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NEW CHARACTERIZATIONS OF WEIGHTED MORREY-CAMPANATO SPACES

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Abstract. Let $\alpha \in (0, \infty)$, $q \in [1, \infty]$, s be a nonnegative integer, $\omega \in A_1(\mathbb{R}^n)$ (the class of Muckenhoupt's weights). In this paper, the authors introduce the weighted Morrey-Campanato space $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ and obtain its equivalence on different $q \in [1, \infty]$ and integers $s \geq \lfloor n\alpha \rfloor$ (the integer part of $n\alpha$). The authors then introduce the weighted Lipschitz space $\wedge(\alpha,\, \omega;\, \mathbb{R}^n)$ and prove that $\wedge(\alpha,\, \omega;\, \mathbb{R}^n) = L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ when $\alpha \in (0, \infty)$, $s \geq \lfloor n\alpha \rfloor$ and $q \in [1, \infty]$. Using this, the authors further establish a new characterization of $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ by using the convolution $\varphi_{t_B} * f$ to replace the minimizing polynomial $P^s_B f$ on any ball B of a function f in its norm when $\alpha \in (0, \infty)$, $s \geq \lfloor n\alpha \rfloor$, $\omega \in A_1(\mathbb{R}^n) \cap RH_{1+1/\alpha}(\mathbb{R}^n)$ and $q \in [1, \infty]$, where φ is an appropriate Schwartz function, t_B denotes the radius of the ball B and $\varphi_{t_B}(\cdot) \equiv t_B^{-n} \varphi(t_B^{-1} \cdot)$.

1. Introduction

It is well-known that the classical Morrey-Campanato spaces play an important role in the study of partial differential equations and harmonic analysis; see, for example, [4, 13, 15, 1, 2, 3, 11, 10]. Let $\alpha \in [0, \infty)$, $q \in [1, \infty]$ and s be an integer that is no less than $\lfloor n\alpha \rfloor$, where and in what follows, $\lfloor s \rfloor$ denotes the *maximal integer no more than* s. It was proved by Taibleson and Weiss [17] that the classical Morrey-Campanato spaces $L(\alpha,\, q,\, s;\, \mathbb{R}^n)$ are dual spaces of Hardy spaces on $\mathbb{R}^n$. It was also pointed out by Janson, Taibleson and Weiss in [10] that for $\alpha = 0$, the spaces $L(\alpha,\, q,\, s;\, \mathbb{R}^n)$ are variants of $\mathrm{BMO}\,(\mathbb{R}^n)$ (see [11]) and for $\alpha \in (0, \infty)$, they are variants of homogenous Besov-Lipschitz spaces with

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smoothness of order $n\alpha$ (see [9]). Throughout the whole paper, we only consider the case when $\alpha \in (0, \infty)$.

For $\alpha \in (0, 1/n)$, $q \in [1, \infty]$, $s = 0$ and $\omega \in A_1(\mathbb{R}^n)$, Tang [18] recently introduced the weighted Morrey-Campanato spaces $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ and obtained their certain equivalent characterizations. Here and in what follows, $A_1(\mathbb{R}^n)$ denotes the class of Muckenhoupt's weights. We should point out that when $n = 1$, the weighted Morrey-Campanato spaces $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ were essentially introduced by García-Cuerva [6] as the dual spaces of the corresponding weighted Hardy spaces.

In this paper, for $\alpha \in (0, \infty)$, $q \in [1, \infty]$, s being a nonnegative integer and $\omega \in A_1(\mathbb{R}^n)$, we introduce and investigate the weighted Morrey-Campanato spaces $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$, which generalize the classical Morrey-Campanato spaces by taking $\omega \equiv 1$ and $s \geq \lfloor n\alpha \rfloor$, the weighted Morrey-Campanato spaces introduced in [18] by taking $s = 0$ and $\alpha \in (0, 1/n)$ and the corresponding results in [6] to any $n \in \mathbb{N}$. Some equivalent characterizations of $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ are also obtained. These results essentially improve the known results in [6, 17, 10, 5, 18].

To be precise, we first recall the notion of the classical Morrey-Campanato space $L(\alpha,\, q,\, s;\, \mathbb{R}^n)$. Let $q \in [1, \infty)$ and $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ denote the set of all locally integrable functions on $\mathbb{R}^n$. Let $\alpha \in (0, \infty)$, $q \in [1, \infty)$ and s be an integer no less than $\lfloor n\alpha \rfloor$. Following [10] (see also [17]), the classical Morrey-Campanato space $L(\alpha,\, q,\, s;\, \mathbb{R}^n)$ is defined to be the set of all functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{L(\alpha,\, q,\, s;\, \mathbb{R}^n)} \equiv \sup_{B \subset \mathbb{R}^n} |B|^{-\alpha} \left[|B|^{-1} \int_B |f(x) - P^s_B f(x)|^q \, dx \right]^{1/q} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$, and $P^s_B f$ denotes the minimizing polynomial of f on the ball B with degree at most s, namely, for all multi-indices $\theta \in (\mathbb{N} \cup \{0\})^n$ with $0 \leq |\theta| \leq s$,

$$\int_B [f(x) - P^s_B f(x)] x^\theta \, dx = 0. \tag{1.1}$$

It is well-known that if f is locally integrable, then $P^s_B f$ uniquely exists; see, for example, [17] or [12]. In the case $q = \infty$ and $\alpha \in (0, \infty)$, the space $L(\alpha,\, \infty,\, s;\, \mathbb{R}^n)$ in [10] is defined to be the set of all functions $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{L(\alpha,\, \infty,\, s;\, \mathbb{R}^n)} \equiv \sup_{B \subset \mathbb{R}^n} \operatorname*{esssup}_{x \in B} |B|^{-\alpha} |f(x) - P^s_B f(x)| < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

In what follows, for any $\omega \in A_1(\mathbb{R}^n)$, let $\omega(E) \equiv \int_E \omega(x)\, dx$, where E is a measurable set in $\mathbb{R}^n$. We introduce the following weighted Morrey-Campanato spaces.

Definition 1.1. Let $\alpha \in (0,\infty)$, s be a nonnegative integer and $\omega \in A_1(\mathbb{R}^n)$. When $q \in [1,\infty)$, the weighted Morrey-Campanato space $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ is defined to be the set of all functions $f \in L^1_{\rm loc}(\mathbb{R}^n)$ such that

$$\begin{aligned} &\|f\|_{L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)} \\ &\equiv \sup_{B\subset\mathbb{R}^n} \frac{1}{[\omega(B)]^\alpha}\left[\frac{1}{\omega(B)}\int_B |f(x)-P_B^s f(x)|^q[\omega(x)]^{1-q}\,dx\right]^{1/q} < \infty, \end{aligned} \tag{1.2}$$

where the supremum is taken over all balls $B\subset\mathbb{R}^n$ and $P_B^s f$ is as in (1.1). When $q=\infty$, the weighted Morrey-Campanato space $L(\alpha,\,\infty,\,s,\,\omega;\,\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^1_{\rm loc}(\mathbb{R}^n)$ such that

$$\|f\|_{L(\alpha,\,\infty,\,s,\,\omega;\,\mathbb{R}^n)} \equiv \sup_{B\subset\mathbb{R}^n}\ \operatorname*{esssup}_{x\in B} \frac{1}{[\omega(B)]^\alpha}\frac{|f(x)-P_B^s f(x)|}{\omega(x)} < \infty, \tag{1.3}$$

where the supremum is taken over all balls $B\subset\mathbb{R}^n$ and $P_B^s f$ is as in (1.1).

We point out that the weighted Morrey-Campanato space $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ is actually defined as a space of equivalence classes modulo the polynomials with degree at most s. Moreover, obviously, if $\omega\equiv 1$ and $s\ge\lfloor n\alpha\rfloor$, then $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ is just the classical Morrey-Campanato space $L(\alpha,\, q,\, s;\, \mathbb{R}^n)$ in [10]. We also remark that the space $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ in Definition 1.1 when $n=1$ was essentially introduced by García-Cuerva [6, p. 29] and when $s=0$, $n\in\mathbb{N}$ and $\alpha\in(0,1/n)$ by Tang in [18]. In Section 2 of this paper, we obtain the equivalence of the spaces $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ with respect to different $q\in[1,\infty]$ and integers $s\ge\lfloor n\alpha\rfloor$, which generalizes the corresponding result on [6, p. 29] to all $n\in\mathbb{N}$ and completely covers [10, Theorem 1] and [17, p. 132, (8.17)] when $\alpha\in(0,\infty)$ and [10, Theorem 2] by taking $\omega\equiv1$, and [18, Theorem 2.1] by taking $\alpha\in(0,1/n)$ and $s=0$; see Theorems 2.1 and 2.2 below.

In what follows, let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$. Choose $\Psi\in\mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{\Psi}\subset\{\xi\in\mathbb{R}^n:\ 1/2\le|\xi|\le2\}$, $\widehat{\Psi}(\xi)\ge1/4$ when $3/5\le|\xi|\le5/3$, and $\sum_{j=-\infty}^{\infty}\widehat{\Psi}(2^{-j}\xi)=1$ for all $\xi\in\mathbb{R}^n\setminus\{0\}$, where and in what follows, $\widehat{f}$ denotes the Fourier transform of f, namely, for all $\xi\in\mathbb{R}^n$, $\widehat{f}(\xi)\equiv\int_{\mathbb{R}^n}e^{-2\pi i\xi x}f(x)\,dx$. For all $j\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, let $\Psi_{2^{-j}}(x)\equiv2^{jn}\Psi(2^jx)$.

Following Triebel's [19], set

$$\mathcal{S}_\infty(\mathbb{R}^n)\equiv\left\{\varphi\in\mathcal{S}(\mathbb{R}^n):\int_{\mathbb{R}^n}\varphi(x)x^\gamma\,dx=0 \text{ for all multi-indices } \gamma\in\mathbb{Z}_+^n\right\}$$

and let $\mathcal{S}'_\infty(\mathbb{R}^n)$ be the topological dual of $\mathcal{S}_\infty(\mathbb{R}^n)$. We now introduce the weighted Lipschitz space as follows.

Definition 1.2. Let Ψ be as above. For any $\alpha \in (0,\infty)$ and $\omega \in A_1(\mathbb{R}^n)$, the weighted Lipschitz space $\wedge(\alpha,\, \omega;\, \mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ satisfying that there exists a nonnegative constant C such that for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,

$$|\Delta_j f(x)| \le C[\omega(B(x, 2^{-j}))]^{1+\alpha} 2^{jn}, \tag{1.4}$$

where Δ_j is given by $\Delta_j f \equiv f * \Psi_{2^{-j}}$. Moreover, the smallest bound C in (1.4) is defined to be the norm of f in $\wedge(\alpha,\, \omega;\, \mathbb{R}^n)$ and denoted by $\|f\|_{\wedge(\alpha,\omega;\,\mathbb{R}^n)}$.

Obviously, the space $\wedge(\alpha,\, 1;\, \mathbb{R}^n)$ is just the classical Lipschitz space; see, for example, [8, Theorem 6.3.6]. The only difference is that here, following [19, Chapter 5], we use $\mathcal{S}'_\infty(\mathbb{R}^n)$ to replace $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, where and in what follows, $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of Schwartz distributions and $\mathcal{P}(\mathbb{R}^n)$ the set of all polynomials on $\mathbb{R}^n$. We also remark that the space $\wedge(\alpha,\, \omega;\, \mathbb{R}^n)$ was defined to be the set of all locally integrable functions such that (1.4) hold by Tang in [18]. Thus, Definition 1.2 generalizes the notion of the corresponding weighted Lipschitz spaces in [18]. In Section 3 below, we prove that for $\alpha \in (0,\infty)$, s being an integer satisfying $s \ge \lfloor n\alpha \rfloor$ and $q \in [1,\infty]$, $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n) = \wedge(\alpha,\, \omega;\, \mathbb{R}^n)$ with equivalent norms, which implies that the space $\wedge(\alpha,\, \omega;\, \mathbb{R}^n)$ is independent of the choices of Ψ. Recall that Tang in [18, Theorem 2.2] showed that when $\alpha \in (0, 1/n)$ and $s = 0$, if f is locally integrable and has compact support, then $f \in L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ if and only if $f \in \wedge(\alpha,\, \omega;\, \mathbb{R}^n)$. Thus, Theorem 3.1 below essentially improves Theorem 2.2 in [18]. A new ingredient appearing in the proof of Theorem 3.1 below is that we invoke the useful Calderón reproducing formula on $\mathcal{S}'_\infty(\mathbb{R}^n)$ obtained in [20, Lemma 2.1].

Now we choose $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that the following two conditions hold:

(A_1) There exists an $s \in \mathbb{Z}_+$ such that

$$\int_{\mathbb{R}^n} \varphi(x) x^\theta dx = \begin{cases} 1, & \text{when } \theta = 0; \\ 0, & \text{when } 0 < |\theta| \le s, \end{cases} \tag{1.5}$$

where $\theta = (\theta_1, \ldots, \theta_n) \in (\mathbb{N} \cup \{0\})^n$ and $|\theta| = \theta_1 + \cdots + \theta_n$.

(A_2) The function $\Phi \equiv \varphi - \varphi * \varphi$ satisfies the Tauberian condition, namely, for all $\xi \in \mathbb{R}^n \setminus \{0\}$, there exists a $t \in (0,\infty)$ such that $\widehat{\Phi}(t\xi) \ne 0$.

In what follows, for any ball B, t_B denotes the radius of the ball B. Motivated by Deng, Duong and Yan [5], we introduce the weighted Morrey-Campanato-type space $L_\varphi(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ by using $\varphi_{t_B} * f$ to replace the minimizing polynomial $P^s_B f$ in Definition 1.1, where $\varphi_{t_B}(x) = \frac{1}{t_B^n}\varphi(\frac{x}{t_B})$ for all $x \in \mathbb{R}^n$.

Definition 1.3. Let $\alpha \in (0,\infty)$, s be a nonnegative integer, $\omega \in A_1(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. When $q \in [1,\infty)$, the weighted Morrey-Campanato-type space

$L_\varphi(\alpha, q, s, \omega; \mathbb{R}^n)$ is defined to be the set of all functions $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ such that for all balls $B \subset \mathbb{R}^n$,

$$\|f\|_{L_\varphi(\alpha,q,s,\omega;\mathbb{R}^n)} \equiv \sup_{B\subset\mathbb{R}^n} \frac{1}{[\omega(B)]^\alpha}\left[\frac{1}{\omega(B)}\int_B |f(x)-\varphi_{t_B}*f(x)|^q[\omega(x)]^{1-q}dx\right]^{1/q} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. When $q=\infty$, the weighted Morrey-Campanato-type space $L_\varphi(\alpha, \infty, s, \omega; \mathbb{R}^n)$ is defined to be the set of all functions $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ such that for all balls $B \subset \mathbb{R}^n$,

$$\|f\|_{L_\varphi(\alpha,\infty,s,\omega;\mathbb{R}^n)} \equiv \sup_{B\subset\mathbb{R}^n} \operatorname*{esssup}_{x\in B} \frac{1}{[\omega(B)]^\alpha}\frac{|f(x)-\varphi_{t_B}*f(x)|}{\omega(x)} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

When $\omega \equiv 1$, the space $L_\varphi(\alpha, q, s, 1; \mathbb{R}^n)$ was initially introduced by Deng, Duong and Yan in [5], but with $L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ replaced by $\mathcal{N}_s$; where $\mathcal{N}_s$ is the set of all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} |f(x)|[1+|x|]^{-(n+s+1)}\,dx < \infty$ (see [5, Definition 2.1]). Deng, Duong and Yan proved in [5, Theorem 3.4] that if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies the conditions (A_1) and (A_2), then $L_\varphi(\alpha, q, s, 1; \mathbb{R}^n) = L(\alpha, q, s, 1; \mathbb{R}^n)$ with equivalent norms. In Section 4 of this paper, for any $\alpha \in (0,\infty)$, s being a nonnegative integer and $\omega \in A_1(\mathbb{R}^n)$, we establish the equivalence of the spaces $L_\varphi(\alpha, q, s, \omega; \mathbb{R}^n)$ on different $q \in [1,\infty]$ if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies the condition (A_1), which completely covers [18, Theorem 3.4]; see Theorem 4.1 below. Here and in what follows, given $p \in (1,\infty)$, we say that $\omega \in RH_p(\mathbb{R}^n)$ if ω satisfies a reverse Hölder condition with exponent p, namely, there exists a positive constant C such that for all balls $B \subset \mathbb{R}^n$, $\left(\frac{1}{|B|}\int_B[\omega(x)]^p\,dx\right)^{1/p} \le C\frac{1}{|B|}\int_B \omega(x)\,dx$. Using Theorem 4.1, for any $\omega \in A_1(\mathbb{R}^n) \cap RH_{1+1/\alpha}(\mathbb{R}^n)$ and $\alpha \in (0,\infty)$, s being an integer satisfying $s \ge \lfloor n\alpha \rfloor$ and $q \in [1,\infty]$, we prove that if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies the conditions (A_1) and (A_2), then $L_\varphi(\alpha, q, s, \omega; \mathbb{R}^n) = L(\alpha, q, s, \omega; \mathbb{R}^n)$ with equivalent norms, which completely covers [5, Theorem 3.4] by taking $\omega \equiv 1$. Recall that Tang in [18, Theorem 4.2] showed that when $\alpha \in (0, 1/n)$, $s=0$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies the conditions (A_1) and (A_2), if f is locally integrable and has compact support, then $f \in L(\alpha, q, s, \omega; \mathbb{R}^n)$ if and only if $f \in L_\varphi(\alpha, q, s, \omega; \mathbb{R}^n)$. Thus, Theorem 4.3 below essentially generalizes Theorem 4.2 in [18].

Finally we make some conventions on notation. Throughout the whole paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $A \lesssim B$ means that $A \le CB$. If

$A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For any given "normed" spaces $\mathcal{A}$ and $\mathcal{B}$, the symbol $\mathcal{A} \subset \mathcal{B}$ means that for all $f \in \mathcal{A}$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{A}}$. For a measurable set E, denote by χ_E the characteristic function of E. We also set $\mathbb{N} \equiv \{1, 2, \cdots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. For a multi-indices $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{Z}_+^n$, we let $|\theta| \equiv \theta_1 + \cdots + \theta_n$, $\theta! \equiv \theta_1 \cdots \theta_n$ and $\partial_x^\theta \equiv \frac{\partial^{|\theta|}}{\partial x_1^{\theta_1} \cdots \partial x_n^{\theta_n}}$.

2. Some Characterizations of $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$

First we recall some notation and properties of the Muckenhoupt weights. In what follows, $B(x,t)$ denotes the ball centered at x and of the radius t. Given $B \equiv B(x,t)$ and $\lambda \in (0,\infty)$, we write $\lambda B \equiv B(x, \lambda t)$.

A nonnegative function ω on $\mathbb{R}^n$ is called a *weight* if it is locally integrable. A weight ω is said to belong to the *Muckenhoupt class* $A_1(\mathbb{R}^n)$ if there exists a positive constant C such that for almost all $x \in \mathbb{R}^n$, $M(\omega)(x) \le C\omega(x)$, where M denotes the Hardy-Littlewood maximal operator on $\mathbb{R}^n$; see, for example, [7, 8]. The following facts on $A_1(\mathbb{R}^n)$ can be found in [7, 8]. If $\omega \in A_1(\mathbb{R}^n)$, then there exists a positive constant C such that for all balls $B_1,\, B_2 \subset \mathbb{R}^n$ with $B_1 \subset B_2$,

$$\frac{\omega(B_2)}{\omega(B_1)} \le C\frac{|B_2|}{|B_1|}; \tag{2.1}$$

also there exists a positive constant C and $\delta \in (0,1)$ such that for all balls $B_1,\, B_2 \subset \mathbb{R}^n$ with $B_1 \subset B_2$,

$$\frac{\omega(B_1)}{\omega(B_2)} \le C\left(\frac{|B_1|}{|B_2|}\right)^\delta. \tag{2.2}$$

In particular, if $\omega \in A_1(\mathbb{R}^n)$, by (2.1), we have that there exists a positive constant C such that for all balls $B \subset \mathbb{R}^n$,

$$\omega(2B) \le C\omega(B). \tag{2.3}$$

On the equivalence of the spaces $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ with respect to different q, we have the following conclusion.

Theorem 2.1. *Let* $\omega \in A_1(\mathbb{R}^n)$, $\alpha \in (0,\infty)$ *and* $s \in \mathbb{Z}_+$. *Then the following propositions* (I), (II) *and* (III) *are equivalent:*

(*I*) $f \in L(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n)$*;*

(*II*) $f \in L(\alpha,\, \infty,\, s,\, \omega;\, \mathbb{R}^n)$*;*

(*III*) *For any given* $q \in [1,\infty)$, $f \in L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$.

Moreover, their norms are equivalent with equivalent constants independent of f.

Proof. (I) $\Rightarrow$ (II). In this case, we adopt some ideas from [10]. Let $p \equiv \frac{1}{\alpha+1}$. Then $p \in (0,1)$. First we recall the notion of (p,s)-atoms on $\mathbb{R}^n$. A function g is called a (p,s)-atom on $\mathbb{R}^n$, if g is supported on a ball B, $\|g\|_{L^\infty(B)} \le |B|^{-1/p}$ and $\int_{\mathbb{R}^n} g(x)x^\theta\,dx = 0$ for all multi-indices $\theta \in \mathbb{Z}_+^n$ with $0 \le |\theta| \le s$.

Let $f \in L(\alpha,\,1,\,s,\,\omega;\,\mathbb{R}^n)$. Without loss of generality, by homogeneity, we may assume that $\|f\|_{L(\alpha,\,1,\,s,\,\omega;\,\mathbb{R}^n)} = 1$. We first observe that if g is a (p,s)-atom and $f \in L(\alpha,\,1,\,s,\,\omega;\,\mathbb{R}^n)$, then $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| &= \left|\int_{\mathbb{R}^n}[f(x)-P_B^s f(x)]g(x)\,dx\right| \\ &\lesssim |B|^{-1/p}[\omega(B)]^{1+\alpha} \sim \frac{[\omega(B)]^{1+\alpha}}{|B|^{1+\alpha}}.\end{aligned} \tag{2.4}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that φ is supported on $B(0,1)$ and φ satisfies (1.5). Set $\varphi_t(x) \equiv t^{-n}\varphi(x/t)$ for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$. We have the following facts (i) and (ii).

(i) If $p \in (0,1)$ and $0 < t/2 \le \tau \le t$, then there exists a (p,s)-atom g and a $\lambda \in \mathbb{R}$ satisfying $|\lambda| \lesssim t^{n(1/p-1)}$ such that $\varphi_t - \varphi_\tau = \lambda g$;

In fact, by the definition of φ_t, we know that $\varphi_t - \varphi_\tau$ is supported on $B(0,t)$, and there exists a positive constant $\widetilde{C}$ such that $\|\varphi_t - \varphi_\tau\|_{L^\infty(B(0,t))} \le (1+2^n)t^{-n}\|\varphi\|_{L^\infty(B(0,1))} \le \widetilde{C}t^{n(1/p-1)}|B(0,t)|^{-1/p}$. Letting $g = \frac{\varphi_t-\varphi_\tau}{\widetilde{C}t^{n(1/p-1)}}$ and $\lambda = \widetilde{C}t^{n(1/p-1)}$ then yields (i).

(ii) If $f \in L(\alpha,\,1,\,s,\,\omega;\,\mathbb{R}^n)$, then for almost all $x \in \mathbb{R}^n$, we have $f*\varphi_t(x) \to f(x)$ as $t \to 0$.

In fact, since $f \in L(\alpha,\,1,\,s,\,\omega;\,\mathbb{R}^n)$, then f is locally integrable. Thus, for almost all $x \in \mathbb{R}^n$, $f*\varphi_t \to f$ as $t \to 0$. Hence, (ii) holds.

Fix $B \equiv B(x_0,r)$. By the fact (ii), we know that for almost all $x \in B$,

$$f(x) = f*\varphi_r(x) + \sum_{k=1}^\infty [f*\varphi_{r2^{-k}}(x) - f*\varphi_{r2^{-(k-1)}}(x)]. \tag{2.5}$$

By [10, p. 111], we know that there exists a kernel $g_B(t;x)$ such that for all $x \in B$, $g_B(\cdot;x)$ is supported on B and

$$P_B^s f(x) = \int_B f(t)g_B(t;x)\,dt. \tag{2.6}$$

In addition, we have $|g_B(t;x)| \lesssim |B|^{-1}$.

By [10, pp. 111-112], there exist functions λ_0 on $\mathbb{R}^n$ and a_0 on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying that for each fixed $x \in B$, $|\lambda_0(x)| \lesssim |B|^\alpha$ and $a_0(\cdot; x)$ is a (p, s)-atom supported on $2B$ such that $\varphi_r(x-t) - g_B(t; x) \equiv \lambda_0(x) a_0(t; x)$. Also, by the previous fact (i), for each $k \in \mathbb{N}$, there exist functions λ_k on $\mathbb{R}^n$ and a_k on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying that for each fixed $x \in B$, $|\lambda_k(x)| \lesssim |B|^\alpha 2^{-kn\alpha}$ and $a_k(\cdot; x)$ is a (p, s)-atom supported on $B(x, r2^{-(k-1)})$ such that $\varphi_{2^{-k}r}(x-t) - \varphi_{2^{-(k-1)}r}(x-t) \equiv \lambda_k(x) a_k(t; x)$. These facts together with (2.4), (2.5), (2.6), (2.2), (2.3) and the definition of $\omega \in A_1(\mathbb{R}^n)$ yield that for almost all $x \in B$,

$$
\begin{aligned}
|f(x) - P_B^s f(x)| &= \left| \int_{\mathbb{R}^n} f(t)[\varphi_r(x-t) - g_B(t; x)]\, dt \right. \\
&\quad \left. + \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f(t)[\varphi_{r2^{-k}}(x-t) - \varphi_{r2^{-(k-1)}}(x-t)]\, dt \right| \\
&= \left| \sum_{k=0}^{\infty} \lambda_k(x) \int_{\mathbb{R}^n} f(t) a_k(t; x)\, dt \right| \\
&\lesssim |B|^\alpha \left\{ \sum_{k=1}^{\infty} 2^{-kn\alpha} \frac{[\omega(B(x, 2^{-(k-1)}r))]^{1+\alpha}}{|B(x, 2^{-(k-1)}r)|^{1+\alpha}} + \frac{[\omega(B(x_0, 2r))]^{1+\alpha}}{|B(x_0, 2r)|^{1+\alpha}} \right\} \\
&\lesssim [\omega(B(x, r))]^\alpha \omega(x) \left\{ \sum_{k=1}^{\infty} 2^{-kn\alpha\delta} + 1 \right\} \lesssim [\omega(B(x, r))]^\alpha \omega(x),
\end{aligned}
$$

which implies $f \in L(\alpha, \infty, s, \omega; \mathbb{R}^n)$. Hence, we obtain (II).

(II) $\Rightarrow$ (III). In this case, let $q \in [1, \infty)$ and f satisfy (II). Without loss of generality, by homogeneity, we may assume that $\|f\|_{L(\alpha, \infty, s, \omega; \mathbb{R}^n)} = 1$. By (2.3), we then have that for any ball $B \equiv B(x_0, r)$,

$$
\begin{aligned}
&\frac{1}{[\omega(B)]^\alpha} \left[\frac{1}{\omega(B)} \int_B |f(x) - P_B^s f(x)|^q [\omega(x)]^{1-q}\, dx \right]^{1/q} \\
&\lesssim \frac{1}{[\omega(B)]^\alpha} \left[\frac{1}{\omega(B)} \int_B [\omega(x)]^q [\omega(B(x, r))]^{\alpha q} [\omega(x)]^{1-q}\, dx \right]^{1/q} \lesssim 1,
\end{aligned}
$$

which implies that $f \in L(\alpha, q, s, \omega; \mathbb{R}^n)$ and $\|f\|_{L(\alpha, q, s, \omega; \mathbb{R}^n)} \lesssim \|f\|_{L(\alpha, \infty, s, \omega; \mathbb{R}^n)}$. Thus, (III) holds.

(III) $\Rightarrow$ (I). In this case, from Hölder's inequality and (III), we immediately deduce (I), which completes the proof of Theorem 2.1. ∎

Remark 2.1.

(i) Theorem 2.1 when $n = 1$ was essentially obtained by García-Cuerva [6, p. 29] via the duality and equivalence of weighted atomic Hardy spaces, which is different from the method used in the above proof of Theorem 2.1.

(ii) We remark that it is easy to show that when $\omega \in A_1(\mathbb{R}^n)$, $s = 0$ and $\alpha \in (0, 1/n)$, $f \in L(\alpha, \infty, s, \omega; \mathbb{R}^n)$ if and only if there exists a positive constant C such that for almost all $x, y \in \mathbb{R}^n$, $|f(x)-f(y)| \le C[\omega(B(x, |x-y|))]^{\alpha}[\omega(x)+\omega(y)]$, which is [18, Theorem 2.1(II)]. Thus, Theorem 2.1 when $\omega \equiv 1$ and $s \ge \lfloor n\alpha \rfloor$ covers [10, Theorem 2] and when $\alpha \in (0, 1/n)$ and $s = 0$ covers [18, Theorem 2.1].

(iii) We point out that when $\omega \equiv 1$, $q \in [1, \infty]$, $s = 0$ and $n\alpha > 1$, then the space $L(\alpha, q, s, \omega; \mathbb{R}^n) = \mathbb{C}$. In fact, in this case, letting $f \in L(\alpha, q, s, \omega; \mathbb{R}^n)$, by Theorem 2.1, we have that for any ball $B \subset \mathbb{R}^n$ and almost all $x \in B$, $|f(x) - f_B| \lesssim \|f\|_{L(\alpha, q, s, \omega; \mathbb{R}^n)}|B|^{\alpha}$, where $f_B = \frac{1}{|B|}\int_B f(x)\,dx$. For any $x, y \in \mathbb{R}^n$, taking a ball B such that $x, y \in B$ and $r_B = |x-y|$, then $|f(x) - f(y)| \le |f(x) - f_B| + |f(y) - f_B| \lesssim |B|^{\alpha} \sim |x-y|^{n\alpha}$. Since $n\alpha > 1$, this implies that f equals to a constant. Hence, the above claim holds.

However, when $\omega \equiv 1$, for $q \in [1, \infty]$, $s \in \mathbb{Z}_+$ and $\alpha \in (0, \infty)$ such that $s + 1 = n\alpha$, we have $\{\mathcal{S}(\mathbb{R}^n) \cup \mathcal{P}^s(\mathbb{R}^n)\} \subset L(\alpha, q, s, \omega; \mathbb{R}^n)$. In fact, let $f \in \mathcal{S}(\mathbb{R}^n)$. For any ball $B \equiv B(x_0, r)$, let $P_{x_0}f$ be the Taylor polynomial of f about x_0 with degree s. By Lemma 2.1(ii) and (iii) below, we have that

$$\begin{aligned}\frac{1}{|B|^{1+\alpha}}\int_B |f(x) - P_B^s(x)|\,dx &\lesssim \frac{1}{|B|^{1+\alpha}}\int_B |f(x) - P_{x_0}f(x)|\,dx\\ &\lesssim \frac{1}{|B|^{1+\alpha}}\int_B |x - x_0|^{s+1}\,dx \lesssim 1.\end{aligned}$$

Then by Theorem 2.1, we know that $f \in L(\alpha, q, s, \omega; \mathbb{R}^n)$. Thus, the claim is true.

From the above discussion, we see that when $0 \le s < \lfloor n\alpha \rfloor$, the space $L(\alpha, q, s, \omega; \mathbb{R}^n)$ depends on s.

Next, we give two properties of $L(\alpha, q, s, \omega; \mathbb{R}^n)$. First, we recall some necessary facts on the minimizing polynomials; see, for example, [12, p. 55].

Lemma 2.1. *Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$, $B \subset \mathbb{R}^n$ be a ball, $s \in \mathbb{Z}_+$ and $P_B^s f$ be the minimizing polynomial of f on B with degree at most s.*

(*i*) *Let $\{\varphi_l^B : l \in \mathbb{Z}_+^n, |l| \le s\}$ denote the Gram-Schmidt orthonormalization of $\{x^l : l \in \mathbb{Z}_+^n, |l| \le s\}$ on B with respect to the weight $1/|B|$, then for all $x \in \mathbb{R}^n$,*

$$\begin{aligned}P_B^s f(x) &= \sum_{\{l \in \mathbb{Z}_+^n:\, |l| \le s\}} \langle f, \varphi_l^B\rangle \varphi_l^B(x)\\ &\equiv \sum_{\{l \in \mathbb{Z}_+^n:\, |l| \le s\}} \left\{\frac{1}{|B|}\int_B \varphi_l^B(y) f(y)\,dy\right\} \varphi_l^B(x).\end{aligned}$$

(*ii*) *There exists a positive constant* C *such that for all* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and balls* $B \subset \mathbb{R}^n$,

$$\sup_{x \in B} |P^s_B f(x)| \le \frac{C}{|B|} \int_B |f(x)|\,dx.$$

(*iii*) *If* P *is a polynomial with degree at most* s, *then for all* $x \in \mathbb{R}^n$, $P^s_B P(x) = P(x)$.

Proposition 2.1. *Let* $\omega \in A_1(\mathbb{R}^n)$, $\alpha \in (0, \infty)$, s *be an integer satisfying* $s > \lfloor n\alpha \rfloor$ *and* $f \in L(\alpha, 1, s, \omega; \mathbb{R}^n)$. *Then there exists a polynomial* $P_f \in \mathcal{P}^s(\mathbb{R}^n)$ *such that* $f - P_f \in L(\alpha, 1, s-1, \omega; \mathbb{R}^n)$ *and* $\|f - P_f\|_{L(\alpha, 1, s-1, \omega; \mathbb{R}^n)} \le C\|f\|_{L(\alpha, 1, s, \omega; \mathbb{R}^n)}$, *where* $\mathcal{P}^s(\mathbb{R}^n)$ *denotes the set of all polynomials on* $\mathbb{R}^n$ *with degree at most* s *and* C *is a positive constant independent of* f.

Proof. We prove this proposition by following a procedure used in the proof of [10, Theorem 1]. First we fix $s > \lfloor n\alpha \rfloor$ and $f \in L(\alpha, 1, s, \omega; \mathbb{R}^n)$. By homogeneity, without loss of generality, we may assume that $\|f\|_{L(\alpha, 1, s, \omega; \mathbb{R}^n)} = 1$. For any fixed $x_0 \in \mathbb{R}^n$ and $r_0 \in (0, \infty)$, let $B \equiv B(x_0, r_0)$. We write

$$P^s_B f(x) \equiv \sum_{\{\nu \in \mathbb{Z}^n_+ :\, |\nu| \le s\}} a_\nu(B)(x - x_0)^\nu.$$

To prove Proposition 2.1, we need the following fact. For any balls $B_1, B_2 \subset \mathbb{R}^n$ with $B_1 \subset B_2$ and that the radius of B_2 is at most twice the radius of B_1, any $\nu \in \mathbb{Z}^n_+$ with $|\nu| = s$, by a slight modification of [10, Lemma 2.1], we have that

$$|a_\nu(B_1) - a_\nu(B_2)| \lesssim |B_1|^{-(s/n+1)}[\omega(B_1)]^{1+\alpha}. \tag{2.7}$$

For any $k \in \mathbb{N}$, let $B_1 \equiv B(x_0, 2^k)$ and $B_2 \equiv B(x_0, 2^{k+1})$. By (2.7), we have that

$$|a_\nu(B_1) - a_\nu(B_2)| \lesssim \frac{[\omega(B_1)]^{1+\alpha}}{|B_1|^{s/n+1}}.$$

Similarly, if $2^k < r \le 2^{k+1}$, then

$$|a_\nu(B(x_0, r)) - a_\nu(B_1)| \lesssim \frac{[\omega(B(x_0, r))]^{1+\alpha}}{|B(x_0, r)|^{s/n+1}}.$$

Hence, by (2.1), we have that

$$|a_\nu(B(x_0, r)) - a_\nu(B_1)| \lesssim [\omega(B(x_0, 1))]^{1+\alpha} r^{-(s-n\alpha)}.$$

Consequently, $a_\nu(x_0) \equiv \lim_{r\to\infty} a_\nu(B(x_0, r))$ exists and by (2.7) and (2.1),

$$
(2.8)\qquad \begin{aligned}
|a_\nu(x_0)-a_\nu(B)| &= \left|\sum_{k=0}^{\infty}[a_\nu(2^{k+1}B)-a_\nu(2^kB)]\right|\\
&\lesssim \frac{[\omega(B)]^{1+\alpha}}{|B|^{s/n+1}}\sum_{k=0}^{\infty}2^{-k(s-n\alpha)}\lesssim \frac{[\omega(B)]^{1+\alpha}}{|B|^{s/n+1}}.
\end{aligned}
$$

Next we show that $a_\nu(x_0)$ is independent of x_0. If $x_0\neq y_0$, we let $\widetilde{B}_1\equiv B(x_0,\widetilde{r})$ and $\widetilde{B}_2\equiv B(y_0,2\widetilde{r})$ with $\widetilde{r}>|y_0-x_0|$. Then $\widetilde{B}_1\subset\widetilde{B}_2$. By (2.7), (2.1) and $s>\lfloor n\alpha\rfloor$, we have that

$$
|a_\nu(\widetilde{B}_1)-a_\nu(\widetilde{B}_2)|\lesssim[\omega(B(x_0,1))]^{1+\alpha}\widetilde{r}^{-(s-n\alpha)}\to 0,
$$

as $\widetilde{r}\to\infty$. Thus, $a_\nu(x_0)=a_\nu(y_0)$. Therefore, $\lim_{r\to\infty}a_\nu(B(x_0,r))\equiv a_\nu$ exists, which is independent of x_0, and by (2.8),

$$
(2.9)\qquad |a_\nu(B)-a_\nu|\lesssim\frac{[\omega(B)]^{1+\alpha}}{|B|^{s/n+1}}.
$$

For all $x\in\mathbb{R}^n$, let $P_f(x)\equiv\sum_{\{\nu\in\mathbb{Z}_+^n:\,|\nu|=s\}}a_\nu x^\nu$. Notice that $P_f\in\mathcal{P}^s(\mathbb{R}^n)$. Let $Q_B(x)\equiv P_B^sf(x)-P_f(x)-\sum_{\{\nu\in\mathbb{Z}_+^n:\,|\nu|=s\}}[a_\nu(B)-a_\nu](x-x_0)^\nu$. Observe that $Q_B\in\mathcal{P}^{s-1}(\mathbb{R}^n)$ and for all $x\in\mathbb{R}^n$,

$$
[f(x)-P_f(x)]-Q_B(x)=[f(x)-P_B^sf(x)]+\sum_{\{\nu\in\mathbb{Z}_+^n:\,|\nu|=s\}}[a_\nu(B)-a_\nu](x-x_0)^\nu.
$$

Thus, by (2.9), we have

$$
\begin{aligned}
&\frac{1}{[\omega(B)]^{1+\alpha}}\int_B|[f(x)-P_f(x)]-Q_B(x)|\,dx\\
&\le\frac{1}{[\omega(B)]^{1+\alpha}}\int_B|[f(x)-P_B^sf(x)|\,dx\\
&\quad+\sup_{x\in B}\sum_{\{\nu\in\mathbb{Z}_+^n:\,|\nu|=s\}}\frac{|B|}{[\omega(B)]^{1+\alpha}}|a_\nu(B)-a_\nu||x-x_0|^{|\nu|}\lesssim 1,
\end{aligned}
$$

which together with Lemma 2.1(ii) and (iii) implies that

$$
\begin{aligned}
&\frac{1}{[\omega(B)]^{1+\alpha}}\int_B|[f(x)-P_f(x)]-P_B^{s-1}(f-P_f)(x)|\,dx\\
&\lesssim\frac{1}{[\omega(B)]^{1+\alpha}}\int_B|[f(x)-P_f(x)]-Q_B(x)|\,dx\lesssim 1.
\end{aligned}
$$

This finishes the proof of Proposition 2.1. ■

From Proposition 2.1, it further follows the equivalence of $L(\alpha,\,q,\,s,\,\omega;\,\mathbb{R}^n)$ on different s as below.

Theorem 2.2. *Let $\omega \in A_1(\mathbb{R}^n)$, $\alpha \in (0,\infty)$, s be an integer satisfying $s \ge \lfloor n\alpha \rfloor$ and $q \in [1,\infty]$. Then $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n) = L(\alpha,\, 1,\, \lfloor n\alpha \rfloor,\, \omega;\, \mathbb{R}^n)$ with equivalent norms.*

Proof. By Theorem 2.1, we only need to prove $L(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n) = L(\alpha,\, 1,\, \lfloor n\alpha \rfloor,\, \omega;\, \mathbb{R}^n)$. Notice that if $f \in L(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n)$ and $P_f \in \mathcal{P}^s(\mathbb{R}^n)$ is as in Proposition 2.1, then f and $f - P_f$ represent the same element of $L(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n)$. This observation combined with Proposition 2.1 implies that $L(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n) \subset L(\alpha,\, 1,\, \lfloor n\alpha \rfloor,\, \omega;\, \mathbb{R}^n)$ when $s \ge \lfloor n\alpha \rfloor$. On the other hand, by Lemma 2.1(ii) and (iii), we easily obtain that when $s \ge \lfloor n\alpha \rfloor$, $L(\alpha,\, 1,\, \lfloor n\alpha \rfloor,\, \omega;\, \mathbb{R}^n) \subset L(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n)$, which completes the proof of Theorem 2.2. ∎

Remark 2.2. Theorem 2.2 completely covers [10, Theorem 1] and [17, p. 132, (8.17)] when $\alpha \in (0,\infty)$ and [10, Theorem 2] by taking $\omega \equiv 1$, and [18, Theorem 2.1] by taking $\alpha \in (0, 1/n)$ and $s = 0$. Notice that when $\omega \equiv 1$, Remark 2.1(iii) implies that the restriction that $s \ge \lfloor n\alpha \rfloor$ in Theorem 2.2 is sharp.

Proposition 2.2. *Let $\omega \in A_1(\mathbb{R}^n)$, $\alpha \in (0,\infty)$, $s \in \mathbb{Z}_+$ and $\epsilon > \max\{\frac{s}{n}, \alpha\}$. Then there exists a positive constant C such that for all $f \in L(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n)$ and all balls $B \equiv B(x_0, r)$ with $x_0 \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\int_{\mathbb{R}^n} \frac{r^{n\epsilon}}{(r + |x - x_0|)^{n(1+\epsilon)}} |f(x) - P_B^s f(x)|\, dx \le C|B|^{-1}[\omega(B)]^{1+\alpha} \|f\|_{L(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n)}.$$

The proof of Proposition 2.2 is an obvious modification of [12, p. 59, Proposition 4.1] and we omit the details. As an application of Proposition 2.2, we have the following Remark 2.3, which is used in the proof of Theorem 3.1 below.

Remark 2.3. Let all the notation be the same as in Proposition 2.2 and $B \equiv B(0,1)$. Then by Proposition 2.2, for all $f \in L(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \frac{1}{(1 + |x|)^{n(1+\epsilon)}} |f(x) - P_B^s f(x)|\, dx \lesssim [\omega(B)]^{1+\alpha} \|f\|_{L(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n)},$$

which implies that $(1 + |x|)^{-n(1+\epsilon)} f \in L^1(\mathbb{R}^n)$ and hence, $f \in \mathcal{S}'(\mathbb{R}^n)$.

3. Equivalence between $L(\alpha,\, q,\, s, \omega;\, \mathbb{R}^n)$ and $\wedge(\alpha,\, \omega;\, \mathbb{R}^n)$

In this section, we establish the equivalence between $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ and $\wedge(\alpha,\, \omega;\, \mathbb{R}^n)$. Here is the main result of this section.

Theorem 3.1. *Let $\omega \in A_1(\mathbb{R}^n)$, $\alpha \in (0,\infty)$, s be an integer satisfying $s \ge \lfloor n\alpha \rfloor$ and $q \in [1,\infty]$. Then $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n) = \wedge(\alpha,\, \omega;\, \mathbb{R}^n)$ with equivalent norms.*

Proof. By Theorem 2.2, it suffices to show that $L(\alpha,\,1,\,\lfloor n\alpha\rfloor,\,\omega;\,\mathbb{R}^n) = \wedge(\alpha,\,\omega;\,\mathbb{R}^n)$ with equivalent norms. For simplicity, we write $s_0 \equiv \lfloor n\alpha\rfloor$ in the remaining part of this proof.

We first prove $L(\alpha,\,1,\,s_0,\,\omega;\,\mathbb{R}^n) \subset \wedge(\alpha,\,\omega;\,\mathbb{R}^n)$. Let $f \in L(\alpha,\,1,\,s_0,\,\omega;\,\mathbb{R}^n)$. By homogeneity, without loss of generality, we may assume that $\|f\|_{L(\alpha,\,1,\,s_0,\,\omega;\,\mathbb{R}^n)} = 1$. By Remark 2.3, we know that $f \in \mathcal{S}'(\mathbb{R}^n)$. To show that $f \in \wedge(\alpha,\,\omega;\,\mathbb{R}^n)$, we still need to prove that f satisfies (1.4). For any $j \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^n$, let $B \equiv B(x_0, 2^{-j})$. By the support condition of $\widehat{\Psi}$, it is easy to see that $\int_{\mathbb{R}^n} P(y)\Psi(y)\,dy = 0$ for any polynomials P on $\mathbb{R}^n$. Thus,

$$\Delta_j f(x_0) = f * \Psi_{2^{-j}}(x_0) = \int_{\mathbb{R}^n} [f(y) - P_B^{s_0} f(y)]\Psi_{2^{-j}}(x_0 - y)\,dy.$$

Since $\Psi \in \mathcal{S}(\mathbb{R}^n)$, then $|\Psi(x)| \lesssim (1+|x|)^{-n(2+\alpha)}$ for all $x \in \mathbb{R}^n$. Thus,

$$\begin{aligned} |\Delta_j f(x_0)| &\lesssim 2^{jn} \int_{\mathbb{R}^n} |f(y) - P_B^{s_0} f(y)|(1 + 2^j|x_0 - y|)^{-n(2+\alpha)}\,dy \\ &\lesssim 2^{jn} \int_{|x_0-y|<2^{-j}} |f(y) - P_B^{s_0} f(y)|\,dy \\ &\quad + 2^{jn} \sum_{k=1}^{\infty} 2^{-kn(2+\alpha)} \int_{|x_0-y|<2^{k-j}} |f(y) - P_B^{s_0} f(y)|\,dy \equiv \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

By Theorems 2.1 and 2.2, we have that for all $k \in \mathbb{Z}_+$ and almost all $y \in 2^k B$,

$$|f(y) - P_{2^kB}^{s_0} f(y)| \lesssim \omega(y)[\omega(2^k B)]^{\alpha}. \tag{3.1}$$

Hence, by (3.1) with $k = 0$, we obtain that $\mathrm{I}_1 \lesssim 2^{jn}[\omega(B(x_0, 2^{-j}))]^{1+\alpha}$.

Let $J_k \equiv \int_{|x_0-y|<2^{k-j}} |P_B^{s_0} f(y) - P_{2^kB}^{s_0} f(y)|\,dy$ for $k \in \mathbb{N}$. Then for I_2, by (3.1) with $k \in \mathbb{N}$ and (2.1), we have that

$$\begin{aligned} \mathrm{I}_2 &\lesssim 2^{jn} \sum_{k=1}^{\infty} 2^{-kn(2+\alpha)} \left[\int_{|x_0-y|<2^{k-j}} |f(y) - P_{2^kB}^{s_0} f(y)|\,dy + J_k\right] \\ &\lesssim 2^{jn} \sum_{k=1}^{\infty} 2^{-kn(2+\alpha)} \left\{2^{kn(1+\alpha)}[\omega(B)]^{1+\alpha} + J_k\right\} \\ &\lesssim 2^{jn} \left\{[\omega(B)]^{1+\alpha} + \sum_{k=1}^{\infty} 2^{-kn(2+\alpha)} J_k\right\}. \end{aligned} \tag{3.2}$$

Now we estimate $|P_B^{s_0} f(y) - P_{2^kB}^{s_0} f(y)|$ for almost all $y \in 2^k B$. Let $k \in \mathbb{N}$ and $B_k \equiv 2^k B$. Since $P_{B_k}^{s_0} f \in \mathcal{P}^{s_0}(\mathbb{R}^n)$, we write that for all $x \in \mathbb{R}^n$,

$$P_{B_k}^{s_0} f(x) \equiv \sum_{\{\nu \in \mathbb{Z}_+^n:\,|\nu| \le s_0\}} \frac{a_\nu(x_0, 2^{k-j})}{\nu!}(x - x_0)^\nu,$$

where $a_\nu(x_0, 2^{k-j}) = \partial_x^\nu(P_{B_k}^{s_0} f)(x_0)$. Thus,

$$\begin{aligned}
&\sup_{x\in B_k} |P_{B_k}^{s_0} f(x) - P_B^{s_0} f(x)| \\
&\le \sum_{\{\nu\in\mathbb{Z}_+^n:\,|\nu|\le s_0\}} \frac{2^{(k-j)|\nu|}}{\nu!} |a_\nu(x_0, 2^{k-j}) - a_\nu(x_0, 2^{-j})| \\
&\le \sum_{\{\nu\in\mathbb{Z}_+^n:\,|\nu|\le s_0\}} \frac{2^{(k-j)|\nu|}}{\nu!} \sum_{l=0}^{k-1} |a_\nu(x_0, 2^{l-j+1}) - a_\nu(x_0, 2^{l-j})|.
\end{aligned}$$

By the estimate in [12, p. 61, (4.3)] that for any polynomial $P \in \mathcal{P}^{s_0}(\mathbb{R}^n)$, ball $B_1 \equiv B(y_0, r_0)$ and multi-index ν satisfying $|\nu| \le s_0$,

$$|\partial_x^\nu(P)(y_0)| \le Cr_0^{-|\nu|} \frac{1}{|B_1|} \int_{B_1} |P(x)|\, dx, \tag{3.3}$$

where C is a positive constant only depending on s_0, (3.3) and (2.1), we have

$$\begin{aligned}
&\sup_{x\in B_k} |P_{B_k}^{s_0} f(x) - P_B^{s_0} f(x)| \\
&\lesssim \sum_{\{\nu\in\mathbb{Z}_+^n:\,|\nu|\le s_0\}} \frac{2^{(k-j)|\nu|}}{\nu\,!} \sum_{l=0}^{k-1} 2^{-(l-j)|\nu|} \frac{1}{|B_l|} \int_{B_l} |P_{B_{l+1}}^{s_0} f(y) - P_{B_l}^{s_0} f(y)|\, dy \\
&\lesssim \sum_{\{\nu\in\mathbb{Z}_+^n:\,|\nu|\le s_0\}} \frac{2^{(k-j)|\nu|}}{\nu!} \sum_{l=0}^{k-1} 2^{-(l-j)|\nu|} \frac{1}{|B_l|} \left[\int_{B_l} |P_{B_{l+1}}^{s_0} f(y) - f(y)|\, dy \right.\\
&\quad \left. + \int_{B_l} |P_{B_l}^{s_0} f(y) - f(y)|\, dy\right] \\
&\lesssim \sum_{\{\nu\in\mathbb{Z}_+^n:\,|\nu|\le s_0\}} \frac{2^{(k-j)|\nu|}}{\nu\,!} \sum_{l=0}^{k-1} 2^{-(l-j)|\nu|} \frac{1}{|B_l|} \left([\omega(B_{l+1})]^{1+\alpha} + [\omega(B_l)]^{1+\alpha}\right) \\
&\lesssim \begin{cases} |B|^{-1}[\omega(B)]^{1+\alpha} 2^{kn\alpha}, & n\alpha > s_0 \\ |B|^{-1}[\omega(B)]^{1+\alpha} k 2^{ks_0}, & n\alpha = s_0 \end{cases} \lesssim |B|^{-1}[\omega(B)]^{1+\alpha} k 2^{nk\alpha},
\end{aligned}$$

which together with the previous estimate (3.2) of I_2 yields that

$$\mathrm{I}_2 \lesssim 2^{jn}[\omega(B(x_0, 2^{-j}))]^{1+\alpha} \left\{1 + \sum_{k=1}^{\infty} k 2^{-kn}\right\} \lesssim 2^{jn}[\omega(B(x_0, 2^{-j}))]^{1+\alpha}.$$

By the arbitrariness of $j \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^n$, we obtain that $f \in \wedge(\alpha, \omega, \mathbb{R}^n)$ and $\|f\|_{\wedge(\alpha,\omega;\mathbb{R}^n)} \lesssim \|f\|_{L(\alpha,1,s_0,\omega;\mathbb{R}^n)}$. Thus, $L(\alpha,\, 1,\, s_0,\, \omega;\, \mathbb{R}^n) \subset \wedge(\alpha,\, \omega;\, \mathbb{R}^n)$.

Next we show that $\wedge(\alpha, \omega; \mathbb{R}^n) \subset L(\alpha, 1, s_0, \omega; \mathbb{R}^n)$ by borrowing some ideas from [14]. Let $f \in \wedge(\alpha, \omega; \mathbb{R}^n)$. By homogeneity, without loss of generality, we may assume that $\|f\|_{\wedge(\alpha, \omega; \mathbb{R}^n)} = 1$. By the Calderón reproducing formula obtained in [20, Lemma 2.1], we have $f = \sum_{j=-\infty}^{\infty} \Delta_j f$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$. Notice that $\Delta_j = \Delta_j(\Delta_{j-1} + \Delta_j + \Delta_{j+1})$. We further have

$$f = \sum_{j=-\infty}^{\infty} \Delta_j f_j \quad \text{in} \quad \mathcal{S}'_\infty(\mathbb{R}^n), \tag{3.4}$$

where $f_j \equiv (\Delta_{j-1} + \Delta_j + \Delta_{j+1})f$. From (1.4), it follows that for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,

$$|f_j(x)| \lesssim [\omega(B(x, 2^{-j}))]^{1+\alpha} 2^{jn}. \tag{3.5}$$

For all multi-indices ν, $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, by (3.5), (2.2), (2.1) and (2.3), we have

$$\begin{aligned}
&|\partial_x^\nu(\Delta_j f_j)(x)| \\
&\le \int_{\mathbb{R}^n} |f_j(y)|\, |\partial_x^\nu(\Psi_{2^{-j}})(x-y)|\, dy \\
&\lesssim 2^{j|\nu|} 2^{2jn} \int_{\mathbb{R}^n} [\omega(B(y, 2^{-j}))]^{1+\alpha} (1 + 2^j|x-y|)^{-n(2+\alpha)}\, dy \\
&\lesssim 2^{j|\nu|} 2^{2jn} \left\{ \int_{|x-y|<2^{-j}} [\omega(B(y, 2^{-j}))]^{1+\alpha}\, dy \right. \\
&\quad \left. + \sum_{k=1}^{\infty} 2^{-kn(2+\alpha)} \int_{|x-y|<2^{k-j}} [\omega(B(y, 2^{-j}))]^{1+\alpha}\, dy \right\} \\
&\lesssim 2^{j|\nu|} 2^{jn} [\omega(B(x, 2^{-j}))]^{1+\alpha} \left\{ 1 + \sum_{k=1}^{\infty} 2^{-kn\delta} \right\} \\
&\lesssim 2^{j|\nu|} 2^{jn} [\omega(B(x, 2^{-j}))]^{1+\alpha},
\end{aligned} \tag{3.6}$$

where δ is as in (2.2).

Now, for any given $x_0 \in \mathbb{R}^n$ and $m \in \mathbb{Z}$, and all $x \in \mathbb{R}^n$, let

$$\begin{aligned}
h_{x_0,m}(x) &\equiv \sum_{j=-\infty}^{-m} [\Delta_j f_j(x) - P_j^{x_0} f_j(x)] + \sum_{j=-(m-1)}^{\infty} \Delta_j f_j(x) \\
&\equiv \mathrm{I}_1(x) + \mathrm{I}_2(x),
\end{aligned} \tag{3.7}$$

where $P_j^{x_0} f_j$ denotes the Taylor polynomial of $\Delta_j f_j$ about $x - x_0$ with degree s_0. We now claim that for all $x_0 \in \mathbb{R}^n$ and $m \in \mathbb{Z}$,

$$\int_{B(x_0, 2^m)} |h_{x_0,m}(x)|\, dx \lesssim [\omega(B(x_0, 2^m))]^{1+\alpha}. \tag{3.8}$$

First we estimate $\mathrm{I}_1(x)$. For all $x \in B(x_0, 2^m)$, by the mean value theorem, (3.6), (2.1), (2.3) and $s_0 + n + 1 > n(1+\alpha)$, we have that

$$\begin{aligned}
|\mathrm{I}_1(x)| &\le \sum_{j=-\infty}^{-m} \left|\Delta_j f_j(x) - P_j^{x_0} f_j(x)\right| \\
&\lesssim \sum_{j=-\infty}^{-m} \sum_{\{\nu\in\mathbb{Z}_+^n:\ |\nu|=s_0+1\}} |\partial_x^\nu(\Delta_j f_j)(x_j)|\,|x-x_0|^{s_0+1} \\
&\lesssim 2^{m[s_0+1-n(1+\alpha)]}[\omega(B(x_0,2^m))]^{1+\alpha} \sum_{j=-\infty}^{-m} 2^{j(s_0+1-n\alpha)} \\
&\lesssim \frac{[\omega(B(x_0,2^m))]^{1+\alpha}}{|B(x_0,2^m)|},
\end{aligned}$$

where $x_j = \theta_j x + (1-\theta_j)x_0$ for certain $\theta_j \in (0,1)$. To estimate $\mathrm{I}_2(x)$, by (1.4), (2.2) and the definition of $\omega \in A_1(\mathbb{R}^n)$, we obtain that for almost all $x \in B(x_0, 2^m)$,

$$\begin{aligned}
|\mathrm{I}_2(x)| &\le \sum_{j=-(m-1)}^{\infty} |\Delta_j f_j(x)| \lesssim \sum_{j=-(m-1)}^{\infty} 2^{jn}[\omega(B(x,2^{-j}))]^{1+\alpha} \\
&\lesssim 2^{-mn\delta\alpha}[\omega(B(x,2^m))]^{\alpha}\omega(x) \sum_{j=-(m-1)}^{\infty} 2^{-jn\delta\alpha} \lesssim [\omega(B(x,2^m))]^{\alpha}\omega(x),
\end{aligned}$$

where δ is as in (2.2).

Thus,

$$\begin{aligned}
\int_{B(x_0,2^m)} |\mathrm{I}_1(x)+\mathrm{I}_2(x)|\,dx &\le \int_{B(x_0,2^m)} |\mathrm{I}_1(x)|\,dx + \int_{B(x_0,2^m)} |\mathrm{I}_2(x)|\,dx \\
&\lesssim [\omega(B(x_0,2^m))]^{1+\alpha},
\end{aligned}$$

namely, (3.8) holds.

By (3.4), we obtain that for all $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$\langle f, \varphi\rangle = \sum_{j=-\infty}^{\infty} \langle \Delta_j f_j, \varphi\rangle = \sum_{j=-\infty}^{-m} \langle \Delta_j f_j - P_j^{x_0} f_j, \varphi\rangle + \sum_{j=-(m-1)}^{\infty} \langle \Delta_j f_j, \varphi\rangle,$$

namely, $f = h_{x_0,m}$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ for all $x_0 \in \mathbb{R}^n$ and $m \in \mathbb{Z}$.

Next we claim that there exists a locally integrable function h such that $f = h$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and for all $x_0 \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\text{(3.9)} \qquad \frac{1}{[\omega(B(x_0,r))]^{1+\alpha}} \int_{B(x_0,r)} \left|h(x) - P_{B(x_0,r)}^{s_0} h(x)\right|\,dx \lesssim 1.$$

In this sense, we have that $f \in L(\alpha, 1, s_0, \omega; \mathbb{R}^n)$ and $\|f\|_{L(\alpha,1,s_0,\omega;\mathbb{R}^n)} \lesssim \|f\|_{\wedge(\alpha,\omega;\mathbb{R}^n)}$.

In fact, by taking $h \equiv h_{0,1}$, we then prove that h satisfies (3.9). Obviously, $f = h_{0,1}$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ by the construction of $h_{0,1}$ as above.

By (3.8) and Lemma 2.1(ii), we obtain that for any $y_0 \in \mathbb{R}^n$ and $m \in \mathbb{Z}$,

$$\int_{B(y_0,2^m)} \left| h_{y_0,m}(x) - P^{s_0}_{B(y_0,2^m)} h_{y_0,m}(x) \right| dx \lesssim [\omega(B(y_0, 2^m))]^{1+\alpha}. \tag{3.10}$$

For any $r \in (0, \infty)$, there exists $m_0 \in \mathbb{Z}$ such that $2^{m_0-1} \le r < 2^{m_0}$. By the constructions (3.7) of $h_{0,1}$ and h_{0,m_0}, we know that there exists a polynomial $P^{s_0}_{m_0}$ with degree s_0 such that $h_{0,1} = h_{0,m_0} + P^{s_0}_{m_0}$. By (3.8), (3.10), Lemma 2.1(ii) and (iii) and (2.3), we have

$$\begin{aligned}
&\frac{1}{[\omega(B(0,r))]^{1+\alpha}} \int_{B(0,r)} \left| h_{0,1}(x) - P^{s_0}_{B(0,r)} h_{0,1}(x) \right| dx \\
&= \frac{1}{[\omega(B(0,r))]^{1+\alpha}} \int_{B(0,r)} \left| h_{0,m_0}(x) - P^{s_0}_{B(0,r)} h_{0,m_0}(x) \right| dx \\
&\le \frac{1}{[\omega(B(0,r))]^{1+\alpha}} \int_{B(0,2^{m_0})} \left| h_{0,m_0}(x) - P^{s_0}_{B(0,2^{m_0})} h_{0,m_0}(x) \right| dx \\
&\quad + \frac{1}{[\omega(B(0,r))]^{1+\alpha}} \int_{B(0,r)} \left| P^{s_0}_{B(0,2^{m_0})} h_{0,m_0}(x) - P^{s_0}_{B(0,r)} h_{0,m_0}(x) \right| dx \\
&\lesssim \frac{[\omega(B(0,2^{m_0}))]^{1+\alpha}}{[\omega(B(0,2^{m_0-1}))]^{1+\alpha}} \lesssim 1.
\end{aligned}$$

For any fixed $x_0 \in \mathbb{R}^n$, from the constructions of $h_{0,1}$ and h_{x_0,m_0}, it follows that

$$h_{0,m_0} - h_{x_0,m_0} = \sum_{j=-\infty}^{-m_0} \left(P^{x_0}_j f_j - P^0_j f_j \right).$$

Now we show that $\sum_{j=-\infty}^{-m_0} (P^{x_0}_j f_j - P^0_j f_j)$ is a polynomial with degree at most s_0. Similarly to the estimate of $\mathrm{I}_1(x)$, we have that for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\sum_{j=-\infty}^{-m_0} \left| P^{x_0}_j f_j(x) - P^0_j f_j(x) \right| \\
&\le \sum_{j=-\infty}^{-m_0} \left| \Delta_j f_j(x) - P^{x_0}_j f_j(x) \right| + \sum_{j=-\infty}^{-m_0} \left| \Delta_j f_j(x) - P^0_j f_j(x) \right| \\
&\lesssim 2^{-m_0(s_0+n+1)} \left\{ [\omega(B(x^1_j, 2^{m_0}))]^{1+\alpha} |x - x_0|^{s_0+1} + [\omega(B(x^2_j, 2^{m_0}))]^{1+\alpha} |x|^{s_0+1} \right\},
\end{aligned}$$

where $x^1_j = \theta^1_j x_0 + (1 - \theta^1_j) x$, $x^2_j = \theta^2_j x$, $\theta^1_j \in (0,1)$ depends on x_0 and x, and $\theta^2_j \in (0,1)$ depends on x. By the estimate just above, we know that $\sum_{j=-\infty}^{-m_0} [P^{x_0}_j f_j(x) -$

$P_j^0 f_j(x)]$ is absolutely convergent and locally integrable. Hence, for any ball $B \subset \mathbb{R}^n$, by Lemma 2.1(i) and (iii), and the Lebesgue dominated convergence theorem, we obtain that for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
&P_B^{s_0}\left(\sum_{j=-\infty}^{-m_0}[P_j^{x_0}f_j - P_j^0 f_j]\right)(x)\\
&= \sum_{\{l\in\mathbb{Z}_+^n:\,|l|\le s_0\}}\left\{\frac{1}{|B|}\int_B \sum_{j=-\infty}^{-m_0}[P_j^{x_0}f_j(y) - P_j^0 f_j(y)]\varphi_l^B(y)\,dy\right\}\varphi_l^B(x)\\
&= \sum_{j=-\infty}^{-m_0}\sum_{\{l\in\mathbb{Z}_+^n:\,|l|\le s_0\}}\left\{\frac{1}{|B|}\int_B [P_j^{x_0}f_j(y) - P_j^0 f_j(y)]\varphi_l^B(y)\,dy\right\}\varphi_l^B(x)\\
&= \sum_{j=-\infty}^{-m_0} P_B^{s_0}\left(P_j^{x_0}f_j - P_j^0 f_j\right)(x) = \sum_{j=-\infty}^{-m_0}\left[P_j^{x_0}f_j(x) - P_j^0 f_j(x)\right].
\end{aligned}$$

Thus, $\sum_{j=-\infty}^{-m_0}(P_j^{x_0}f_j - P_j^0 f_j)$ is a polynomial with degree at most s_0. Let

$$P_{x_0,0}^{m_0} \equiv \sum_{j=-\infty}^{-m_0}\left(P_j^{x_0}f_j - P_j^0 f_j\right).$$

Then, $h_{0,m_0} - h_{x_0,m_0} = P_{x_0,0}^{m_0}$ for all $x_0 \in \mathbb{R}^n$. Thus, $h_{0,1} = h_{0,m_0} + P_{m_0}^{s_0} = h_{x_0,m_0} + P_{x_0,0}^{m_0} + P_{m_0}^{s_0}$, which together with Lemma 3.1(iii), (3.10) and (2.3) further implies that for all $x_0 \in \mathbb{R}^n$,

$$\begin{aligned}
&\frac{1}{[\omega(B(x_0,r))]^{1+\alpha}}\int_{B(x_0,r)}\left|h_{0,1}(x) - P_{B(x_0,r)}^{s_0}h_{0,1}(x)\right|\,dx\\
&= \frac{1}{[\omega(B(x_0,r))]^{1+\alpha}}\int_{B(x_0,r)}\left|h_{x_0,m_0}(x) - P_{B(x_0,r)}^{s_0}h_{x_0,m_0}(x)\right|\,dx \lesssim 1.
\end{aligned}$$

Thus, (3.9) holds, which completes the proof of Theorem 3.1. ■

4. Equivalence between $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ and $L_\varphi(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$

We begin with two technical lemmas, which are from [18, Propositions 3.1 and 3.2], respectively.

Lemma 4.1. *Let* $\alpha \in (0,\infty)$, $s \in \mathbb{Z}_+$, $\omega \in A_1(\mathbb{R}^n)$ *and* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *satisfy the condition* (A_1). *Then there exists a positive constant* C *such that for all* $t \in (0,\infty)$, $K \in (1,\infty)$, $f \in L_\varphi(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n)$ *and almost all* $x \in \mathbb{R}^n$,

$$|\varphi_t * f(x) - \varphi_{Kt} * f(x)| \le C[\omega(B(x,Kt))]^\alpha \frac{\omega(B(x,t))}{|B(x,t)|}\|f\|_{L_\varphi(\alpha,\,1,\,s,\,\omega;\,\mathbb{R}^n)}.$$

Lemma 4.2. *Let $\alpha,\ s,\ \omega$ and φ be the same as in Lemma 4.1. Then there exists a positive constant C such that for all $t \in (0,\infty)$, $f \in L_{\varphi}(\alpha,\ 1,\ s,\ \omega;\ \mathbb{R}^n)$ and almost all $x \in \mathbb{R}^n$,*

$$|\varphi_t * (|f-\varphi_t * f|)(x)| \le C\frac{[\omega(B(x,t))]^{1+\alpha}}{|B(x,t)|}\|f\|_{L_{\varphi}(\alpha,\,1,\,s,\,\omega;\,\mathbb{R}^n)}.$$

On the equivalence of the spaces $L_{\varphi}(\alpha,\ q,\ s,\ \omega;\ \mathbb{R}^n)$ with respect to different q, we have the following conclusion.

Theorem 4.1. *Let $\alpha,\ s,\ \omega$ and φ be the same as in Lemma 4.1. Then the following propositions* (I), (II) *and* (III) *are equivalent:*

(*I*) $f \in L_{\varphi}(\alpha,\ 1,\ s,\ \omega;\ \mathbb{R}^n)$;

(*II*) $f \in L_{\varphi}(\alpha,\ \infty,\ s,\ \omega;\ \mathbb{R}^n)$;

(*III*) *For any given* $q \in [1,\infty)$, $f \in L_{\varphi}(\alpha,\ q,\ s,\ \omega;\ \mathbb{R}^n)$.

Moreover, their norms are equivalent with equivalent constants independent of f.

Proof. (I) $\Rightarrow$ (II). Let $f \in L_{\varphi}(\alpha,\ 1,\ s,\ \omega;\ \mathbb{R}^n)$. By homogeneity, without loss of generality, we may assume that $\|f\|_{L_{\varphi}(\alpha,\,1,\,s,\,\omega;\,\mathbb{R}^n)} = 1$. For any ball B, let $x \in B$ be a Lebesgue point of both f and ω. Let $B_0 \equiv B$ and $B_j \equiv B(x, 2^{-j}t_B)$ for $j \in \mathbb{N}$. Then

$$\begin{aligned}
|f(x)-\varphi_{t_B} * f(x)| &= \lim_{j\to\infty}\frac{1}{|B_j|}\int_{B_j}|f(y)-\varphi_{t_B} * f(y)|\,dy \\
&\le \limsup_{j\to\infty}\frac{1}{|B_j|}\int_{B_j}|f(y)-\varphi_{t_{B_j}} * f(y)|\,dy \\
&\quad + \limsup_{j\to\infty}\frac{1}{|B_j|}\int_{B_j}|\varphi_{t_B} * f(y)-\varphi_{t_{B_j}} * f(y)|\,dy \equiv \mathrm{I}_1+\mathrm{I}_2.
\end{aligned}$$

For I_1, by the definition of the Lebesgue point, we have

$$\mathrm{I}_1 \lesssim \limsup_{j\to\infty}\frac{[\omega(B_j)]^{1+\alpha}}{|B_j|} = 0.$$

Next, we estimate I_2. By Lemma 4.2 and (2.3), we know that for all $l \in \{1,\cdots,j\}$ and almost all $y \in B_j$,

$$|\varphi_{t_{B_l}} * f(y)-\varphi_{t_{B_{l-1}}} * f(y)| \lesssim \frac{[\omega(B(y,t_{B_{l-1}}))]^{1+\alpha}}{|B(y,t_{B_{l-1}})|} \lesssim \frac{[\omega(B(x,t_{B_{l-1}}))]^{1+\alpha}}{|B(x,t_{B_{l-1}})|},$$

which together with (2.2) and the definition of $\omega \in A_1(\mathbb{R}^n)$ yields that

$$\begin{aligned}
&\frac{1}{|B_j|}\int_{B_j} |\varphi_{t_B} * f(y) - \varphi_{t_{B_j}} * f(y)|\,dy \\
&\le \sum_{l=1}^{j} \frac{1}{|B_j|}\int_{B_j} |\varphi_{t_{B_l}} * f(y) - \varphi_{t_{B_{l-1}}} * f(y)|\,dy \lesssim \sum_{l=1}^{j} \frac{[\omega(B(x,t_{B_{l-1}}))]^{1+\alpha}}{|B(x,t_{B_{l-1}})|} \\
&\lesssim [\omega(B(x,t_B))]^{\alpha}\omega(x)\sum_{l=1}^{j} 2^{-(l-1)n\delta} \lesssim [\omega(B(x,t_B))]^{\alpha}\omega(x),
\end{aligned}$$

where δ is as in (2.2). Thus, we obtain that $\mathrm{I}_2 \lesssim [\omega(B(x,t_B))]^{\alpha}\omega(x)$. Therefore, for almost all $x \in B$, $|f(x) - \varphi_{t_B} * f(x)| \lesssim [\omega(B(x,t_B))]^{\alpha}\omega(x)$, which implies that $f \in L_{\varphi}(\alpha, \infty, s, \omega; \mathbb{R}^n)$ and $\|f\|_{L_{\varphi}(\alpha, \infty, s, \omega; \mathbb{R}^n)} \lesssim \|f\|_{L_{\varphi}(\alpha, 1, s, \omega; \mathbb{R}^n)}$. Hence, we obtain (II).

(II) $\Rightarrow$ (III). In this case, let $q \in [1,\infty)$ and f satisfy (II). By homogeneity, without loss of generality, we may assume that $\|f\|_{L_{\varphi}(\alpha, \infty, s, \omega; \mathbb{R}^n)} = 1$. By (2.3), we then have that for any ball $B \equiv B(x_0, r)$,

$$\begin{aligned}
&\frac{1}{[\omega(B)]^{\alpha}}\left[\frac{1}{\omega(B)}\int_B |f(x) - \varphi_{t_B} * f(x)|^q [\omega(x)]^{1-q}\,dx\right]^{1/q} \\
&\lesssim \frac{1}{[\omega(B)]^{\alpha}}\left[\frac{1}{\omega(B)}\int_B [\omega(x)]^q [\omega(B(x,r))]^{\alpha q} [\omega(x)]^{1-q}\,dx\right]^{1/q} \lesssim 1,
\end{aligned}$$

which implies that $f \in L_{\varphi}(\alpha, q, s, \omega; \mathbb{R}^n)$ and $\|f\|_{L_{\varphi}(\alpha,q,s,\omega;\mathbb{R}^n)} \lesssim \|f\|_{L_{\varphi}(\alpha,\infty,s,\omega;\mathbb{R}^n)}$. Hence, we obtain (III).

(III) $\Rightarrow$ (I). In this case, from Hölder's inequality and (III), we immediately deduce (I), which completes the proof of Theorem 4.1. ∎

Remark 4.4. Theorem 4.1 when $\alpha \in (0, 1/n)$ and $s = 0$ completely covers [18, Theorem 3.4].

Next, we clarify the relations between $L(\alpha, q, s, \omega; \mathbb{R}^n)$ and $L_{\varphi}(\alpha, q, s, \omega; \mathbb{R}^n)$. We first establish the following embedding theorem.

Theorem 4.2. *Let $\alpha \in (0,\infty)$, $\omega \in A_1(\mathbb{R}^n)$ and $s \in \mathbb{Z}_+$. Assume that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies the condition (A_1). Then $L(\alpha, q, s, \omega; \mathbb{R}^n) \subset L_{\varphi}(\alpha, q, s, \omega; \mathbb{R}^n)$ for all $q \in [1,\infty]$.*

Proof. By Theorem 2.1 and Theorem 4.1, it suffices to prove that $L(\alpha, 1, s, \omega; \mathbb{R}^n) \subset L_{\varphi}(\alpha, 1, s, \omega; \mathbb{R}^n)$.

Let $f \in L(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n)$ and $B \equiv B(x_0, t_B)$ be a fixed ball centered at x_0 and of radius t_B. By homogeneity, without loss of generality, we may assume that $\|f\|_{L(\alpha,\,1,\,s,\,\omega;\,\mathbb{R}^n)} = 1$. Since φ satisfies the condition (A_1), we obtain that for all $t \in (0,\infty)$, $x \in \mathbb{R}^n$ and multi-indices θ with $0 \le |\theta| \le s$,

$$\int_{\mathbb{R}^n} \varphi_t(x-y) y^\theta\, dy = \int_{\mathbb{R}^n} \varphi(y)(x-ty)^\theta\, dy = x^\theta.$$

This gives $\varphi_{t_B} * P_B^s f \equiv P_B^s f$. Thus,

$$|f(x) - \varphi_{t_B} * f(x)| \le |f(x) - P_B^s f(x)| + |\varphi_{t_B} * (f - P_B^s f)(x)|.$$

Notice that for all $x \in B$,

$$\begin{aligned} |\varphi_{t_B} * (f - P_B^s f)(x)| &\le \int_{\mathbb{R}^n} |\varphi_{t_B}(x-y)||f(y) - P_B^s f(y)|\, dy \\ &\le \int_{\mathbb{R}^n} \left\{ \frac{t_B^{\epsilon n}}{(t_B + |x-y|)^{n(1+\epsilon)}} |f(y) - P_B^s f(y)| \right\} \\ &\quad \times \left\{ \frac{(t_B + |x-y|)^{n(1+\epsilon)}}{t_B^{\epsilon n}} |\varphi_{t_B}(x-y)| \right\} dy, \end{aligned} \tag{4.1}$$

where ϵ is a positive constant satisfying $\epsilon > \max\{\frac{s}{n},\, \alpha\}$. By Proposition 2.2, we have

$$\int_{\mathbb{R}^n} \frac{t_B^{\epsilon n}}{(t_B + |x-y|)^{n(1+\epsilon)}} |f(y) - P_B^s f(y)|\, dy \lesssim |B|^{-1}[\omega(B)]^{1+\alpha}.$$

By $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and the fact that $\frac{(t_B+|x-y|)^{n(1+\epsilon)}}{t_B^{\epsilon n}} |\varphi_{t_B}(x-y)| = (1+\frac{|x-y|}{t_B})^{n(1+\epsilon)} |\varphi(\frac{|x-y|}{t_B})|$, we obtain that

$$\operatorname*{esssup}_{y \in \mathbb{R}^n} \frac{(t_B + |x-y|)^{n(1+\epsilon)}}{t_B^{\epsilon n}} |\varphi_{t_B}(x-y)| \lesssim 1.$$

This combined with (4.1) and the definition of $\omega \in A_1(\mathbb{R}^n)$ implies that for almost all $x \in B$, we have that $|\varphi_{t_B} * (f - P_B^s f)(x)| \lesssim |B|^{-1}[\omega(B)]^{1+\alpha} \lesssim [\omega(B)]^\alpha \omega(x)$. From this, it further follows that for almost all $x \in B$,

$$|f(x) - \varphi_{t_B} * f(x)| \le |f(x) - P_B^s f(x)| + |\varphi_{t_B} * (f - P_B^s f)(x)| \lesssim [\omega(B)]^\alpha \omega(x).$$

Then applying Theorem 4.1 yields that $f \in L_\varphi(\alpha,\, 1,\, s,\, \omega;\, \mathbb{R}^n)$ and $\|f\|_{L_\varphi(\alpha,\,1,\,s,\,\omega;\,\mathbb{R}^n)} \lesssim \|f\|_{L(\alpha,\,1,\,s,\,\omega;\,\mathbb{R}^n)}$, which completes the proof of Theorem 4.2. ■

Next, we show that the equivalence between $L(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$ and $L_\varphi(\alpha,\, q,\, s,\, \omega;\, \mathbb{R}^n)$.

Theorem 4.3. *Let* $\alpha \in (0, \infty)$, $s \in \mathbb{Z}_+$ *satisfy* $s \geq \lfloor n\alpha \rfloor$, $\omega \in A_1(\mathbb{R}^n) \cap RH_{1+1/\alpha}(\mathbb{R}^n)$ *and* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *satisfy the conditions* (A_1) *and* (A_2). *Then for any* $q \in [1, \infty]$, $L(\alpha, q, s, \omega; \mathbb{R}^n) = L_\varphi(\alpha, q, s, \omega; \mathbb{R}^n)$ *with equivalent norms.*

Proof. By Theorem 4.2, we know that $L(\alpha, q, s, \omega; \mathbb{R}^n) \subset L_\varphi(\alpha, q, s, \omega; \mathbb{R}^n)$. It remains to prove $L_\varphi(\alpha, q, s, \omega; \mathbb{R}^n) \subset L(\alpha, q, s, \omega; \mathbb{R}^n)$. By Theorems 2.1 and 3.1, it suffices to prove that $L_\varphi(\alpha, 1, s, \omega; \mathbb{R}^n) \subset \wedge(\alpha, \omega; \mathbb{R}^n)$. By the fact that $\mathcal{S}'(\mathbb{R}^n)$ is a subset of $\mathcal{S}'_\infty(\mathbb{R}^n)$, it suffices to prove that if $f \in L_\varphi(\alpha, 1, s, \omega; \mathbb{R}^n)$, then it satisfies (1.4), whose proof is as in the proof of [18, Theorem 4.2] and we omit the details. This finishes the proof of Theorem 4.3. ∎

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