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AN EQUALITY CONDITION OF ARHIPPAINEN-MÜLLER'S ESTIMATE AND ITS RELATED PROBLEM

Sin-Ei Takahasi, Takeshi Miura and Takahiro Hayata

Dedicated to Professor Wataru Takahashi for his retirement
from TIT with respect and affection

Abstract. J. Arhippainen and V. Müller showed that $\|a+\lambda e\|_1 \leq 3\|a+\lambda e\|_{\text{op}}$ for all $a+\lambda e \in \tilde{A}$, where $\tilde{A}$ is the unitization of an arbitrary non-unital Banach algebra A with regular norm. In this note, we present a necessary and sufficient condition for the equality of this estimate and its related problem.

1. Introduction

Let A be a Banach algebra with regular norm $\|\cdot\|$. Here we shall say that $\|\cdot\|$ is regular if

$$\|a\| = \sup\{\|ax\|, \|xa\| : x \in A, \|x\| = 1\}$$

holds for all $a \in A$. Let $\tilde{A} = \{a + \lambda e : a \in A, \lambda \in \mathbb{C}\}$ be the unitization of A with the identity element denoted by e. The norm $\|\cdot\|$ of A can be extended to $\tilde{A}$ in many ways, however, all regular extensions are equivalent. Denote by $\|\cdot\|_1$ the ℓ_1-norm on $\tilde{A}$: $\|a+\lambda e\|_1 = \|a\| + |\lambda|$, and by $\|\cdot\|_{\text{op}}$ the operator seminorm on $\tilde{A}$:

$$\|a+\lambda e\|_{\text{op}} = \sup\{\|ax+\lambda x\|, \|xa+\lambda x\| : x \in A, \|x\| = 1\}.$$

Note that $\|\cdot\|_{\text{op}}$ becomes a norm if and only if A has no identity element. In fact, suppose that $\|\cdot\|_{\text{op}}$ becomes a norm and A has an identity element e_A. Then

$$\|e_A - e\|_{\text{op}} = \sup\{\|e_A x - x\|, \|x e_A - x\| : x \in A, \|x\| = 1\} = 0$$

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and hence $e_A - e = 0$, so $e \in A$, a contradiction. Conversely, suppose that $\|\cdot\|_{\rm op}$ does not become a norm. Then there exists a nonzero element $c+\mu e \in \tilde{A}$ such that $\|c+\mu e\|_{\rm op} = 0$, that is, $cx = xc = -\mu x$ for all $x \in A$. If $\mu = 0$, then $cx = xc = 0$ for all $x \in A$ and hence $c = 0$ since $\|\cdot\|$ is regular. This contradicts $c + \mu e \neq 0$. Therefore, we have $\mu \neq 0$ and we see that $-c/\mu$ becomes an identity element of A.

In the remainder of this note, we assume that A is non-unital. In 1991, A. K. Gaur and Z. V. Kovárík [3] showed that the ℓ_1-norm on $\tilde{A}$ is the maximal and the operator norm on $\tilde{A}$ is the minimal among all regular extensions of $\|\cdot\|$. Moreover, they [4] showed in 1993 that

$$\|a+\lambda e\|_1 \leq 6\exp(1)\,\|a+\lambda e\|_{\rm op}$$

holds for all $a+\lambda e \in \tilde{A}$ and that if A is a C^*-algebra, then

$$\|a+\lambda e\|_1 \leq 3\,\|a+\lambda e\|_{\rm op}$$

holds for all Hermitian element $a \in A$ and $\lambda \in \mathbb{C}$ and the constant 3 is best possible. Also T. W. Palmer [5] pointed out that $\|a+\lambda e\|_1 \leq (1+\exp(1))\,\|a+\lambda e\|_{\rm op}$ holds for all $(a, \lambda) \in \tilde{A}$ in his review on Gaur-Kovárík's paper. In 1997, J. Arhippainen and V. Müller [1] showed the following final version.

Theorem A-M. *Let A be a non-unital Banach algebra with regular norm. Then $\|a+\lambda e\|_1 \leq 3\,\|a+\lambda e\|_{\rm op}$ holds for all $a+\lambda e \in \tilde{A}$.*

In this note, our purpose is to present a necessary and sufficient condition for the equality of Arhippainen-Müller's estimate above and its related problem.

2. Main Results

The following is our main result and this can be shown by applying a technique used in [1].

Proposition 2.1. *Let A be a non-unital Banach algebra with regular norm $\|\cdot\|$ and $a+\lambda e \in \tilde{A}$. Then $\|a+\lambda e\|_1 = 3\,\|a+\lambda e\|_{\rm op}$ if and only if $\|a\| = 2\,\|a+\lambda e\|_{\rm op} = 2|\lambda|$.*

Proof. Since A is a proper two-sided ideal of the Banach algebra $(\tilde{A}, \|\cdot\|_{\rm op})$ with identity element e, it follows that $\|b-e\|_{\rm op} \geq 1$ for all $b \in A$ and hence we have

$$|\lambda| \leq \|a+\lambda e\|_{\rm op}. \tag{2.1}$$

By (2.1), we have $\|a+\lambda e\|_{\rm op} \geq \|a\|_{\rm op} - \|\lambda e\|_{\rm op} = \|a\| - |\lambda| \geq \|a\| - \|a+\lambda e\|_{\rm op}$ and hence

$$\|a\| \leq 2\,\|a+\lambda e\|_{\rm op}\,. \tag{2.2}$$

By (2.1) and (2.2), we have $\|a+\lambda e\|_1 \leq 3\,\|a+\lambda e\|_{\rm op}$ and the equality holds if and only if $|\lambda| = \|a+\lambda e\|_{\rm op}$ and $\|a\| = 2\,\|a+\lambda e\|_{\rm op}$. This completes the proof. ■

We want to say that the equality condition $\|a\| = 2\,\|a+\lambda e\|_{\rm op} = 2|\lambda|$ in the above proposition asserts that a point $-a/\lambda \in A$ belongs to the intersection of the sphere with center 0 and radius 2 and the sphere with center e and radius 1 in $(\tilde{A}, \|\cdot\|_{\rm op})$. Also the proper two-sided ideal A of $\tilde{A}$ is the kernel of the homomorphism φ_A of $\tilde{A}$ onto $\mathbb{C}$ defined by $\varphi_A(a+\lambda e) = \lambda$ for each $a+\lambda e \in \tilde{A}$. Therefore, we shall pose the following.

Question. Given a unital Banach algebra B with identity element e_B and a homomorphism φ of B onto $\mathbb{C}$, can one find an element $x \in B$ having the property

$$\varphi(x) = 0, \quad \|x\| = 2 \quad \text{and} \quad \|x - e_B\| = 1? \tag{$\sharp$}$$

We can not find an element of $\tilde{A}$ which has the property ($\sharp$) for the pair $\{(\tilde{A}, \|\cdot\|_1), \varphi_A\}$. In fact, suppose that there exists an element $x_0 + \lambda_0 e \in \tilde{A}$ having the property ($\sharp$). Then $\lambda_0 = \varphi_A(x_0+\lambda_0 e) = 0$ and hence $\|x_0\| = \|x_0+\lambda_0 e\|_1 = 2$. Also we have $1 = \|x_0+\lambda_0 e - e\|_1 = \|x_0 - e\|_1 = \|x_0\| + 1$ and then $x_0 = 0$, a contradiction. However, the following result gives an affirmative answer to the above question in cases of unital C^*-algebra or unital group algebra.

Proposition 2.2. *Let B be a C^*-algebra or group algebra of a discrete Abelian group G and φ be a homomorphism of B onto $\mathbb{C}$. Then there exists an element $x \in B$ having the property ($\sharp$): $\varphi(x) = 0$, $\|x\| = 2$ and $\|x - e_B\| = 1$, where e_B is the identity element of B.*

Proof.

(1) Suppose B is a unital C^*-algebra and φ be a homomorphism of B onto $\mathbb{C}$. Take a norm one positive element $b \in B$ with $\varphi(b) = 0$. Denote by $\sigma(b)$ the spectrum of $b \in B$. Then $0, 1 \in \sigma(b)$ and $\sigma(b)$ is contained in the closed unit interval $[0,1]$ in $\mathbb{R}$. Also the C^*-subalgebra of B generated by b and the identity element e_B is isometrically isomorphic to the commutative C^*-algebra of all continuous complex-valued functions on $\sigma(b)$. Set $f(t) = 2t$ for each $t \in \sigma(a)$. Then $f(0) = 0$, $\|f\|_\infty = 2$ and $\|f - 1\|_\infty = 1$, where $\|\cdot\|_\infty$ denotes the supremum norm on $\sigma(b)$. Therefore the corresponding element x of B has the property ($\sharp$).

(2) Let B be a group algebra $L^1(G)$ of a discrete Abelian group G and φ be a homomorphism of B onto $\mathbb{C}$. Let γ be the corresponding character of G and then $\ker\varphi = \{f \in L^1(G) : \hat{f}(\gamma) = 0\}$. Take an element $x_0 \in G$ which is different from 0 and define

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ -\gamma(x_0) & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases} .$$

Then $\|f\|_1 = 1 + |-\gamma(x_0)| = 2$. Let δ_0 be the Dirac measure at $0 \in G$. Then δ_0 is an identity element of $L^1(G)$ and we have $\|f - \delta_0\|_1 = |-\gamma(x_0)| = 1$. Also we have

$$\begin{aligned} \hat{f}(\gamma) &= \sum_{x \in G} f(x)\overline{\gamma(x)} = f(0)\overline{\gamma(0)} + f(x_0)\overline{\gamma(x_0)} \\ &= 1 - \gamma(x_0)\overline{\gamma(x_0)} = 1 - |\gamma(x_0)|^2 = 0 \end{aligned}$$

and then the function f has the property ($\sharp$). ■

Remark 1. The regular norms are sometimes defined as those satisfying

$$\|a\| = \sup\{\|ax\| : x \in A, \|x\| = 1\}$$

for all $a \in A$. Both Theorem A-M and Proposition 2.1 remain true for this definition without any change (cf. [1, 3, 4]).

Remark 2. If we can find an element of $\tilde{A}$ which has the property ($\sharp$) for the pair $\{(\tilde{A}, \|\cdot\|_{\text{op}}), \varphi_A\}$, then the constant 3 in Theorem A-M is best possible. In fact, suppose that there exists $a_0 + \lambda_0 e \in \tilde{A}$ such that $\varphi_A(a_0 + \lambda_0 e) = 0$, $\|a_0 + \lambda_0 e\|_{\text{op}} = 2$ and $\|a_0 + \lambda_0 e - e\|_{\text{op}} = 1$. Then we have $\lambda_0 = 0$, hence $\|a_0\| = \|a_0\|_{\text{op}} = 2$ and $\|a_0 - e\|_{\text{op}} = 1$. Therefore, $\|a_0 - e\|_1 = 3\,\|a_0 - e\|_{\text{op}}$ by Proposition 2.1. This implies that the constant 3 in Theorem A-M is best possible.

Remark 3. Let A be a non-unital C^*-algebra and put

$$\|a + \lambda e\|_{lm} = \sup\{\|ax + \lambda x\| : x \in A, \|x\| = 1\}$$

for each $a + \lambda e \in \tilde{A}$. Then $(\tilde{A}, \|\cdot\|_{lm})$ becomes a unital C^*-algebra by [2, 1.3.8]. Similarly, we can show that $(\tilde{A}, \|\cdot\|_{\text{op}})$ becomes a C^*-algebra and then we have from the uniqueness of C^*-norm that

$$\|a + \lambda e\|_{lm} = \|a + \lambda e\|_{\text{op}}$$

for each $a + \lambda e \in \tilde{A}$. Therefore we see from Proposition 2.2 and Remark 2 that the constant 3 in Theorem A-M is best possible in case C^*-algebras. This improves Gaur-Kovárík's result [4] stated above.

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Sin-Ei Takahasi
Department of Applied Mathematics and Physics,
Graduate School of Science and Engineering,
Yamagata University,
Yonezawa 992-8510,
Japan
E-mail: sin-ei@emperor.yz.yamagata-u.ac.jp

Takeshi Miura
Department of Applied Mathematics and Physics,
Graduate School of Science and Engineering,
Yamagata University,
Yonezawa 992-8510,
Japan
E-mail: miura@yz.yamagata-u.ac.jp

Takahiro Hayata
Department of Informatics,
Graduate School of Science and Engineering,
Yamagata University,
Yonezawa 992-8510,
Japan
E-mail: hayata@yz.yamagata-u.ac.jp