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ON THE COMPLETE CONVERGENCE FOR NEGATIVELY ASSOCIATED RANDOM FIELDS

Mi-Hwa Ko

Abstract. The aim of this note is to establish almost sure Marcinkiewicz-Zygmund type result for a field of negatively associated random variables indexed by $\mathbb{Z}_+^d$ $(d \geq 2)$, the positive d-dimensional lattice points.

1. Introduction and Main Result

Let $\mathbb{Z}_+^d$ $(d \geq 2)$ denote the positive integer d-dimensional lattice with coordinatewise partial ordering $\leq$. The notation $\mathbf{m} \leq \mathbf{n}$, where $\mathbf{m} = (m_1, m_2, \cdots, m_d)$ and $\mathbf{n} = (n_1, n_2, \cdots, n_d)$, thus means that $m_k \leq n_k$ for $k = 1, 2, \cdots, d$. We also use $|\mathbf{n}|$ for $\prod_{k=1}^d n_k$, $\mathbf{n} \to \infty$ is to be interpreted as $n_k \to \infty$ for $k = 1, 2, \cdots, d$ and $\mathbf{1} = (1, 1, \cdots, 1)$. For $d = 1$ we use the notation $\mathbb{Z}_+$ instead of $\mathbb{Z}_+^1$. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a field of random variables on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Set $S_{\mathbf{n}} = \sum_{\mathbf{j} \leq \mathbf{n}} X_{\mathbf{j}}$ for $\mathbf{j} \in \mathbb{Z}_+^d$. Then $S_{\mathbf{n}}$ is simply a sum of $|\mathbf{n}|$ random variables. The field $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ is called negatively associated, if for every pair of disjoint subsets A, B of $\mathbb{Z}_+^d$ and any pair of coordinatewise increasing functions $f(X_{\mathbf{i}}; \mathbf{i} \in A), g(X_{\mathbf{j}}; \mathbf{j} \in B)$ with $Ef^2(X_{\mathbf{i}}; \mathbf{i} \in A) < \infty$ and $Eg^2(X_{\mathbf{j}}; \mathbf{j} \in B) < \infty$, it holds that $Cov(f(X_{\mathbf{i}}; \mathbf{i} \in A), g(X_{\mathbf{j}}; \mathbf{j} \in B)) \leq 0$. The concept of NA was introduced by Alam and Saxena (1981) and Joag-Dev and Proschan (1983). As pointed out and proved by Joag-Dev and Proschan (1983), a number of well-known multivariated distributions possess the NA property, such as multinomial distribution, multivariate hypergeometric distribution, Dirichlet distribution, negatively correlated normal distribution, permutation distribution, and joint distribution of ranks. Because of their wide applications in multivariate statistical analysis and reliability

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theory the concept of negatively associated random variables has received extensive attention recently. We refer to Joag-Dev and Proschan (1983) for fundamental properties. In the case of $d = 1$, we refer to Newman (1984) for the central limit theorem, Matula (1992) for the three series theorem, Shao (2000) for the Rosenthal-type inequality and complete convergence, Liang (2004) for complete convergence of weighted sums, Li and Zhang (2004) for complete moment convergence, Fu and Zhang (2007) for precise rate in the law of logarithm. In the case of $d \geq 2$, Roussas (1994) studied the central limit theorems for weak stationary NA random fields and Zhang (2000) obtained the weak convergence under condition that the $2+\delta$th moment is finite. Zhang and Wen (2001) investigated the weak convergence for a centered stationary NA random field under finite second moments. Complete convergence gives a convergence rate with respect to the strong law of large numbers. One can refer to Hsu and Robbin (1947) and Baum and Katz (1965) for details. Applying the maximal inequality for NA random variables, one can get the following result easily.

Theorem A. (Shao [2000]). *Let* $\frac{1}{2} < \alpha \leq 1,\ p\alpha \geq 1$ *and let* $\{X_n; n \geq 1\}$ *be a negatively associated sequence of identically distributed random variables with* $EX_1 = 0$. *Then the following statements are equivalent:*

(i) $E|X_1|^p < \infty$,

(ii) $\sum_{n=1}^{\infty} n^{p\alpha-2} P(\max_{1\leq j\leq n} |S_j| > \epsilon n^{\alpha})$ for all $\epsilon > 0.$

Inspired by Peligrad and Gut (1999) we wish to consider theorem A in the random fields as follows.

Theorem 1.1. *Let* $p\alpha > 1$ *and* $\alpha > \frac{1}{2}$ *and let* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^d\}$ *be a field of identically distributed NA random variables with* $EX_{\mathbf{1}} = 0$. *Then the following statements are equivalent:*

(i) $E|X_{\mathbf{1}}|^p(\log^+ |X_{\mathbf{1}}|)^{d-1} < \infty$,

(ii) $\sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2} P(\max_{\mathbf{1}\leq\mathbf{k}\leq\mathbf{n}} |S_{\mathbf{k}}| > \epsilon|\mathbf{n}|^{\alpha}) < \infty$ for all $\epsilon > 0$,

where $\log^+ x = \max\{1, \log x\}$.

Theorem 1.2. *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ *be a field of identically distributed NA random variables with* $EX_{\mathbf{1}} = 0$. *Then* $E|X_{\mathbf{1}}|(\log^+ |X_{\mathbf{1}}|)^{d-1} < \infty$ *implies that* $\frac{S_{\mathbf{n}}}{|\mathbf{n}|} \to 0\ a.s.$

2. PROOFS

Lemma 2.1. (Zhang and Wen [2001]). *Let* $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ *be a field of centered NA random variables. Then for all* $q \geq 2$, *there exists a positive constant* $C = C_q$ *such that*

$$E|S_{\mathbf{n}}|^q \leq C\{\sum_{\mathbf{1}\leq\mathbf{k}\leq\mathbf{n}} EX_{\mathbf{k}}^2 + \sum_{\mathbf{1}\leq\mathbf{k}\leq\mathbf{n}} E|X_{\mathbf{k}}|^q\}, \tag{2.1.a}$$

$$\begin{aligned} E\max_{\mathbf{1}\leq\mathbf{k}\leq\mathbf{n}}|S_{\mathbf{k}}|^q &\leq C\{(E\max_{\mathbf{1}\leq\mathbf{k}\leq\mathbf{n}}|S_{\mathbf{k}}|)^q \\ &+ \left(\sum_{\mathbf{1}\leq\mathbf{k}\leq\mathbf{n}} EX_{\mathbf{k}}^2\right)^{q/2} + \sum_{\mathbf{1}\leq\mathbf{k}\leq\mathbf{n}} E|X_{\mathbf{k}}|^q\}, \end{aligned} \tag{2.1.b}$$

$\forall\ \mathbf{n} \in \mathbb{Z}_+^d$.

The following quantities and their asymptotic behavior turn out to be of importance(see e.g. Gut (1978) for details and further references).

$$M(j) = Card\{\mathbf{k} : |\mathbf{k}| \leq j\} \ll j(\log j)^{d-1}\ as\ j \to \infty \tag{2.2}$$

and

$$d(j) = Card\{\mathbf{k} : |\mathbf{k}| = j\} = o(j^\delta) \text{ for any } \delta > 0\ as\ j \to \infty. \tag{2.3}$$

Lemma 2.2. (Gut [1978]). *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a field of identically distributed random variables with $EX_{\mathbf{1}} = 0$. Then*

$$\sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-1} P(|X_{\mathbf{1}}| > |\mathbf{n}|^\alpha) < \infty \text{ if and only if } E|X_{\mathbf{1}}|^p(\log^+|X_{\mathbf{1}}|)^{d-1} < \infty.$$

Proof of Theorem 1.1. In order to prove that $(i) \to (ii)$ we truncate at the level $|\mathbf{n}|^\alpha$, and set

$$\begin{aligned} Y_{\mathbf{i}} &= -|\mathbf{n}|^\alpha I(X_{\mathbf{i}} < -|\mathbf{n}|^\alpha) + X_{\mathbf{i}} I(|X_i| \leq |\mathbf{n}|^\alpha) \\ &\quad +|\mathbf{n}|^\alpha I(X_{\mathbf{i}} > |\mathbf{n}|^\alpha) \text{ for } \mathbf{1} \leq \mathbf{i} \leq \mathbf{n} \end{aligned}$$

and $X_{\mathbf{i}}' = Y_{\mathbf{i}} - EY_{\mathbf{i}},\ S_{\mathbf{n}}' = \sum_{\mathbf{1}\leq\mathbf{i}\leq\mathbf{n}} X_{\mathbf{i}}'$ and $S_{\mathbf{n}} = \sum_{\mathbf{1}\leq\mathbf{i}\leq\mathbf{n}} X_{\mathbf{i}}$.

In view of the fact that $EX_{\mathbf{1}} = 0$ we obtain

$$EX_{\mathbf{1}} I(|X_{\mathbf{1}}| \leq |\mathbf{n}|^\alpha) = -EX_{\mathbf{1}} I(|X_{\mathbf{1}}| > |\mathbf{n}|^\alpha). \tag{2.4}$$

First, we obtain

$$
\begin{aligned}
&\sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2} P(\max_{\mathbf{1}\le \mathbf{j}\le \mathbf{n}} |S_{\mathbf{j}}| > \epsilon |\mathbf{n}|^{\alpha}) \\
\le &\sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2} P(\max_{\mathbf{1}\le \mathbf{j}\le \mathbf{n}} |X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}) + \sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2} P(\max_{\mathbf{1}\le \mathbf{j}\le \mathbf{n}} |\sum_{\mathbf{1}\le \mathbf{i}\le \mathbf{j}} Y_{\mathbf{i}}| > \epsilon |\mathbf{n}|^{\alpha}) \\
\le &\sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2} P(\max_{\mathbf{1}\le \mathbf{j}\le \mathbf{n}} |X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}) \\
&+ \sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2} P(\max_{\mathbf{1}\le \mathbf{j}\le \mathbf{n}} |\sum_{\mathbf{1}\le \mathbf{i}\le \mathbf{j}} (Y_{\mathbf{i}} - EY_{\mathbf{i}})| > \epsilon |\mathbf{n}|^{\alpha} - \max_{\mathbf{1}\le \mathbf{j}\le \mathbf{n}} |\sum_{\mathbf{1}\le \mathbf{i}\le \mathbf{j}} EY_{\mathbf{i}}|) \\
\le &\sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-1} P(|X_{\mathbf{1}}| > |\mathbf{n}|^{\alpha}) \\
&+ \sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2} P(\max_{\mathbf{1}\le \mathbf{j}\le \mathbf{n}} |\sum_{\mathbf{1}\le \mathbf{i}\le \mathbf{j}} (Y_{\mathbf{i}} - EY_{\mathbf{i}})| > \epsilon |\mathbf{n}|^{\alpha} - \max_{\mathbf{1}\le \mathbf{j}\le \mathbf{n}} |\sum_{\mathbf{1}\le \mathbf{i}\le \mathbf{j}} EY_{\mathbf{i}}|) \\
= &I_1 + I_2.
\end{aligned}
$$

By Lemma 2.2 $I_1 < \infty$. Hence it remains to show that $I_2 < \infty$.

Since $p\alpha > 1$ clearly, we also obtain

$$
|\mathbf{n}| E|X_{\mathbf{1}}| I(|X_{\mathbf{1}}| > |\mathbf{n}|^{\alpha}) = o(|\mathbf{n}|^{\alpha}) \ as \ \mathbf{n} \to \infty. \tag{2.5}
$$

It follows from (2.4) and (2.5) that

$$
\begin{aligned}
&\max_{\mathbf{1}\le \mathbf{j}\le \mathbf{n}} |\sum_{\mathbf{1}\le \mathbf{i}\le \mathbf{j}} EY_{\mathbf{i}}| \\
&= \max_{\mathbf{1}\le \mathbf{j}\le \mathbf{n}} |\sum_{\mathbf{1}\le \mathbf{i}\le \mathbf{j}} E\{X_{\mathbf{i}} I(|X_{\mathbf{i}}| \le |\mathbf{n}|^{\alpha}) - |\mathbf{n}|^{\alpha} I(X_{\mathbf{i}} < -|\mathbf{n}|^{\alpha}) \\
&\qquad + |\mathbf{n}|^{\alpha} I(X_{\mathbf{i}} > |\mathbf{n}|^{\alpha})\}| \\
&\le \sum_{\mathbf{1}\le \mathbf{i}\le \mathbf{n}} E|X_{\mathbf{i}}| I(|X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha}) + \sum_{\mathbf{1}\le \mathbf{i}\le \mathbf{n}} |\mathbf{n}|^{\alpha} EI(|X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha}) \\
&\le 2 \sum_{\mathbf{1}\le \mathbf{i}\le \mathbf{n}} E|X_{\mathbf{i}}| I(|X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha}) \\
&= 2|\mathbf{n}| E|X_{\mathbf{1}}| I(|X_{\mathbf{1}}| > |\mathbf{n}|^{\alpha}) = o(|\mathbf{n}|^{\alpha}) \ as \ \mathbf{n} \to \infty.
\end{aligned} \tag{2.6}
$$

Hence, it follows from (2.6) that for $|\mathbf{n}|$ large enough

$$
I_2 \le \sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2} P(\max_{\mathbf{1}\le \mathbf{j}\le \mathbf{n}} |\sum_{\mathbf{1}\le \mathbf{i}\le \mathbf{j}} (Y_{\mathbf{i}} - EY_{\mathbf{i}}) > \frac{\epsilon}{2} |\mathbf{n}|^{\alpha}).
$$

By Lemma 2.1 and Chebyshev's inequality for suitable large k which will be determined later, we first observe that

$$\begin{aligned}
I_2 &\le \sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2} P(\max_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} |S'_{\mathbf{j}}| > \frac{\epsilon}{2}|\mathbf{n}|^{\alpha}) \\
&\le \sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2-\alpha k} E(\max_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} |S'_{\mathbf{j}}|)^k \\
&\le \sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2-\alpha k}(|\mathbf{n}|^k(E|X'_{\mathbf{1}}|)^k + |\mathbf{n}|^{\frac{k}{2}}(E|X'_{\mathbf{1}}|^2)^{\frac{k}{2}} + |\mathbf{n}|(E|X'_{\mathbf{1}}|^k)) = I_3 + I_4 + I_5.
\end{aligned}$$

For I_3, we have

$$I_3 \le \sum_{\mathbf{n}}^{\infty} |\mathbf{n}|^{p\alpha-2-\alpha k+k} \le \sum_{m=1}^{\infty}\sum_{|\mathbf{n}|=m} |\mathbf{n}|^{p\alpha-2-\alpha k+k} = \sum_{m=1}^{\infty} d(m) m^{p\alpha-2-\alpha k+k} \tag{2.7}$$

which, in view of (2.3), is convergent if k is selected such that $k > (p\alpha-1)/(\alpha-1)$.

Letting $b_j = P(j \le |X_{\mathbf{1}}| < j+1)$ we use partial summation and (2.2) to obtain

$$\begin{aligned}
I_5 &\le \sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-1-\alpha k} \sum_{j\le|\mathbf{n}|^{\alpha}} j^k b_j = \sum_{m=1}^{\infty}\sum_{|\mathbf{n}|=m} |\mathbf{n}|^{p\alpha-1-\alpha k} \sum_{j\le m^{\alpha}} j^k b_j \\
&= \sum_{m=1}^{\infty} d(m) m^{p\alpha-1-\alpha k} \sum_{j\le m^{\alpha}} j^k b_j = \sum_{j=1}^{\infty}\{ \sum_{m\ge j^{\frac{1}{\alpha}}}^{\infty} d(m) m^{p\alpha-1-\alpha k} j^k b_j\} \\
&\le \sum_{j=1}^{\infty} M(j^{\frac{1}{\alpha}}) j^{(p\alpha-1-\alpha k)/\alpha} j^k b_j \qquad (2.8) \\
&\le \sum_{j=1}^{\infty} j^{\frac{1}{\alpha}}(\log^+(j^{\frac{1}{\alpha}}))^{d-1} j^{(p-k-\frac{1}{\alpha})} j^k b_j \\
&= \sum_{j=1}^{\infty} j(\log^+ j)^{d-1} b_j < \infty,
\end{aligned}$$

since $E|X_{\mathbf{1}}|^p(\log^+|X_{\mathbf{1}}|)^{d-1} < \infty$.

As, for I_4 we distinguish two cases.

(1) $p \ge 2$, in which case

$$\begin{aligned}
I_4 &\le \sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2-\alpha k+\frac{k}{2}} E(X'_{\mathbf{1}})^2)^{\frac{k}{2}} \le \sum_{m=1}^{\infty}\sum_{|\mathbf{n}|=m} |\mathbf{n}|^{p\alpha-2-\alpha k+\frac{k}{2}} \\
&= \sum_{m=1}^{\infty} d(m) m^{p\alpha-2-\alpha k+\frac{k}{2}}
\end{aligned} \tag{2.9}$$

which, in view of (2.3), is convergent if k is selected such that $k > (p\alpha-1)/(\alpha-\frac{1}{2})$.

(2) $1 \le p < 2$, in which case

$$I_4 \le \sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2-\alpha k+\frac{k}{2}} E(X_{\mathbf{1}}')^2 \le \sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2-\alpha k+\frac{k}{2}+\alpha(2-p)(\frac{k}{2})} \tag{2.10}$$

which is convergent for $k > 2$, provided $p\alpha > 1$.

Hence from (2.7), (2.8), (2.9) and (2.10) we derive that $I_2 < \infty$ and thus (ii) follows.

Now we prove that (ii) implies (i). Obviously (ii) implies that

$$\sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-2} P(\max_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} |X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}) < \infty \tag{2.11}$$

and that

$$P(\max_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} |X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}) \to 0. \tag{2.12}$$

Let $A_{\mathbf{j}} = \{|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha},\ \max_{\mathbf{i}\ne\mathbf{j},\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| \le |\mathbf{n}|^{\alpha}\}$. Then $\{A_{\mathbf{j}}\}$ are disjoint subsets. Since

$$\begin{aligned} P(\max_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} |X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}) &= \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} P(|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}, \max_{\mathbf{i}\ne\mathbf{j},\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| \le |\mathbf{n}|^{\alpha}) \\ &= P(\cup_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}}\{|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}, \max_{\mathbf{i}\ne\mathbf{j},\mathbf{i}\in\mathbf{n}} |X_{\mathbf{i}}| \le |\mathbf{n}|^{\alpha}\}), \end{aligned}$$

we deduce, in view of the equidistribution , that

$$\begin{aligned} &|\mathbf{n}|P(|X_{\mathbf{1}}| > |\mathbf{n}|^{\alpha}) = \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} P(|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}) \\ &= \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} P(|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}, \max_{\mathbf{i}\ne\mathbf{j},\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| \le |\mathbf{n}|^{\alpha}) \\ &\quad + \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} P(|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}, \max_{\mathbf{i}\ne\mathbf{j},\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha}) \\ &= P(\max_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} |X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}) + \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} P(|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}, \max_{\mathbf{i}\ne\mathbf{j},\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha}). \end{aligned} \tag{2.13}$$

By centering the second term we get

$$\begin{aligned} &\sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} P(|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}, \max_{\mathbf{i}\ne\mathbf{j},\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha}) \\ &= \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} E[I(|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}) I(\max_{\mathbf{i}\ne\mathbf{j},\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha})] \\ &\le E \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} [I(|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}) - P(|X_{\mathbf{1}}| > |\mathbf{n}|^{\alpha})] I(\max_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha}) \\ &\quad + |\mathbf{n}|P(|X_{\mathbf{1}}| > |\mathbf{n}|^{\alpha}) P(\max_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha}) = I + II. \end{aligned} \tag{2.14}$$

In order to estimate I we apply the Cauchy-Schwarz inequality and relation (2.1.a) with $q = 2$ and we obtain

$$\begin{aligned}
|I| &\le \sqrt{Var\{\sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} I(|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha})\} \cdot P(\max_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha})} \\
&\le \sqrt{2\{Var[\sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} I(X_{\mathbf{j}} > |\mathbf{n}|^{\alpha})] + Var[\sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} I(X_{\mathbf{j}} < -|\mathbf{n}|^{\alpha})]\}} \\
&\quad \cdot \sqrt{P(\max_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha})} \\
&\le \sqrt{2[\sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} P(X_{\mathbf{j}} > |\mathbf{n}|^{\alpha}) + \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} P(X_{\mathbf{j}} < -|\mathbf{n}|^{\alpha})]} \qquad (2.15) \\
&\quad \cdot \sqrt{P(\max_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha})} \\
&= \sqrt{2\sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} P(|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}) \cdot P(\max_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha})} \\
&\le \frac{1}{2}\sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} P(|X_{\mathbf{j}}| > |\mathbf{n}|^{\alpha}) + P(\max_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha}) \ \text{ by } \sqrt{ab} \le \frac{a+b}{2}.
\end{aligned}$$

From (2.13)-(2.15) we have

$$\begin{aligned}
\frac{1}{2}|\mathbf{n}|P(X_{\mathbf{1}} > |\mathbf{n}|^{\alpha}) &\le 2P(\max_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha}) \\
&\quad + |\mathbf{n}|P(|X_{\mathbf{1}}| > |\mathbf{n}|^{\alpha})P(\max_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha})
\end{aligned}$$

and by (2.12) there exists a positive constant C such that

$$|\mathbf{n}|P(|X_{\mathbf{1}}| > |\mathbf{n}|^{\alpha}) \le CP(\max_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}} |X_{\mathbf{i}}| > |\mathbf{n}|^{\alpha}) \qquad (2.16)$$

for sufficiently large $|\mathbf{n}|$. Relations (2.11) and (2.16) finally gives

$$\sum_{\mathbf{n}} |\mathbf{n}|^{p\alpha-1}P(|X_{\mathbf{1}}| > |\mathbf{n}|^{\alpha}) < \infty. \qquad (2.17)$$

Hence, from (2.17), (i) follows by Lemma 2.2.

Proof of Theorem 1.2. Note that $\{X_n, n \in Z_+^d\}$ is a field of pairwise negatively quadrant dependent (NQD) random variables since NA implies NQD. Hence, the proof follows Theorem 2 of Matula (1992).

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Mi-Hwa Ko
Department of Mathematics,
WonKwang University,
Jeonbuk 570-749,
Republic of Korea
E-mail: songhack@wonkwang.ac.kr