TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 1, pp. 229-240, February 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

MAXIMAL REGULARITY FOR INTEGRAL EQUATIONS IN BANACH SPACES

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Abstract. We study maximal regularity in periodic Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ for the integral equations (P): $u(t) = A\int_{-\infty}^t a(t-s)u(s)ds) + B\int_{-\infty}^t b(t-s)u(s)ds + f(t)$ on $[0, 2\pi]$ with periodic boundary condition $u(0) = u(2\pi)$, where A and B are closed operators in a Banach space X, $a,\ b \in L^1(\mathbb{R}_+)$ and f is a given function defined on $[0, 2\pi]$ with values in X. Under suitable assumptions on the kernels $a,\ b$ and the closed operators $A,\ B$, we completely characterize $B^s_{p,q}$-maximal regularity of (P).

1. Introduction

In a series of recent publications operator-valued Fourier multipliers on vector-valued function spaces are studied (see e.g. [1-4, 13, 14]. They are needed to establish existence and uniqueness as well as regularity of differential equations in Banach spaces, and thus also for partial differential equations (see e.g. [1-3, 5-10]. In this paper, we use operator-valued Fourier multiplier result established in [3] to study $B^s_{p,q}$-maximal regularity for the following integral equations:

$$
(1) \qquad \begin{cases} u(t) = A\int_{-\infty}^t a(t-s)u(s)ds \\ \qquad\quad + B\int_{-\infty}^t b(t-s)u(s)ds + f(t), \quad 0 \le t \le 2\pi \\ u(0) = u(2\pi), \end{cases}
$$

here $A,\ B$ are closed linear operators in a complex Banach space X, $f \in B^s_{p,q}(\mathbb{T}, X)$, and $a,\ b \in L^1(\mathbb{R}_+)$.

Received October 31, 2008, accepted July 12, 2009.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: Primary 45N05; Secondary 45D05, 43A15, 47D99.
Key words and phrases: Fourier multiplier, Maximal regularity, Integral equation, Besov spaces.
This work was supported by the NSF of China (10731020) and the specialized research fund for the doctoral program of higher education (200800030059).

Equations of the form (1) has been motivated by Pugliese [11] and Prüss [12, page 235]. L^p-maximal regularity for (1) has been studied by Lizama and Poblete [8], using operator-valued Fourier multiplier result obtained in [2], they completely characterized L^p-maximal regularity for (1) under suitable assumptions on the kernels $a,\ b$ and the operators $A,\ B$.

In this paper, we study the maximal regularity of (1) in periodic Besov spaces $B^s_{p,q}(\mathbb{T}, X)$, where $1 \le p,\ q \le \infty,\ s > 0$. We do not make any parabolicity assumptions on $A,\ B$, not even that A generates a semigroup. Thus semigroup theory is no longer applicable in our situation. The main tool in our study is operator-valued Fourier multiplier results on $B^s_{p,q}(\mathbb{T}, X)$ established in [3]. In fact, we will transform $B^s_{p,q}$-maximal regularity problem of (1) to a problem of whether an operator-valued sequence $(M_k)_{k\in\mathbb{Z}}$ defined by the kernels $a,\ b$ and the operators $A,\ B$ is a $B^s_{p,q}$-multiplier. We will show that the resulting sequence $(M_k)_{k\in\mathbb{Z}}$ satisfies the sufficient conditions given in [3] ensuring an operator-valued sequence to be a $B^s_{p,q}$-multiplier. We notice that the presence of two closed operators A and B makes this verification particularly complicated and more careful computation is needed.

Since our necessary and sufficient condition for (1) to have $B^s_{p,q}$-maximal regularity does not depends on the choice of $1 \le p,\ q \le \infty,\ s > 0$, one immediate consequence of our main result is that under suitable conditions on the kernels $a,\ b$, the problem (1) has $B^s_{p,q}$-maximal regularity for some $1 \le p,\ q \le \infty,\ s > 0$ if and only if it has $B^s_{p,q}$-maximal regularity for all $1 \le p,\ q \le \infty,\ s > 0$. Moreover since periodic Hölder continuous function space $C^\alpha_{per}([0, 2\pi], X)$ is a particular case of the periodic Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ when taking $p = q = \infty$ and $s = \alpha$, our main result gives a characterization of C^α_{per}-maximal regularity for (1). Our result may be applied to the case when A is sectorial and $B = A^\epsilon$ for some $0 < \epsilon < 1$, in this case one can use the functional calculus of A to determine a concrete expression of the resulting sequence $(M_k)_{k\in\mathbb{Z}}$.

2. Preliminaries

Let X be a complex Banach space. For $f \in L^1(\mathbb{T}, X)$, we denote by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e_{-k}(t) f(t) dt$$

the k-th Fourier coefficient of f, where $k \in \mathbb{Z}$, $\mathbb{T} = [0, 2\pi]$ (the points 0 and 2π are identified), and $e_k(t) = e^{ikt}$. For $x \in X$, we denote by $e_k \otimes x$ the X-valued function defined on $\mathbb{T}$ by $(e_k \otimes x)(t) = e_k(t)x$.

Firstly, we briefly recall the definition of periodic Besov spaces in the vector-valued case introduced in [3]. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorms $\|f\|_\alpha = sup_{x\in\mathbb{T}}|f^{(\alpha)}(x)|$ for $\alpha \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Let $\mathcal{D}'(\mathbb{T},X) := \mathcal{L}(\mathcal{D}(\mathbb{T}),X)$ be the space of all bounded linear operator from $\mathcal{D}(\mathbb{T})$ to X. For $k \in \mathbb{Z}$ and $f \in \mathcal{D}'(\mathbb{T},X)$, one defines the k-th Fourier coefficient of f by $\hat{f}(k) := f(e_{-k})$. In order to define periodic Besov spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : \ |t| \le 2\}, \ I_k = \{t \in \mathbb{R} : \ 2^{k-1} < |t| \le 2^{k+1}\}$$

for $k \in \mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ satisfying $supp(\phi_k) \subset \bar{I}_k$ for each $k \in \mathbb{N}_0$,

$$\sum_{k\in\mathbb{N}_0} \phi_k(x) = 1 \qquad \text{for} \quad x \in \mathbb{R},$$

and for each $\alpha \in \mathbb{N}_0$

$$sup_{\substack{x\in\mathbb{R}\\ k\in\mathbb{N}_0}} 2^{k\alpha}|\phi_k^{(\alpha)}(x)| < \infty.$$

Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \in \phi(\mathbb{R})$ be fixed. For $1 \le p,q \le \infty$, $s \in \mathbb{R}$, the X-valued periodic Besov space is defined by

$$B^s_{p,q}(\mathbb{T},X) := \Big\{ f \in \mathcal{D}'(\mathbb{T},X) : \ \|f\|_{B^s_{p,q}} := \Big(\sum_{j\ge 0} 2^{sjq} \Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k) \Big\|_p^q \Big)^{1/q} < \infty \Big\}$$

with the usual modification if $q = \infty$. The space $B^s_{p,q}(\mathbb{T},X)$ is independent from the choice of ϕ and different choices of ϕ lead to equivalent norms $\|\cdot\|_{B^s_{p,q}}$ on $B^s_{p,q}(\mathbb{T},X)$. $B^s_{p,q}(\mathbb{T},X)$ equipped with the norm $\|\cdot\|_{B^s_{p,q}}$ is a Banach space. See [3, Section 2] for more information about the space $B^s_{p,q}(\mathbb{T},X)$. We only recall that when $s > 0$, then $B^s_{p,q}(\mathbb{T},X) \subset L^p(\mathbb{T},X)$ and the inclusion is continuous.

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from X to Y. If $X = Y$, we will simply denote it by $\mathcal{L}(X)$. let $M = (M_k)_{k\in\mathbb{Z}}$ be a sequence in $\mathcal{L}(X,Y)$. We define the first derivative of M as the sequence in $\mathcal{L}(X,Y)$ given by

$$(\Delta M)_k := M_{k+1} - M_k, \qquad (k \in \mathbb{Z}).$$

The second derivative of M is defined by

$$(\Delta^2 M)_k := \big(\Delta(\Delta M)\big)_k = M_{k+2} - 2M_{k+1} + M_k, \qquad (k \in \mathbb{Z}).$$

If $a = (a_k)_{k\in\mathbb{Z}}$ is a scalar sequence, we define the first and second derivatives of a in a similar way.

The main tool in our study of $B^s_{p,q}$-maximal regularity of (1) is the operator-valued Fourier multiplier theory established in [3].

Definition 2.1. Let $X,\ Y$ be Banach spaces, $1 \leq p,\ q \leq \infty,\ s \in \mathbb{R}$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$. We say that $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier, if for each $f \in B^s_{p,q}(\mathbb{T},X)$, there exists $u \in B^s_{p,q}(\mathbb{T},Y)$, such that $\hat{u}(k) = M_k\hat{f}(k)$ for all $k \in \mathbb{Z}$.

It follows from the closed graph theorem that when $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier, then there exists a constant $C \geq 0$, such that for all $f \in B^s_{p,q}(\mathbb{T},X)$, one has $\big\| \sum_{k\in\mathbb{Z}} e_k \otimes M_k\hat{f}(k)\big\|_{B^s_{p,q}} \leq C\|f\|_{B^s_{p,q}}$. In particular, $(M_k)_{k\in\mathbb{Z}}$ must be bounded.

The following result has been obtained in [3]:

Theorem 2.2. *Let $X,\ Y$ be Banach spaces, $1 \leq p,\ q \leq \infty,\ s \in \mathbb{R}$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$. We assume that*

$$\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k(\Delta M)_k\|) < \infty, \tag{2.1}$$

$$\sup_{k\in\mathbb{Z}}\|k^2(\Delta^2 M)_k\| < \infty. \tag{2.2}$$

Then $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Moreover, if X and Y are B-convex, then the first order condition (2.1) is sufficient for $(M_k)_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-multiplier.

Recall that a Banach space X is B-convex if it does not contain l^n_1 uniformly. This is equivalent to say that X has Fourier type $1 < p \leq 2$, i.e., the Fourier transform is a bounded linear operator from $L^p(\mathbb{R},X)$ to $l^q(\mathbb{Z},X)$, where $1/p + 1/q = 1$. It is well known that when $1 < p < \infty$, then $L^p(\mu)$ has Fourier type $\min\{p, \frac{p}{p-1}\}$.

Given $a \in L^1(\mathbb{R}_+)$ and $u \in B^s_{p,q}(\mathbb{T},X)$ (extended by periodicity to $\mathbb{R}$), we define

$$(a * u)(t) := \int_{-\infty}^{t} a(t-s)u(s)ds. \tag{2.3}$$

Let $\tilde{a}(\lambda) = \int_0^{+\infty} e^{-\lambda t}a(t)dt$ be the Laplace transform of a for $\mathrm{Re}\lambda \geq 0$. An easy computation shows that:

$$\widehat{a * u}(k) = \tilde{a}(ik)\hat{u}(k), \quad (k \in \mathbb{Z}). \tag{2.4}$$

It follows that when $u \in B^s_{p,q}(\mathbb{T},X)$, then $a * u \in B^s_{p,q}(\mathbb{T},X)$ and $\|a * u\|_{B^s_{p,q}} \leq \|a\|_{L^1}\|u\|_{B^s_{p,q}}$ by the inequality of Young.

3. A Characterization of $B^s_{p,q}$-Maximal Regularity for (1)

We consider the integral equations

(3.1)
$$\begin{cases} u(t) = A\displaystyle\int_{-\infty}^{t} a(t-s)u(s)ds \\ \qquad + B\displaystyle\int_{-\infty}^{t} b(t-s)u(s)ds + f(t), \quad 0 \le t \le 2\pi \\ u(0) = u(2\pi), \end{cases}$$

where $A,\ B$ are closed linear operators in a complex Banach space X, $f \in B^s_{p,q}(\mathbb{T}, X)$, and $a,\ b \in L^1(\mathbb{R}_+)$. Using the notation (2.4), (3.1) may be written in the more compact form: $u(t) = A(a * u)(t) + B(b * u)(t) + f(t),\ (t \in \mathbb{T}),\ u(0) = u(2\pi)$.

Definition 3.1. Let $1 \le p, q \le \infty,\ s > 0$ and let $f \in B^s_{p,q}(\mathbb{T}, X)$ be given. $u \in B^s_{p,q}(\mathbb{T}, X)$ is called a mild $B^s_{p,q}$-solution of (3.1), if $a * u \in B^s_{p,q}(\mathbb{T}, D(A))$, $b * u \in B^s_{p,q}(\mathbb{T}, D(B))$ and (3.1) holds for a.e. $t \in \mathbb{T}$. Here we consider $D(A)$ and $D(B)$ as Banach spaces equipped with their graph norms. We say that (3.1) has $B^s_{p,q}$-maximal regularity, if for each $f \in B^s_{p,q}(\mathbb{T}, X)$, (3.1) has a unique mild $B^s_{p,q}$-solution.

It follows easily from the closed graph theorem that when (3.1) has $B^s_{p,q}$-maximal regularity, then there exists a constant $C \ge 0$, such that for $f \in B^s_{p,q}(\mathbb{T}, X)$, if u is the unique mild $B^s_{p,q}$-solution of (3.1), then

(3.2)
$$\|u\|_{B^s_{p,q}} + \|A(a * u)\|_{B^s_{p,q}} + \|B(b * u)\|_{B^s_{p,q}} \le C\|f\|_{B^s_{p,q}}.$$

Let $a \in L^1(\mathbb{R}_+)$ be given, for $k \in \mathbb{Z}$ we denote by

$$\tilde{a}_k := \int_0^\infty a(t)e^{-ikt}dt$$

the Laplace transform of a. Let $b \in L^1(\mathbb{R}_+)$ and $A,\ B$ be closed operators in X. We will consider the operator

$$C_k := I - \tilde{a}_k A - \tilde{b}_k B, \quad (k \in \mathbb{Z}).$$

The natural domain of definition $D(C_k)$ of C_k depends on the values of $\tilde{a}_k$ and $\tilde{b}_k$:

(1) if $\tilde{a}_k \ne 0$ and $\tilde{b}_k \ne 0$, then $D(C_k) = D(A) \cap D(B)$;
(2) if $\tilde{a}_k \ne 0$ and $\tilde{b}_k = 0$, then $D(C_k) = D(A)$;
(3) if $\tilde{b}_k \ne 0$ and $\tilde{a}_k = 0$, then $D(C_k) = D(B)$;
(4) if $\tilde{a}_k = \tilde{b}_k = 0$, then $D(C_k) = X$.

We define the resolvent set of $A,\ B$ with respect to $a,\ b$ by

$$\rho_{a,b}(A,B) := \Big\{k \in \mathbb{Z} : \ C_k \text{ is bijective from } D(C_k)$$
$$\text{to } X \text{ and } C_k^{-1}, \ \ \tilde{b}_k B C_k^{-1} \in \mathcal{L}(X)\Big\}.$$

It is clear from the definition that when $k \in \rho_{a,b}(A,B)$, then $\tilde{a}_k A C_k^{-1} \in \mathcal{L}(X)$.

The notion of 1-regular and 2-regular scalar sequences were introduced in [7]. Let $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ be a scalar sequence such that there exists $N \in \mathbb{N}$ such that for $|k| \geq N$, we have $a_k \neq 0$. We say that $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ is 1-regular if

$$\sup_{|k|\geq N} \Big\| \frac{k(\Delta a)_k}{a_k} \Big\|_{k\in\mathbb{Z}} < \infty.$$

It is said to be 2-regular if it is 1-regular and

$$\sup_{|k|\geq N} \Big\| \frac{k^2(\Delta^2 a)_k}{a_k} \Big\|_{k\in\mathbb{Z}} < \infty.$$

It is clear from the definition that when $(a_k)_{k\in\mathbb{Z}}$ is 1-regular, then $\lim_{|k|\to\infty} \frac{a_{k+1}}{a_k} = 1$.

In order to give a characterization of $B^s_{p,q}$-maximal regularity for (3.1), we need the following key preparation.

Theorem 3.2. *Let* $1 \leq p, \ q \leq \infty, \ s > 0$, *let* $a, \ b \in L^1(\mathbb{R}_+)$ *be such that the corresponding sequences* $(\tilde{a}_k)_{k\in\mathbb{Z}}$ *and* $(\tilde{b}_k)_{k\in\mathbb{Z}}$ *be* 2*-regular, and let* $A, \ B$ *be closed operators in a complex Banach space* X. *Assume that* $\rho_{\,a,b}(A,B) = \mathbb{Z}$. *Then* $\big((I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\big)_{k\in\mathbb{Z}}$, $\big(\tilde{b}_k B(I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\big)_{k\in\mathbb{Z}}$ *and* $\big(\tilde{a}_k A(I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\big)_{k\in\mathbb{Z}}$ *are* $B^s_{p,q}$*-multipliers.*

Proof. Since $(\tilde{a}_k)_{k\in\mathbb{Z}}$ and $(\tilde{b}_k)_{k\in\mathbb{Z}}$ are 2-regular by assumption, we have

$$\lim_{|k|\to\infty} \tilde{a}_{k+1}/\tilde{a}_k = \lim_{|k|\to\infty} \tilde{b}_{k+1}/\tilde{b}_k = 1. \tag{3.3}$$

Assume that $\rho_{a,b}(A,B) = \mathbb{Z}$. We let $M_k := (I - \tilde{a}_k A - \tilde{b}_k B)^{-1}$ for $k \in \mathbb{Z}$. Then $(M_k)_{k\in\mathbb{Z}}$, $(\tilde{b}_k B M_k)_{k\in\mathbb{Z}}$ and $(\tilde{a}_k A M_k)_{k\in\mathbb{Z}}$ are bounded in $\mathcal{L}(X)$. Firstly, we show that $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. For this we are going to show that $(M_k)_{k\in\mathbb{Z}}$ satisfies the conditions (2.1) and (2.2). A simple computation gives

$$\begin{aligned} (\Delta M)_k &= M_{k+1}((\Delta\tilde{a})_k A + (\Delta\tilde{b})_k B)M_k \\ k(\Delta M)_k &= M_{k+1}\frac{k(\Delta\tilde{a})_k}{\tilde{a}_k}\tilde{a}_k A M_k + M_{k+1}\frac{k(\Delta\tilde{b})_k}{\tilde{b}_k}\tilde{b}_k B M_k, \end{aligned} \tag{3.4}$$

for large $|k|$. This shows that $\sup_{k\in\mathbb{Z}} \|k(\Delta M)_k\| < \infty$ by assumption. On the other hand by (3.4) we have

$$(3.5)\qquad \begin{aligned}(\Delta^2 M)_k \;=\; & (\Delta M)_{k+1}\big((\Delta\tilde{a})_{k+1}A + (\Delta\tilde{b})_{k+1}B\big)M_{k+1}\\ & +M_{k+1}\big((\Delta^2\tilde{a})_k A + (\Delta^2\tilde{b})_k B\big)M_{k+1}\\ & +M_{k+1}\big((\Delta\tilde{a})_k A + (\Delta\tilde{b})_k B\big)(\Delta M)_k\end{aligned}$$

and thus for large $|k|$

$$\begin{aligned}k^2(\Delta^2 M)_k \;=\; & \big[k(\Delta M)_{k+1}\big]\Big(\frac{k(\Delta\tilde{a})_{k+1}}{\tilde{a}_{k+1}}\tilde{a}_{k+1}AM_{k+1} + \frac{k(\Delta\tilde{b})_{k+1}}{\tilde{b}_{k+1}}\tilde{b}_{k+1}BM_{k+1}\Big)\\ & +M_{k+1}\Big(\frac{k^2(\Delta^2\tilde{a})_k}{\tilde{a}_k}\tilde{a}_kAM_{k+1} + \frac{k^2(\Delta^2\tilde{b})_k}{\tilde{b}_k}\tilde{b}_kBM_{k+1}\Big)\\ & +M_{k+1}\Big(\frac{k(\Delta\tilde{a})_k}{\tilde{a}_k}\tilde{a}_kAM_{k+1} + \frac{k(\Delta\tilde{b})_k}{\tilde{b}_k}\tilde{b}_kBM_{k+1}\Big)\\ & \cdot\Big(\frac{k(\Delta\tilde{a})_k}{\tilde{a}_k}\tilde{a}_kAM_k + \frac{k(\Delta\tilde{b})_k}{\tilde{b}_k}\tilde{b}_kBM_k\Big).\end{aligned}$$

This implies that $\sup_{k\in\mathbb{Z}} \|k^2(\Delta^2 M)_k\| < \infty$ by assumption and (3.3). We have shown that $(M_k)_{k\in\mathbb{Z}}$ satisfies the conditions (2.1) and (2.2). Consequently $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier by Theorem 2.2.

Let $N_k = \tilde{b}_kBM_k$ for $k\in\mathbb{Z}$. Then for large $|k|$

$$(3.6)\qquad \begin{aligned}(\Delta N)_k \;=\; & (\Delta\tilde{b})_kBM_{k+1} + \tilde{b}_kB(\Delta M)_k\\ k(\Delta N)_k \;=\; & \frac{k(\Delta\tilde{b})_k}{\tilde{b}_k}\tilde{b}_kBM_{k+1} + \tilde{b}_kBM_{k+1}\Big(\frac{k(\Delta\tilde{a})_k}{\tilde{a}_k}\tilde{a}_kAM_k\\ & +\frac{k(\Delta\tilde{b})_k}{\tilde{b}_k}\tilde{b}_kBM_k\Big).\end{aligned}$$

Thus $\sup_{k\in\mathbb{Z}} \|k(\Delta N)_k\| < \infty$ by assumption and (3.3). By (3.6)

$$\begin{aligned}k^2(\Delta^2 N)_k \;&= k^2(\Delta^2\tilde{b})_kBM_{k+2} + 2k^2(\Delta\tilde{b})_kB(\Delta M)_{k+1} + \tilde{b}_kB(\Delta^2 M)_k\\ :&= Q_k^{(1)} + Q_k^{(2)} + Q_k^{(3)}.\end{aligned}$$

It is clear from the assumptions and (3.3) that $(Q_k^{(1)})_{k\in\mathbb{Z}}$ is bounded. On the other hand by (3.4) for large $|k|$

$$Q_k^{(2)} = 2\Big[\frac{k(\Delta\tilde{b})_k}{\tilde{b}_k}\tilde{b}_kBM_{k+1}\Big]\Big(\frac{k(\Delta\tilde{a})_k}{\tilde{a}_k}\tilde{a}_kAM_k + \frac{k(\Delta\tilde{b})_k}{\tilde{b}_k}\tilde{b}_kBM_k\Big)$$

and by (3.5)

$$Q_k^{(3)} = \tilde{b}_k BM_{k+1}\Big(\frac{k(\Delta\tilde{a})_k}{\tilde{a}_k}\tilde{a}_k AM_k + \frac{k(\Delta\tilde{b})_k}{\tilde{b}_k}\tilde{b}_k BM_k\Big)$$

$$\cdot\Big(\frac{k(\Delta\tilde{a})_{k+1}}{\tilde{a}_{k+1}}\tilde{a}_{k+1} AM_{k+1} + \frac{k(\Delta\tilde{b})_{k+1}}{\tilde{b}_{k+1}}\tilde{b}_{k+1} BM_{k+1}\Big)$$

$$+ \tilde{b}_k BM_{k+1}\Big(\frac{k^2(\Delta^2\tilde{a})_k}{\tilde{a}_k}\tilde{a}_k AM_{k+1} + \frac{k^2(\Delta^2\tilde{b})_k}{\tilde{b}_k}\tilde{b}_k BM_{k+1}\Big)$$

$$+ \tilde{b}_k BM_{k+1}\Big(\frac{k(\Delta\tilde{a})_k}{\tilde{a}_k}\tilde{a}_k AM_{k+1} + \frac{k^2(\Delta\tilde{b})_k}{\tilde{b}_k}\tilde{b}_k BM_{k+1}\Big)$$

$$\cdot\Big(\frac{k(\Delta\tilde{a})_k}{\tilde{a}_k}\tilde{a}_k AM_k + \frac{k^2(\Delta\tilde{b})_k}{\tilde{b}_k}\tilde{b}_k BM_k\Big).$$

Therefore $\sup_{k\in\mathbb{Z}} \|k^2(\Delta^2 N)_k\| < \infty$ by assumption and (3.3). Hence $(N_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier by Theorem 2.2 as we have shown that $(N_k)_{k\in\mathbb{Z}}$ satisfies (2.1) and (2.2). Similar argument shows that $(\tilde{a}_k AM_k)_{k\in\mathbb{Z}}$ is also a $B^s_{p,q}$-multiplier. This completes the proof. ■

Remark 3.3. It is clear from Theorem 2.2 and the proof of Theorem 3.2 that when the underlying Banach space X is B-convex, then we may replace the assumption that $(\tilde{a}_k)_{k\in\mathbb{Z}}$ and $(\tilde{b}_k)_{k\in\mathbb{Z}}$ are 2-regular sequences in Theorem 3.2 by the weaker assumption that $(\tilde{a}_k)_{k\in\mathbb{Z}}$ and $(\tilde{b}_k)_{k\in\mathbb{Z}}$ are 1-regular.

The following is the main result of this paper.

Theorem 3.4. *Let $a,\ b \in L^1(\mathbb{R}_+)$ be such that $(\tilde{a}_k)_{k\in\mathbb{Z}}$ and $(\tilde{b}_k)_{k\in\mathbb{Z}}$ are 2-regular, $1 \le p,\ q \le \infty,\ s > 0$, and let $A,\ B$ be closed operators in a complex Banach space X. Then the following assertions are equivalent:*

(*i*) *(3.1) has $B^s_{p,q}$-maximal regularity.*

(*ii*) $\rho_{a,b}(A, B) = \mathbb{Z}$.

Proof. (ii)⇒(i): Assume that $\rho_{a,b}(A, B) = \mathbb{Z}$. For $k \in \mathbb{Z}$ we let $M_k := (I - \tilde{a}_k A - \tilde{b}_k B)^{-1}$. Then $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier by Theorem 3.2. Therefore, for $f \in B^s_{p,q}(\mathbb{T}, X)$, there exists $u \in B^s_{p,q}(\mathbb{T}, X)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \tag{3.7}$$

when $k \in \mathbb{Z}$.

The sequence $(\tilde{b}_k)_{k\in\mathbb{Z}}$ is bounded sequence by Riemann-Lebesgue Lemma as $b \in L^1(\mathbb{R}_+)$. This fact together with the assumption that $(\tilde{b}_k)_{k\in\mathbb{Z}}$ is 2-regular implies that $(\tilde{b}_k I)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier by Theorem 2.2. We conclude that $(\tilde{b}_k M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier as the product of two $B^s_{p,q}$-multipliers is still a $B^s_{p,q}$-multiplier. Hence there exists $v \in B^s_{p,q}(\mathbb{T}, X)$ such that

$$\hat{v}(k) = \tilde{b}_k M_k \hat{f}(k), \quad (k \in \mathbb{Z}).$$

This implies by (3.7) that $\hat{v}(k) = \tilde{b}_k \hat{u}(k)$ when $k \in \mathbb{Z}$. We conclude that $v = b * u$ by (2.4) and thus $b * u \in B^s_{p,q}(\mathbb{T}, X)$

$(\tilde{b}_k B M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier by assumption and Theorem 3.2. There exists $h \in B^s_{p,q}(\mathbb{T}, X)$, such that

$$\hat{h}(k) = \tilde{b}_k B M_k \hat{f}(k), \quad (k \in \mathbb{Z}).$$

One deduces that $\hat{h}(k) = \tilde{b}_k B \hat{u}(k)$ when $k \in \mathbb{Z}$ by (3.7). Thus $(b * u)(t) \in D(B)$ and $h(t) = B(b * u)(t)$ for a.e. $t \in \mathbb{T}$ by [2, Lemma 3.1] and (2.4). We have shown that $b * u \in B^s_{p,q}(\mathbb{T}, X)$ and $B(b * u) \in B^s_{p,q}(\mathbb{T}, X)$. Consequently, $b * u \in B^s_{p,q}(\mathbb{T}, D(B))$. A similar argument shows that $a * u \in B^s_{p,q}(\mathbb{T}, D(A))$.

Now from (3.7) we have $(I - \tilde{a}_k A - \tilde{b}_k B)\hat{u}(k) = \hat{f}(k)$ or equivalently $\hat{u}(k) = \tilde{a}_k A \hat{u}(k) + \tilde{b}_k B \hat{u}(k) + \hat{f}(k)$ for $k \in \mathbb{Z}$. We deduce that

$$u(t) = A(a * u)(t) + B(b * u)(t) + f(t)$$

for a.e. $t \in \mathbb{T}$ by the Uniqueness Theorem in [2, page 314]. This shows that a mild $B^s_{p,q}$-solution of (3.1) exists.

It remains to show that the mild $B^s_{p,q}$-solution of (3.1) is unique. For this we assume that $u \in B^s_{p,q}(\mathbb{T}, X)$ is such that $a * u \in B^s_{p,q}(\mathbb{T}, D(A))$, $b * u \in B^s_{p,q}(\mathbb{T}, D(B))$ and $u(t) = A(a * u)(t) + B(b * u)(t)$ for a.e. $t \in \mathbb{T}$. Taking Fourier transform on both sides, we obtain that $(I - \tilde{a}_k A - \tilde{b}_k B)\hat{u}(k) = 0$ for $k \in \mathbb{Z}$. We conclude that $\hat{u}(k) = 0$ as $\rho_{a,b}(A, B) = \mathbb{Z}$ by assumption. Thus $u = 0$. This implies that for each $f \in B^s_{p,q}(\mathbb{T}, X)$, the mild $B^s_{p,q}$-solution of (3.1) is unique. We have shown that (3.1) has $B^s_{p,q}$-maximal regularity.

(i)⇒(ii): We assume that (3.1) has $B^s_{p,q}$-maximal regularity and let $k \in \mathbb{Z}$ be fixed. We are going to show that $k \in \rho_{a,b}(A, B)$.

Assume that $\tilde{a}_k \neq 0$ and $\tilde{b}_k \neq 0$. Let $y \in X$ and let $f \in B^s_{p,q}(\mathbb{T}, X)$ given by $f = e_k \otimes y$. By assumption, there exists $u \in B^s_{p,q}(\mathbb{T}, X)$, such that $a * u \in B^s_{p,q}(\mathbb{T}, D(A))$, $b * u \in B^s_{p,q}(\mathbb{T}, D(B))$ and

$$u(t) = A(a * u)(t) + B(b * u)(t) + f(t) \tag{3.8}$$

for a.e. $t \in \mathbb{T}$. Taking Fourier transform on both sides of (3.8), one obtains that $\hat{u}(k) \in D(A) \cap D(B)$ and by [2, Lemma 3.1]

$$\hat{u}(k) - \tilde{a}_k A \hat{u}(k) - \tilde{b}_k B \hat{u}(k) = y \tag{3.9}$$

and

$$\hat{u}(n) - \tilde{a}_n A\hat{u}(n) - \tilde{b}_n B\hat{u}(n) = 0 \tag{3.10}$$

when $n \neq k$. This implies that $I - \tilde{a}_k A - \tilde{b}_k B$ is surjective from $D(A) \cap D(B)$ to X.

In order to show that $I - \tilde{a}_k A - \tilde{b}_k B$ is also injective, we assume that $x \in D(A) \cap D(B)$ is such that $(I - \tilde{a}_k A - \tilde{b}_k B)x = 0$. Then it is easy to verify that $u = e_k \otimes x$ is the unique mild $B^s_{p,q}$-solution of (3.1) when taking $f = 0$. Thus $x = 0$ by uniqueness. We have shown that $I - \tilde{a}_k A - \tilde{b}_k B$ is injective. Hence $I - \tilde{a}_k A - \tilde{b}_k B$ is bijective from $D(A) \cap D(B)$ to X.

It remains to show that $(I - \tilde{a}_k A - \tilde{b}_k B)^{-1},\ \tilde{b}_k B(I - \tilde{a}_k A - \tilde{b}_k B)^{-1} \in \mathcal{L}(X)$. Let $y \in X,\ \ f = e_k \otimes y \in B^s_{p,q}(\mathbb{T}, X)$ and let u be the unique mild $B^s_{p,q}$-solution of (3.1). Then

$$\hat{u}(n) = \begin{cases} (I - \tilde{a}_k A - \tilde{b}_k B)^{-1} y, & \text{if } n = k \\ 0, & \text{if } n \neq k \end{cases}$$

by (3.9) and (3.10). This gives $u = e_k \otimes (I - \tilde{a}_k A - \tilde{b}_k B)^{-1} y$. By (3.2), there exists a constant $C \geq 0$ independent from f and u such that

$$\|u\|_{B^s_{p,q}} + \|A(a * u)\|_{B^s_{p,q}} + \|B(b * u)\|_{B^s_{p,q}} \leq C\|f\|_{B^s_{p,q}}.$$

Consequently

$$\begin{aligned} &\|(I - \tilde{a}_k A - \tilde{b}_k B)^{-1} y\| + \|\tilde{a}_k A(I - \tilde{a}_k A - \tilde{b}_k B)^{-1} y\| \\ &\qquad + \|\tilde{b}_k B(I - \tilde{a}_k A - \tilde{b}_k B)^{-1} y\| \leq C\|y\|. \end{aligned}$$

This implies that $k \in \rho_{a,b}(A, B)$.

The same argument shows that in case when $\tilde{a}_k = 0$ or $\tilde{b}_k = 0$, we still have $k \in \rho_{a,b}(A, B)$. The proof is completed. ■

Periodic Hölder continuous function space is a particular case of periodic Besov space $B^s_{p,q}(\mathbb{T}, X)$. From [3, Theorem 3.1], we have $B^\alpha_{\infty,\infty}(\mathbb{T}, X) = C^\alpha_{per}(\mathbb{T}, X)$ whenever $0 < \alpha < 1$, where $C^\alpha_{per}(\mathbb{T}, X)$ is the space of all X-valued functions f defined on $\mathbb{T}$ satisfying $f(0) = f(2\pi)$ and $\sup_{x \neq y} \frac{\|f(x) - f(y)\|}{|x-y|^\alpha} < \infty$. Moreover the norm $\|f\|_{C^\alpha_{per}} := \max_{t \in \mathbb{T}} \|f(t)\| + \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{|x-y|^\alpha}$ on $C^\alpha_{per}(\mathbb{T}, X)$ is an equivalent norm of $B^\alpha_{\infty,\infty}(\mathbb{T}, X)$. If $0 < \alpha < 1$, we say that the problem (3.1) has C^α_{per}-maximal regularity if for every $f \in C^\alpha_{per}(\mathbb{T}, X)$, there exists a unique $u \in C^\alpha_{per}(\mathbb{T}, X)$ such that $a * u \in C^\alpha(\mathbb{T}, D(A)),\ \ b * u \in C^\alpha(\mathbb{T}, D(B))$ and equation (3.1) holds true for all $t \in \mathbb{T}$. Theorem 3.4 and Theorem 2.2 have the following immediate corollary.

Corollary 3.5. *Let $a,\ b \in L^1(\mathbb{R}_+)$, $1 \le p,\ q \le \infty,\ s > 0$, and let $A,\ B$ be closed operators in a complex Banach space X. Then*

(i) *if $(\tilde{a}_k)_{k\in\mathbb{Z}}$ and $(\tilde{b}_k)_{k\in\mathbb{Z}}$ are 2-regular, the (3.1) has C^{α}_{per}-maximal regularity if and only if $\rho_{a,b}(A,B) = \mathbb{Z}$.*

(ii) *when X is B-convex, $(\tilde{a}_k)_{k\in\mathbb{Z}}$ and $(\tilde{b}_k)_{k\in\mathbb{Z}}$ are 1-regular, then (3.1) has C^{α}_{per}-maximal regularity if and only if $\rho_{\,a,b}(A,B) = \mathbb{Z}$.*

Remarks 3.6.

(i) We notice that the assertion (ii) in Theorem 3.4 is independent from the choice of $1 \le p,\ q \le \infty$ and $s > 0$. Therefore, under the assumptions of Theorem 3.4, (3.1) has $B^s_{p,q}$-maximal regularity for some $1 \le p,\ q \le \infty$ and $s > 0$ if and only if (3.1) has $B^s_{p,q}$-maximal regularity for all $1 \le p,\ q \le \infty$ and $s > 0$.

(ii) When the underlying Banach space X is B-convex, we may replace the assumption that $(\tilde{a}_k)_{k\in\mathbb{Z}}$ and $(\tilde{b}_k)_{k\in\mathbb{Z}}$ are 2-regular sequences in Theorem 3.4, by the weaker assumption that $(\tilde{a}_k)_{k\in\mathbb{Z}}$ and $(\tilde{b}_k)_{k\in\mathbb{Z}}$ are 1-regular sequences. This follows from Remark 3.3 and the proof of Theorem 3.4.

(iii) L^p-maximal regularity of (3.1) has been studied by Lizama and Poblete [8], they gave a characterization of L^p-maximal regularity for (3.1) under some suitable conditions on the kernels $a,\ b$ and the operators $A,\ B$ [8, Theorem 3.5]. Using the same argument used in the proof of Theorem 3.4, it is easy to verify that the assumption in [8, Theorem 3.5] that $(\tilde{a}_k A, \tilde{b}_k B)$ is coercive pair is not needed.

(iv) We may also consider the maximal regularity for (3.1) in periodic Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$. Using operator-valued Fourier multiplier results established in [4], similar argument used in the proofs of Theorem 3.2 and Theorem 3.4 gives a characterization of $F^s_{p,q}$-maximal regularity for (3.1), but in this case the appropriate assumptions on $a,\ b$ will be that the corresponding sequences $(\tilde{a}_k)_{k\in\mathbb{Z}}$ and $(\tilde{b}_k)_{k\in\mathbb{Z}}$ are 3-regular sequences.

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