda)}.\end{aligned} \tag{62}$$

We now use the definition given by (2) and perform some mathematical manipulations and simplifications. We thus find from (62) that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!} \\ = e^{-X\ln\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n}\frac{(-\alpha)^j}{j!\,(n-j)!}\left(\frac{\ln a}{\ln b-\ln a}\right)^j \\ \cdot E_{n+k-j}^{(\alpha)}(X)\ \frac{(\ln\lambda)^k}{k!}\,T^n.\end{aligned} \tag{63}$$

Upon substituting the values of X and T from (61) into (63), if we compare the coefficients of t^n on both sides of the resulting equation, we get the assertion (59) of Lemma 2. ■

Theorem 7. *The following explicit series representation holds true*:

$$\begin{aligned}&\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c) \\ &= e^{-x\left(\frac{\ln c\ln\lambda}{\ln b-\ln a}\right)}\left[\ln\left(\frac{b}{a}\right)\right]^k\sum_{k=0}^{\infty}\sum_{j=0}^{n}(-\alpha)^j\binom{n}{j}\left(\frac{\ln a}{\ln b-\ln a}\right)^j \\ &\cdot\frac{(\ln\lambda)^k}{k!}\sum_{m=0}^{n+k-j}\binom{n+k-j}{m}\binom{\alpha+m-1}{m}\sum_{\ell=0}^{m}(-1)^{\ell}\binom{m}{\ell}\left(\frac{\ell}{2}\right)^m \\ &\cdot\left(\frac{x\ \ln c}{\ln b-\ln a}+\ell\right)^{n+k-j-m} \\ &\cdot{}_2F_1\left(m-n-k+j,m;m+1;\frac{\ell\ln\left(\frac{b}{a}\right)}{x\ln c+\ell\ln\left(\frac{b}{a}\right)}\right), \\ &\qquad\big(a,b,c\in\mathbb{R}^+\ \ (a\neq b);\ \alpha\in\mathbb{C}\big),\end{aligned} \tag{64}$$

where ${}_2F_1(\alpha,\beta;\gamma;z)$ *denotes the Gaussian hypergeometric function defined by* (48).

Proof. The assertion (64) of Theorem 7 can be proved by applying Lemma 2 in conjunction with the following known explicit series representation given earlier by Luo [24, p. 923, Equation (33)]:

$$
\begin{aligned}
E_n^{(\alpha)}(x) &= \sum_{l=0}^{n} \binom{n}{l} \binom{\alpha+l-1}{l} \\
&\quad \cdot \sum_{k=0}^{l} (-1)^k \binom{l}{k} \left(\frac{k}{2}\right)^l (x+k)^{n-l} \; {}_2F_1\left(l-n, l; l+1; \frac{k}{x+k}\right)
\end{aligned} \tag{65}
$$

in terms of the Gaussian hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$ defined by (48). ∎

Remark 10. By setting

$$a = 1 \qquad \text{and} \quad b = c = e$$

in (64), we obtain a (presumably new) result for the Apostol-Euler polynomials of order $\alpha \in \mathbb{C}$ given by

$$
\begin{aligned}
&\mathcal{E}_n^{(\alpha)}(x; \lambda) \\
&= e^{-x \ln \lambda} \sum_{k=0}^{\infty} \frac{(\ln \lambda)^k}{k!} \sum_{m=0}^{n+k} \binom{n+k}{m} \binom{\alpha+m-1}{m} \\
&\quad \cdot \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left(\frac{j}{2}\right)^m (x+j)^{n+k-m} {}_2F_1\left(m-n-k, m; m+1; \frac{j}{x+j}\right),
\end{aligned} \tag{66}
$$

which, in the further special case when $\lambda = 1$, yields the known result (65).

Remark 11. For the Apostol-Euler numbers $\mathcal{E}_n^{(\alpha)}(\lambda)$, by setting

$$a = 1, \quad b = c = e \qquad \text{and} \qquad x = 0$$

in (64), we obtain the following explicit series representation:

$$\mathcal{E}_n^{(\alpha)}(\lambda) = \sum_{k=0}^{\infty} \frac{(\ln \lambda)^k}{k!} \sum_{m=0}^{n+k} \binom{\alpha+m-1}{m} \left(-\frac{1}{2}\right)^m m! \, S(n+k, m), \tag{67}$$

where we have made use of the celebrated Gauss summation theorem [1, p. 556, Equation (15.1.20)]:

$$
\begin{aligned}
{}_2F_1(\alpha, \beta; \gamma; 1) &= \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \\
&\qquad \big(\alpha, \beta \in \mathbb{C};\ \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(\gamma-\alpha-\beta) > 0\big),
\end{aligned} \tag{68}
$$

which, for

$$\alpha = m - n - k, \quad \beta = m \qquad \text{and} \qquad c = m + 1 \qquad (k, m, n \in \mathbb{N}_0),$$

yields the following special case:

$$
{}_2F_1\,(m-n-k, m; m+1; 1) = \binom{n+k}{m}^{-1} \qquad (m = 0, 1, 2, \cdots, n+k;\; n, k \in \mathbb{N}_0)\,. \tag{69}
$$

4. The Closely-related Family of Generalized Genocchi Polynomials

In this section, motivated essentially by our Definitions 1 and 2, we introduce a new family of generalized Genocchi polynomials by means of a suitable generating function along the lines of the generating functions involved in (22) and (23).

Definition 3. Let $a, b, c \in \mathbb{R}^+$ $(a \neq b)$ and $n \in \mathbb{N}_0$. Then the *generalized Genocchi polynomials* $\mathfrak{G}_n^{(\alpha)}(x; \lambda; a, b, c)$ *of order* $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$
\left(\frac{2t}{\lambda b^t + a^t}\right)^{\alpha} \cdot c^{xt} = \sum_{n=0}^{\infty} \mathfrak{G}_n^{(\alpha)}(x; \lambda; a, b, c)\, \frac{t^n}{n!} \qquad \left(\left| t \ln\left(\frac{b}{a}\right) + \ln\lambda \right| < \pi;\; 1^{\alpha} := 1;\; x \in \mathbb{R}\right). \tag{70}
$$

Clearly, since the parameter $\lambda \in \mathbb{C}$, by comparing Definition 3 with our Definitions 1 and 2, we are led easily to Lemma 3 below.

Lemma 3. *Each of the following relationships holds true*:

$$
\mathfrak{G}_n^{(\alpha)}(x; \lambda; a, b, c) = (-2)^{\alpha}\; \mathfrak{B}_n^{(\alpha)}(x; -\lambda; a, b, c) \qquad (\alpha \in \mathbb{C};\; 1^{\alpha} := 1) \tag{71}
$$

and

$$
\begin{aligned}
\mathfrak{G}_n^{(l)}(x; \lambda; a, b, c) &= (-1)^l\, (-n)_l\; \mathfrak{E}_{n-l}^{(l)}(x; \lambda; a, b, c) \\
&= \frac{n!}{(n-l)!}\; \mathfrak{E}_{n-l}^{(l)} \\
&\quad (x; \lambda; a, b, c)\, (n, l \in \mathbb{N}_0;\; n \geqq l;\; \lambda \in \mathbb{C})\,.
\end{aligned} \tag{72}
$$

Remark 12. The connection formulas (71) and (72) are potentially useful in the study of the generalized Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x; \lambda; a, b, c)$ in the sense that, by appealing appropriately to Lemma 3, their various properties can readily be deduced from those of the generalized Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x; \lambda; a, b, c)$ and the generalized Euler polynomials $\mathfrak{G}_n^{(\alpha)}(x; \lambda; a, b, c)$. For example, the last connection formula (72) in conjunction with Theorems 5 and 6 (with, of course,

$\alpha = l \in \mathbb{N}_0$) would immediately yield the following results.

Theorem 8. *The following relationship holds true*:

$$\begin{aligned} \mathfrak{G}_n^{(l)}(x;\lambda;a,b,c) &= \frac{2^l \cdot n!}{(n-l)!}\left[\ln\left(\frac{b}{a}\right)\right]^{n-l} \\ &\cdot \Phi_l^*\left(-\lambda, l-n, \frac{x\ln c - l\ln a}{\ln b - \ln a}\right), \\ &\left(a,b,c\in\mathbb{R}^+\ (a\neq b);\ l\in\mathbb{N}_0;\ |\lambda|<1;\ x\in\mathbb{R}\setminus\mathbb{Z}_0^-\right) \end{aligned} \tag{73}$$

between the generalized Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x;\lambda;a,b,c)$ *and the generalized Hurwitz-Lerch Zeta function* $\Phi_\mu^*(z,s,a)$ *defined by* (17).

Theorem 9. *Each of the following explicit series representations holds true*:

$$\begin{aligned} &\mathfrak{G}_n^{(l)}(x;\lambda;a,b,c) \\ &= \frac{2^l\cdot n!}{(n-l)!}\sum_{k=0}^{n-l}\binom{n-l}{k}\binom{k+l-1}{k} \\ &\cdot\frac{\lambda^k}{(\lambda+1)^{k+l}}\left[\ln\left(\frac{b}{a}\right)\right]^k (x\ln c - l\ln a)^{n-k-l} \\ &\cdot\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^k\,{}_2F_1\left(k+l-n,1;k+1;-\frac{j\ln\left(\frac{b}{a}\right)}{x\ln c - l\ln a}\right) \\ &\left(a,b,c\in\mathbb{R}^+\ \ (a\neq b);\ l\in\mathbb{N}_0;\ \lambda\in\mathbb{C}\setminus\{1\}\right) \end{aligned} \tag{74}$$

and

$$\begin{aligned} &\mathfrak{G}_n^{(l)}(x;\lambda;a,b,c) \\ &= \frac{2^l\cdot n!}{(n-l)!}\sum_{k=0}^{n-l}\binom{n-l}{k}\binom{k+l-1}{k}\frac{\lambda^k}{(\lambda+1)^{k+l}}\left[\ln\left(\frac{b}{a}\right)\right]^k \\ &\cdot\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^k\cdot\left[x\ln c - l\ln a + j\ln\left(\frac{b}{a}\right)\right]^{n-k-l} \\ &\cdot{}_2F_1\left(k+l-n,k;k+1;-\frac{j\ln\left(\frac{b}{a}\right)}{x\ln c - l\ln a + j\ln\left(\frac{b}{a}\right)}\right) \\ &\left(a,b,c\in\mathbb{R}^+\ \ (a\neq b);\ l\in\mathbb{N}_0;\ \lambda\in\mathbb{C}\setminus\{1\}\right) \end{aligned} \tag{75}$$

where ${}_2F_1(a,b;c;z)$ *denotes the Gaussian hypergeometric function defined by* (48).

We conclude our investigation by observing that each of the connection formulas asserted by Lemmas 1 and 3 can similarly be applied to appropriately translate any given result involving one of the three families of the generalized Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)$, the generalized Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ and the generalized Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x;\lambda;a,b,c)$ into the corresponding result associated with each of the other two families of these polynomials. We choose to leave the details involved in such straightforward derivations as an exercise for the interested reader.

References

1. M. Abramowitz and I. A. Stegun (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Tenth Printing, Applied Mathematics Series **55**, National Bureau of Standards, Washington D.C., 1972.

2. T. M. Apostol, On the Lerch Zeta function, *Pacific J. Math.*, **1** (1951), 161-167.

3. J. Choi, P. J. Anderson and H. M. Srivastava, Some q-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order n and the multiple Hurwitz zeta function, *Appl. Math. Comput.*, **199** (2008), 723-737.

4. J. Choi, P. J. Anderson and H. M. Srivastava, Carlitz's q-Bernoulli and q-Euler numbers and polynomials and a class of q-Hurwitz zeta functions, *Appl. Math. Comput.*, **215** (2009), 1185-1208.

5. L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions* (Translated from the French by J. W. Nienhuys), Reidel, Dordrecht and Boston, 1974.

6. M. Garg, K. Jain and H. M. Srivastava, Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions, *Integral Transforms Spec. Funct.*, **17** (2006), 803-815.

7. S. P. Goyal and R. K. Laddha, On the generalized Riemann Zeta functions and the generalized Lambert transform, *Gaṇita Sandesh*, **11** (1997), 99-108.

8. B.-N. Guo and F. Qi, Generalization of Bernoulli polynomials, *Internat. J. Math. Ed. Sci. Tech.*, **33** (2002), 428-431.

9. A. F. Horadam, Genocchi polynomials, in *Proceedings of the Fourth International Conference on Fibonacci Numbers and Their Applications*, Kluwer Academic Publishers, Dordrecht, Boston and London, 1991, pp. 145-166.

10. A. F. Horadam, Negative order Genocchi polynomials, *Fibonacci Quart.*, **30** (1992), 21-34.

11. A. F. Horadam, Generation of Genocchi polynomials of first order by recurrence relations, *Fibonacci Quart.*, **30** (1992), 239-243.

12. K.-W. Hwang, Y.-H. Kim and T. Kim, Interpolation functions of q-extensions of Apostol's type Euler polynomials, *J. Inequal. Appl.*, **2009** (2009), Article ID 451217, 1-12.

13. L.-C. Jang and T. Kim, On the distribution of the q-Euler polynomials and the q-Genocchi polynomials of higher order, *J. Inequal. Appl.*, **2008** (2008), Article ID 723615, 1-9.

14. T. Kim, On the q-extension of Euler numbers and Genocchi numbers, *J. Math. Anal. Appl.*, **326** (2007), 1458-1465.

15. T. Kim, On the multiple q-Genocchi and Euler numbers, *Russian J. Math. Phys.*, **15** (2008), 481-486.

16. Y.-H. Kim, W.-J. Kim and L.-C. Jang, On the q-extension of Apostol-Euler numbers and polynomials, *Abstr. Appl. Anal.*, **2008** (2008), Article ID 296159, 1-10.

17. B. Kurt, A further symmetric relation on the analogue of the Apostol-Bernoulli and the analogue of the Apostol-Genocchi polynomials, *Appl. Math. Sci.* (*Ruse*), **3** (2009), 2757-2764.

18. V. Kurt, The multiplication formulae for the Genocchi polynomials, *Proc. Jangjeon Math. Soc.*, **13** (2010), 89-96.

19. V. Kurt and M. Cenkci, A new approach to q-Genocchi numbers and polynomials, *Bull. Korean Math. Soc.* **47** (2010), 575-583.

20. S.-D. Lin and H. M. Srivastava, Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations, *Appl. Math. Comput.*, **154** (2004), 725-733.

21. S.-D. Lin, H. M. Srivastava and P.-Y. Wang, Some expansion formulas for a class of generalized Hurwitz-Lerch Zeta functions, *Integral Transforms Spec. Funct.*, **17** (2006), 817-828.

22. H.-M. Liu and W.-P. Wang, Some identities on the Bernoulli, Euler and Genocchi polynomials via power sums and alternate power sums, *Discrete Math.*, **309** (2009), 3346-3363.

23. Q.-M. Luo, On the Apostol-Bernoulli polynomials, *Central European J. Math.*, **2** (2004), 509-515.

24. Q.-M. Luo, Apostol-Euler polynomials of higher order and Gaussian hypergeometric functions, *Taiwanese J. Math.*, **10** (2006), 917-925.

25. Q.-M. Luo, Fourier expansions and integral representations for the Apostol-Bernoulli and Apostol-Euler polynomials, *Math. Comput.*, **78** (2009), 2193-2208.

26. Q.-M. Luo, The multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials of higher order, *Integral Transforms Spec. Funct.*, **20** (2009), 377-391.

27. Q.-M. Luo, Fourier expansions and integral representations for the Genocchi polynomials, *J. Integer Seq.*, **12** (2009), Article 09.1.4, 1-9.

28. Q.-M. Luo, q-Extensions for the Apostol-Genocchi polynomials, *Gen. Math.*, **17** (2009), 113-125.

29. Q.-M. Luo, Some formulas for Apostol-Euler polynomials associated with Hurwitz zeta function at rational arguments, *Appl. Anal. Discrete Math.*, **3** (2009), 336-346.

30. Q.-M. Luo, Extension for the Genocchi polynomials and its Fourier expansions and integral representations, *Osaka J. Math.*, (to appear).

31. Q.-M. Luo, B.-N. Guo, F. Qi and L. Debnath, Generalizations of Bernoulli numbers and polynomials, *Internat. J. Math. Math. Sci.*, **2003** (59) (2003), 3769-3776.

32. Q.-M. Luo, F. Qi and L. Debnath, Generalizations of Euler numbers and polynomials, *Internat. J. Math. Math. Sci.*, **2003** (61) (2003), 3893-3901.

33. Q.-M. Luo and H. M. Srivastava, Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials, *J. Math. Anal. Appl.*, **308** (2005), 290-302.

34. Q.-M. Luo and H. M. Srivastava, Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials, *Comput. Math. Appl.*, **51** (2006), 631-642.

35. N. E. Nörlund, *Vorlesungen über Differentzenrechnung*, Springer-Verlag, Berlin, 1924; Reprinted by Chelsea Publishing Company, Bronx, New York, 1954.

36. H. Ozden, Y. Simsek and H. M. Srivastava, A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials, *Comput. Math. Appl.*, **60** (2010), 2779-2787.

37. M. Prévost, Padé approximation and Apostol-Bernoulli and Apostol-Euler polynomials, *J. Comput. Appl. Math.*, **233** (2010), 3005-3017.

38. J. Sándor and B. Crstici, *Handbook of Number Theory*, Vol. II, Kluwer Academic Publishers, Dordrecht, Boston and London, 2004.

39. H. M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, *Math. Proc. Cambridge Philos. Soc.*, **129** (2000), 77-84.

40. H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

41. H. M. Srivastava, M. Garg and S. Choudhary, A new generalization of the Bernoulli and related polynomials, *Russian J. Math. Phys.*, **17** (2010), 251-261.

42. H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

43. H. M. Srivastava and Á. Pintér, Remarks on some relationships between the Bernoulli and Euler polynomials, *Appl. Math. Lett.*, **17** (2004), 375-380.

44. H. M. Srivastava and P. G. Todorov, An explicit formula for the generalized Bernoulli polynomials, *J. Math. Anal. Appl.*, **130** (1988), 509-513.

45. W.-P. Wang, C.-Z. Jia and T.-M. Wang, Some results on the Apostol-Bernoulli and Apostol-Euler polynomials, *Comput. Math. Appl.*, **55** (2008), 1322-1332.

46. W.-P. Wang and W.-W. Wang, Some results on power sums and Apostol-type polynomials, *Integral Transforms Spec. Funct.*, **21** (2010), 307-318.

47. Z.-Z. Zhang and J.-Y. Jin, Some identities involving generalized Genocchi polynomials and generalized Fibonacci-Lucas sequences, *Fibonacci Quart.*, **36** (1998), 329-334.

48. Z.-Z. Zhang and H.-Q. Yang, Several identities for the generalized Apostol-Bernoulli polynomials, *Comput. Math. Appl.*, **56** (2008), 2993-2999.

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