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WEIGHTED BLOCH, LIPSCHITZ, ZYGMUND, BERS, AND GROWTH SPACES OF THE BALL: BERGMAN PROJECTIONS AND CHARACTERIZATIONS

H. Turgay Kaptanoğlu and Serdar Tülü

Abstract. We determine precise conditions for the boundedness of Bergman projections from Lebesgue classes onto the spaces in the title, which are members of the same one-parameter family of spaces. The projections provide integral representations for the functions in the spaces. We obtain many properties of the spaces as straightforward corollaries of the projections, integral representations, and isometries among the spaces. We solve the Gleason problem and an extremal problem for point evaluations in each space. We establish maximality of these spaces among those that exhibit Möbius-type invariances and possess decent functionals. We find new Hermitian non-Kählerian metrics that characterize half of these spaces by Lipschitz-type inequalities.

1. Introduction

Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^N$ with respect to the usual hermitian inner product $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_N\overline{w}_N$ and the norm $|z| = \sqrt{\langle z, z\rangle}$. Let $H(\mathbb{B})$ denote the space of holomorphic functions on $\mathbb{B}$ and H^∞ its subclass of bounded functions.

We let ν be the Lebesgue measure on $\mathbb{B}$ normalized so that $\nu(\mathbb{B}) = 1$, which is the normalized area measure on the unit disc $\mathbb{D}$ when $N = 1$. For $q \in \mathbb{R}$, we also define on $\mathbb{B}$ the measures

$$d\nu_q(z) = (1 - |z|^2)^q \, d\nu(z).$$

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For $0 < p < \infty$, we denote the Lebesgue classes with respect to ν_q by L^p_q. The Lebesgue class of essentially bounded functions on $\mathbb{B}$ with respect to any ν_q is the same (see [10, Proposition 2.3]); we denote it by L^∞. For $\alpha \in \mathbb{R}$, we also define the weighted classes

$$L^\infty_\alpha = \{\, \varphi \text{ measurable on } \mathbb{B} : (1 - |z|^2)^\alpha\, \varphi(z) \in L^\infty \,\}.$$

Let's also call the subspace of L^∞_α consisting of holomorphic functions H^∞_α.

We use $\mathcal{C}_0$ to denote the space of continuous functions on the closure $\overline{\mathbb{B}}$ and $\mathcal{C}_{00}$ its subspace of those that vanish on the boundary $\partial\mathbb{B}$. We also define

$$\mathcal{C}_\alpha = \{\, \varphi \in \mathcal{C}_0 : (1 - |z|^2)^\alpha\, \varphi(z) \in \mathcal{C}_0 \,\}$$

and

$$\mathcal{C}_{\alpha 0} = \{\, \varphi \in \mathcal{C}_0 : (1 - |z|^2)^\alpha\, \varphi(z) \in \mathcal{C}_{00} \,\}.$$

Further, the *ball algebra* is $A(\mathbb{B}) = H(\mathbb{B}) \cap \mathcal{C}_0$.

Almost all results in this work depend on certain radial differential operators D^t_s of order $t \in \mathbb{R}$ for any $s \in \mathbb{R}$ that map $H(\mathbb{B})$ to itself defined in detail in [10, Definition 3.1]. Consider the linear transformation I^t_s defined for $f \in H(\mathbb{B})$ by

$$I^t_s f(z) = (1 - |z|^2)^t\, D^t_s f(z).$$

Definition 1.1. For any $\alpha \in \mathbb{R}$, we define the *weighted Bloch space* $\mathcal{B}_\alpha$ to consist of all $f \in H(\mathbb{B})$ for which $I^t_s f$ belongs to L^∞_α for some s, t satisfying

$$\alpha + t > 0. \tag{1}$$

The *weighted little Bloch space* $\mathcal{B}_{\alpha 0}$ is the subspace of $\mathcal{B}_\alpha$ consisting of those f for which $I^t_s f$ lies in $\mathcal{C}_{\alpha 0}$ for some s, t satisfying (1).

Condition (1) ascertains that all $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha 0}$ contain the polynomials and therefore are nontrivial. The spaces $\mathcal{B}_0$ and $\mathcal{B}_{00}$ are the usual Bloch and little Bloch spaces. In notation concerning $\mathcal{C}$ and $\mathcal{B}$, a single subscript indicates boundedness, and double subscripts, the second of which is always 0, indicate vanishing on the boundary.

Our use of α is nontraditional, follows [12, Section 8], conforms with the notation of closely related Besov spaces, and is more logical in view of the operators I^t_s. Most other authors use $\alpha - 1$ where we use α.

Definition 1.1 can be shown to be independent of s, t satisfying (1) using the methods of [2, p. 41]; note that s does not affect the order of the radial differential operator D^t_s. Similarly, these spaces can be defined using other kinds of derivatives; see also [26, Chapter 7]. We show independence from s and t essentially under (1) and (2) in Corollaries 3.4 and 3.5 below as easy consequences of Bergman projections and other similar integral operators.

Definition 1.2. For $s \in \mathbb{R}$ and $z, w \in \mathbb{B}$, the generalized *Bergman-Besov kernels* are

$$K_s(z,w) = \begin{cases} \dfrac{1}{(1-\langle z,w\rangle)^{N+1+s}} = \displaystyle\sum_{k=0}^{\infty} \frac{(N+1+s)_k}{k!} \langle z,w\rangle^k, & \text{if } s > -(N+1); \\ \dfrac{{}_2F_1(1,1;1-N-s;\langle z,w\rangle)}{-N-s} = \displaystyle\sum_{k=0}^{\infty} \frac{k!\,\langle z,w\rangle^k}{(-N-s)_{k+1}}, & \text{if } s \le -(N+1); \end{cases}$$

and the extended *Bergman projections* are

$$P_s\varphi(z) = \int_{\mathbb{B}} K_s(z,w)\,\varphi(w)\,d\nu_s(w)$$

for suitable φ.

Above, ${}_2F_1$ is the hypergeometric function, and $(a)_b$ is the *Pochhammer symbol* given by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function Γ. The presentations of K_s and P_s follow those in [2, Section 1] and [10]. Note that $K_s(\cdot,w) \in H(\mathbb{B})$ and thus $P_s f \in H(\mathbb{B})$ whenever the integral exists. Throughout, s and t can take complex values too as done in [6] and [10].

The following is our first main result.

Theorem 1.3. *$P_s : L^\infty_\alpha \to \mathcal{B}_\alpha$ is bounded if and only if*

$$\alpha < s + 1. \tag{2}$$

Given an s satisfying (2)*, if t satisfies* (1)*, then for $f \in \mathcal{B}_\alpha$,*

$$P_s I_s^t f = \frac{N!}{(1+s+t)_N} f =: \frac{1}{C_{s+t}} f. \tag{3}$$

Also either $P_s : \mathcal{C}_\alpha \to \mathcal{B}_{\alpha 0}$ or $P_s : \mathcal{C}_{\alpha 0} \to \mathcal{B}_{\alpha 0}$ is bounded if and only if (2) *holds.*

Note that (1) and (2) together imply $s+t > -1$ so that C_{s+t} makes sense. Thus $P_s : L^\infty_\alpha \to \mathcal{B}_\alpha$ is surjective and $I_s^t : \mathcal{B}_\alpha \to L^\infty_\alpha$ is an imbedding. For s, t satisfying (2) and (1), each of $C_{s+t} I_s^t$ is a right inverse for P_s on L^∞_α, and $C_{s+t}^{-1} P_s$ is a left inverse for each of I_s^t on $\mathcal{B}_\alpha$. Similar statements hold for the "little" spaces. Moreover, (3) is a family of integral representations for $f \in \mathcal{B}_\alpha$ which take the form

$$f(z) = \frac{(1+s+t)_N}{N!} \int_{\mathbb{B}} K_s(z,w)\,(1-|w|^2)^{s+t}\,D_s^t f(w)\,d\nu(w) \qquad (z \in \mathbb{B}) \tag{4}$$

when written explicitly.

The case $\alpha = 0$ of Theorem 1.3 has been treated earlier in [4, Theorem 2], [10, Corollary 5.3], and [13, Corollary 5.3]. Bergman projections P_s for $s > -1$ from Lebesgue classes of the form $L^\infty_\alpha \cap L^p_s$ with $1 \le p < \infty$ and $\alpha > 0$ to $\mathcal{B}_\alpha$, and between $\mathcal{B}_\alpha$ with $-1 < \alpha < 0$, have been considered in [3]. Bergman-type projections, in which K_{s_1} is used with ν_{s_2}, from L^∞ or $\mathcal{C}_0$ to $\mathcal{B}_\alpha$ or $\mathcal{B}_{\alpha 0}$ have been considered in several theorems dealing with various ranges of α in [26, Chapter 7]. Neither of the last two sources gives a necessary condition on the parameters or a right inverse. Only [26] handles $s_1 \le -(N+1)$, but with the restriction that it is an integer. Further, [8, Theorem 3] obtains the case $s = \alpha > -1$ and $t = 1$ of (4) with a method that does not make use of the idea of a Bergman projection.

Having integral representations is very fruitful, and we exploit (4) and Theorem 1.3 to extract many properties of the weighted Bloch spaces in Sections 3-6, giving also easier proofs of a few known facts. But several properties of these spaces require other considerations. A recurring theme is using kernels to define a new concept on one space and then using radial differential operators to carry them to the remaining spaces.

Consider the *extremal problem* of determining

$$S_\alpha(b) := \sup \left\{ f(b) > 0 \, : \, f \in \mathcal{B}_\alpha, \ \|f\|_{\mathcal{B}_\alpha} = \|I^t_s f\|_{L^\infty_\alpha} = 1 \right\}, \tag{5}$$

for each $b \in \mathbb{B}$ and if possible finding a function realizing it. Note that $S_\alpha(b)$ also depends on s, t satisfying (1). There is also the problem of determining $S_{\alpha 0}(b)$ in which f is allowed to vary only in $\mathcal{B}_{\alpha 0}$.

Theorem 1.4. *For any α, the extremal function attaining $S_\alpha(b)$ exists and is unique. This solution is also the solution for $S_{\alpha 0}(b)$.*

The proof of Theorem 1.4 depends on the following construction. For $\alpha \ge 0$, define the linear transformations

$$T^\alpha_\psi f(z) = f(\psi(z)) \, (J\psi(z))^{2\alpha/(N+1)}, \tag{6}$$

where ψ is a holomorphic automorphism of $\mathbb{B}$, J denotes the complex Jacobian, and an appropriate fixed branch of the logarithm is used for the fractional power. We extend T^α_ψ to all α by setting

$$W^\alpha_\psi = D^{-t}_{s+t} T^{\alpha+t}_\psi D^t_s, \tag{7}$$

where s, t satisfy (1).

Definition 1.5. Let $(X, \|\cdot\|)$ be a Banach space of holomorphic functions on $\mathbb{B}$ containing the constants. We call X an *α-Möbius-invariant* space if $W^\alpha_\psi f \in X$ for some s, t satisfying (1) whenever $f \in X$, $\|W^\alpha_\psi f\| \le C \, \|f\|$, and the action $\psi \longmapsto W^\alpha_\psi f$ is continuous for $f \in X$ and unitary ψ.

Theorem 1.6. *The space $\mathcal{B}_\alpha$ contains with continuous inclusion those α-Möbius-invariant spaces that possess a decent linear functional.*

All our results mentioned so far including their consequences and applications are general and cover all real values of α. If no range for α is specified, that means $\alpha \in \mathbb{R}$ is arbitrary. Most of the results are completely new for the spaces $\mathcal{B}_\alpha$ with $\alpha < 0$, which are actually the Lipschitz spaces $\Lambda_{-\alpha}$ as explained in the next section. The original definition of these spaces for $-1 < \alpha < 0$ states that $\Lambda_{-\alpha}$ is the space of holomorphic functions f on $\mathbb{B}$ satisfying the so-called *Lipschitz condition* $|f(z) - f(w)| \leq C\,|z - w|^{-\alpha}$ for all $z, w \in \mathbb{B}$. For $\alpha = 0$, the corressponding equivalent condition is $|f(z) - f(w)| \leq C\,\rho_0(z, w)$, where ρ_0 is the Bergman metric; see [19, Theorem 3.4 (3)]. We extend this condition to all $\alpha > 0$ by finding new metrics ρ_α in $\mathbb{B}$ in place of the Euclidean or the Bergman metrics.

Theorem 1.7. *For each $\alpha > 0$, there exists a complete Hermitian non-Kählerian metric ρ_α on $\mathbb{B}$ such that if $f \in \mathcal{B}_\alpha$, then $|f(z) - f(w)| \leq C\,\rho_\alpha(z, w)$. The converse also holds for $N = 1$.*

The next section gives some preparatory material on the spaces under consideration and the tools to be used. We prove Theorem 1.3 in Section 3. In Sections 4 and 5, we apply Theorem 1.3 and (3) to a solution of the Gleason problem in $\mathcal{B}_\alpha$, to the growth of functions in $\mathcal{B}_\alpha$ near $\partial\mathbb{B}$, and to the growth of their Taylor series coefficients. In Section 6, we exhibit pairings that yield (pre)duality relationship between the Besov spaces B^1_q and the spaces $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha 0}$, and find the complex interpolation space between two weighted Bloch spaces, again by applying Theorem 1.3. In Section 7, we prove Theorem 1.4 by determining explicitly the extremal functions. We prove Theorem 1.6 in Section 8, where a decent linear functional is also defined. In the final Section 9, we define some new Hermitian metrics similar to the hyperbolic metric that are specific to the $\mathcal{B}_\alpha$ spaces and prove Theorem 1.7.

2. Preliminaries

Stirling formula gives

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b} \qquad \text{and} \qquad \frac{(a)_c}{(b)_c} \sim c^{a-b} \qquad (\operatorname{Re} c \to \infty), \tag{8}$$

where $x \sim y$ means that $|x/y|$ is bounded above and below by two positive constants. Such constants are always independent any parameters or functions in the formulas and are all denoted by the generic upper case C.

We occasionally use multi-index notation in which $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{N}^N$ is an N-tuple of nonnegative integers, $|\lambda| = \lambda_1 + \cdots + \lambda_N$, $\lambda! = \lambda_1! \cdots \lambda_N!$, $0^0 = 1$, and $z^\lambda = z_1^{\lambda_1} \cdots z_N^{\lambda_N}$.

We follow the notation and results of [16, Section 2.2] regarding the automorphism group $\operatorname{Aut}(\mathbb{B})$. If $\psi \in \operatorname{Aut}(\mathbb{B})$ and $b = \psi^{-1}(0)$, then the complex Jacobian of ψ is

$$(9) \qquad J\psi(z) = \eta_1 \left(\frac{1 - |\psi(z)|^2}{1 - |z|^2} \right)^{(N+1)/2} = \eta_2 \, \frac{(1 - |b|^2)^{(N+1)/2}}{(1 - \langle z, b \rangle)^{N+1}},$$

where η_1, η_2 are complex numbers of modulus 1. The group $\operatorname{Aut}(\mathbb{B})$ is generated by the involutive Möbius transformations φ_b exchanging 0 and $b \in \mathbb{B}$ and unitary operators U on $\mathbb{C}^N$. For these special kinds of automorphisms, we write T_b^α and T_U^α in place of T_ψ^α defined in (6). When $\psi = \varphi_b$, then $\eta_2 = (-1)^N$. To avoid the annoying appearance of η_2 in calculations, we redefine T_b^α as

$$(10) \qquad T_b^\alpha f(z) = \frac{(1 - |b|^2)^\alpha}{(1 - \langle z, b \rangle)^{2\alpha}} \, f(\varphi_b(z)).$$

Let $f \in H(\mathbb{B})$ and $f = \sum_{k=0}^\infty f_k$ be its homogeneous expansion, where f_k is a holomorphic homogeneous polynomial of degree k. The action of D_s^t on f is that of a coefficient multiplier in the form

$$(11) \qquad D_s^t f = \sum_{k=0}^\infty d_k f_k,$$

where d_k depends on s, t in such a way that $d_k \sim k^t$ as $k \to \infty$ for any s. So D_s^t is a continuous operator on $H(\mathbb{B})$. In particular, $D_s^t z^\lambda = d_{|\lambda|} z^\lambda$ for any multi-index λ. An important property of our particular d_k is that $d_k \neq 0$ for all $k = 0, 1, 2, \dots$, and this makes D_s^t invertible on $H(\mathbb{B})$. Coupled with the facts that $D_s^0 = I$, the identity, and $D_{s+t}^u D_s^t = D_s^{u+t}$, we obtain the two-sided inverse

$$(12) \qquad (D_s^t)^{-1} = D_{s+t}^{-t}$$

for any $s, t \in \mathbb{R}$.

The differential operators D_s^t relate well with the Bergman-Besov kernels K_s for

$$(13) \qquad D_s^t K_s(z, w) = K_{s+t}(z, w)$$

for any $s, t \in \mathbb{R}$, where differentiation is performed on the holomorphic variable z. All the above properties of D_s^t are taken from [10, Section 3]. Moreover, if $s > -1$ and $f \in H(\mathbb{B})$, then for any t,

$$(14) \qquad D_s^t f(z) = C_s \lim_{r \to 1^-} \int_{\mathbb{B}} K_{s+t}(z, w) \, (1 - |w|^2)^s \, f(rw) \, d\nu(w);$$

see [10, Lemma 5.1].

The space L^∞_α is normed with

$$\|\varphi\|_{L^\infty_\alpha} := \operatorname*{ess\,sup}_{z\in\mathbb{B}} (1-|z|^2)^\alpha\, |\varphi(z)|.$$

The norm on $\mathcal{C}_\alpha$ is given by the same formula. For any s, t satisfying (1), there is induced the norm

$$\|f\|_{\mathcal{B}_\alpha} := \|I^t_s f\|_{L^\infty_\alpha} \tag{15}$$

on $\mathcal{B}_\alpha$. This is a genuine norm, because D^t_s is an invertible operator. Different s, t satisfying (1) give equivalent norms as mentioned above, mainly as a consequence of Corollary 3.4. It is clear from Definition 1.1 that

$$\mathcal{B}_\alpha \subset \mathcal{B}_{\beta 0} \subset \mathcal{B}_\beta \qquad (\alpha < \beta), \tag{16}$$

and the inclusion is continuous.

Proposition 2.1. *For any α and s, t, $D^t_s(\mathcal{B}_\alpha) = \mathcal{B}_{\alpha+t}$ is an isomorphism, and an isometry when appropriate norms are used in the two spaces.*

Proof. Let $f \in \mathcal{B}_\alpha$ and put $g = D^t_s f$. Take u so large that $\alpha + (t+u) > 0$. Then $D^u_{s+t} g = D^u_{s+t} D^t_s f = D^{t+u}_s f$ and $I^{t+u}_s f \in L^\infty_\alpha$. This is equivalent to $I^u_{s+t} g \in L^\infty_{\alpha+t}$. Hence $g \in \mathcal{B}_{\alpha+t}$ and the norms $\|f\|_{\mathcal{B}_\alpha} = \|I^{t+u}_s f\|_{L^\infty_\alpha}$ and $\|g\|_{\mathcal{B}_{\alpha+t}} = \|I^u_{s+t} g\|_{L^\infty_{\alpha+t}}$ are equal. Since $D^{-t}_s(\mathcal{B}_{\alpha+t}) = \mathcal{B}_\alpha$ the same way, both claims are established. ∎

Example 2.2. A "typical" function in $\mathcal{B}_0$ is known and can be checked by Definition 1.1 to be $f_0(z) = \log(1-z_1)^{-1} \sim \sum_k z_1^k/k$. By Proposition 2.1, a "typical" function in $\mathcal{B}_\alpha$ is $f_\alpha(z) = \sum_k k^{\alpha-1} z_1^k$. Using the series expansion in Definition 1.2 and (8), $f_\alpha(z)$ essentially is $(1-z_1)^{-\alpha}$ for $\alpha > 0$, and a hypergeometric function for $\alpha < 0$. The same reasoning shows that all the inclusions in (16) are strict. Letting $\beta - \alpha = 2\varepsilon > 0$ and using a large enough t in Definition 1.1, it is easy to see that $f_\beta \in \mathcal{B}_\beta \setminus \mathcal{B}_{\beta 0}$ and $f_{\alpha+\varepsilon} \in \mathcal{B}_{\beta 0} \setminus \mathcal{B}_\alpha$. Thus all $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha 0}$ spaces are different.

By [9, Definition 4.13], the space $\mathcal{B}_\alpha$ for $\alpha > 0$ is the *growth space* $\mathcal{A}^{-\alpha}$. The space $\mathcal{B}_2$ is also called the *Bers space*. A growth space does not require any derivative in its definition since now $t = 0$ satisfies (1). So an $f \in H(\mathbb{B})$ belongs to $\mathcal{B}_\alpha$ for $\alpha > 0$ whenever $f(z) \leq C\,(1-|z|^2)^{-\alpha}$ for all $z \in \mathbb{B}$. Thus for $\alpha > 0$, the spaces $\mathcal{B}_\alpha$ and H^∞_α coincide.

Also by [26, Theorems 7.17 and 7.18], the space $\mathcal{B}_\alpha$ for $\alpha < 0$ is the *holomorphic Lipschitz space* $\Lambda_{-\alpha}$. Proposition 2.1 for $\alpha < 0$ and $t = -\alpha$ appears in [26, Theorems 7.19 and 7.20].

Proposition 2.1 is the usual way to extend the definition of Lipschitz spaces $\Lambda_{-\alpha}$ beyond $-1 \leq \alpha < 0$. The space $\mathcal{B}_{-1} = \Lambda_1$ is called the *Zygmund class*. It is traditionally defined via a second-order difference quotient as opposed to first-order difference quotients for $-1 < \alpha < 0$; see [15, Section 8.8]. This is no surprise, because the least integer value of t specified by (1) is 1 for $-1 < \alpha < 0$ and is 2 for $\alpha = -1$. Hence the case $\alpha = 0$ and $t = -1$ of Proposition 2.1 is in [1, 3.5.2].

Remark 2.3. All the statements in this section for the $\mathcal{B}_\alpha$ spaces, including Proposition 2.1, have obvious counterparts for the $\mathcal{B}_{\alpha 0}$ spaces.

Definition 2.4. For $q \in \mathbb{R}$ and $0 < p < \infty$, the *Besov space* B^p_q consists of all $f \in H(\mathbb{B})$ for which $I^t_s f$ belongs to L^p_q for some s, t satisfying

$$q + pt > -1. \tag{17}$$

This definition too is independent of s, t under (17), and thus we have the equivalent norms $\|f\|_{B^p_q} = \|I^t_s f\|_{L^p_q}$ on B^p_q. Bergman projections on Besov spaces have been characterized in [10, Theorem 1.2].

Theorem 2.5. *For* $1 \leq p < \infty$, $P_s : L^p_q \to B^p_q$ *is bounded if and only if*

$$q + 1 < p\,(s + 1). \tag{18}$$

Given s *satisfying* (18)*, if* t *satisfies* (17)*, then* (3) *holds for* $f \in B^{\,p}_q$.

We use α as a subscript on $\mathcal{B}$ rather than the usual superscript, because this not only follows the notation for L^∞ and $\mathcal{C}$, but also follows the notation for the Besov spaces B^p_q, where the upper parameter is for the power on the function, and the lower parameter is for the power on the weight $1 - |z|^2$. The power on a function in a $\mathcal{B}_\alpha$ space, if anything, is ∞ and not shown.

Remark 2.6. There are close connections between the Besov and weighted Bloch families of spaces. It is explained in [12, Section 8] that $\mathcal{B}_\alpha$ is the limiting case of $B^p_{\alpha p}$ (or of $B^p_{\beta+\alpha p}$) at $p = \infty$. This is further reflected in the inequalities; (1) is the $p = \infty$ case of (17) with $q = \alpha p$, and (2) is the $p = \infty$ case of (18) with $q = \alpha p$. The set of (p, q) in the right half plane satisfying $q = \beta + \alpha p$ is a ray with slope α and q-intercept β.

For another connection between the two families, see [12, Theorem 8.3], where the Carleson measures of Besov spaces characterize the functions in weighted Bloch spaces, which yields different proofs of Corollaries 3.4 and 3.5 below.

3. Projections

We now prove Theorem 1.3 and indicate several immediate corollaries.

Proof of Theorem 1.3. Fix α throughout the proof.

Let $\varphi \in L^\infty_\alpha$. Take t to satisfy (1), and for the moment so large that also $s+t > -(N+1)$ holds. To show $P_s\varphi \in \mathcal{B}_\alpha$, we need to show $I^t_s(P_s\varphi) \in L^\infty_\alpha$. Using (13) and the assumptions on s, t, we obtain

$$\begin{aligned}(1-|z|^2)^\alpha\, |I^t_s(P_s\varphi)(z)| &= (1-|z|^2)^{\alpha+t} \left| D^t_s \int_{\mathbb{B}} K_s(z,w)\,(1-|w|^2)^s\,\varphi(w)\,d\nu(w)\right| \\ &\le \|\varphi\|_{L^\infty_\alpha}\,(1-|z|^2)^{\alpha+t} \int_{\mathbb{B}} \frac{(1-|w|^2)^{s-\alpha}}{|1-\langle z,w\rangle|^{N+1+s+t}}\,d\nu(w),\end{aligned}$$

and the last integral is finite if and only if (2) holds. Then [16, Proposition 1.4.10] yields

$$(1-|z|^2)^\alpha\,|I^t_s(P_s\varphi)(z)| \le C\,\|\varphi\|_{L^\infty_\alpha} \qquad (z \in \mathbb{B}).$$

Thus $P_s\varphi \in \mathcal{B}_\alpha$ and this proves the first claim on when $P_s : L^\infty_\alpha \to \mathcal{B}_\alpha$ is bounded. As noted earlier, (1) and (2) together imply $s+t > -1$; so the momentary assumption on t above is superfluous.

Now let $f \in \mathcal{B}_\alpha$ and s, t satisfy (2) and (1). Then $I^t_s f \in L^\infty_\alpha$ and $P_s(I^t_s f) \in \mathcal{B}_\alpha$ by the first part. Also $s+t > -1$, and using (14) and (12), we see that

$$\begin{aligned}P_s(I^t_s f)(z) &= \int_{\mathbb{B}} K_s(z,w)\,(1-|w|^2)^{s+t}\,D^t_s f(w)\,d\nu(w) \\ &= \frac{N!}{(1+s+t)_N}\,D^{-t}_{s+t}D^t_s f(z) = \frac{N!}{(1+s+t)_N}\,f(z),\end{aligned}$$

and this proves the second claim.

Next, let s satisfy (2). To show that P_s maps $\mathcal{C}_\alpha$ into $\mathcal{B}_{\alpha 0}$, it suffices to consider $\varphi(w) = (1-|w|^2)^{-\alpha}\,w^\lambda\,\overline{w}^\mu \in \mathcal{C}_\alpha$, since polynomials in w and $\overline{w}$ are dense in $\mathcal{C}_0$. For t satisfying (1), we have

$$(1-|z|^2)^\alpha\, I^t_s(P_s\varphi)(z) = (1-|z|^2)^{\alpha+t} \int_{\mathbb{B}} K_{s+t}(z,w)\,(1-|w|^2)^{s-\alpha}\,w^\lambda\,\overline{w}^\mu\,d\nu(w).$$

We use the series expansion of K_{s+t} in simplified form as $K_{s+t}(z,w) = \sum_\tau c_\tau\,z^\tau\,\overline{w}^\tau$. In the integral above, the only nonzero term is the one with $\tau = \lambda - \mu$ by the orthogonality in [6, Proposition 2.4]. Then that integral is finite by (2) and is

$$c_{\lambda-\mu}\,z^{\lambda-\mu} \int_{\mathbb{B}} |w^\lambda|^2\,(1-|w|^2)^{s-\alpha}\,d\nu(w) = C\,z^{\lambda-\mu}$$

with $|\lambda-\mu| \ge 0$. Thus $(1-|z|^2)^\alpha\,|I^t_s(P_s\varphi)(z)| \to 0$ as $|z| \to 0$ again by (1). Hence $P_s\varphi \in \mathcal{B}_{\alpha 0}$ and P_s is bounded from either of $\mathcal{C}_\alpha$ and $\mathcal{C}_{\alpha 0}$ into $\mathcal{B}_{\alpha 0}$.

If $f \in \mathcal{B}_{\alpha 0}$, then by Definition 1.1, $I^t_s f \in \mathcal{C}_{\alpha 0}$ if t satisfies (1). Now for s satisfying (2), (3) shows that $P_s(I^t_s f) = C\,f$ lies in $\mathcal{B}_{\alpha 0}$. This shows that P_s is onto $\mathcal{B}_{\alpha 0}$ from either of the continuous function classes.

To prove the necessity of (2), put $\varphi_1(w) = w_1\,(1-|w|^2)^{-\alpha}\,[\log(1-|w|^2)]^{-1}$. Clearly $\varphi_1 \in \mathcal{C}_{\alpha 0}$. The integral

$$P_s\varphi_1(z) = \int_{\mathbb{B}} K_s(z,w)\,(1-|w|^2)^{s-\alpha}\,\frac{w_1}{\log(1-|w|^2)}\,d\nu(w)$$

diverges if (2) is violated.

This completes the proof. ■

Corollary 3.1. *The space $\mathcal{B}_{\alpha 0}$ is the closure of the holomorphic polynomials in the norm $\|f\|_{\mathcal{B}_\alpha} = \|I_s^t f\|_{L_\alpha^\infty}$ for any s, t satisfying* (2) *and* (1), *and thus is separable and complete.*

Proof. Theorem 1.3 shows that P_s maps $\mathcal{C}_\alpha$ onto $\mathcal{B}_{\alpha 0}$. Similar to its proof, if $\varphi_{\lambda\mu}(w) = (1-|w|^2)^{-\alpha}\,w^\lambda\,\overline{w}^\mu \in \mathcal{C}_\alpha$, then $P_s\varphi_{\lambda\mu}(z) = C\,z^{\lambda-\mu}$ with $|\lambda-\mu| \geq 0$. The space $\mathcal{C}_\alpha$ is the closure of finite linear combinations of functions of the form $\varphi_{\lambda\mu}$ with λ, μ having rational components. Consequently, $\mathcal{B}_{\alpha 0}$ is the closure of finite linear combinations of functions of the form z^τ. ■

Remark 3.2. The inseparability of $\mathcal{B}_\alpha$ can also be seen using Theorem 1.3. For $\alpha > 0$, take $s = \alpha$, and for other α, take $s = 0$. Also take $\varphi(w) = (1-|b|^2)^{-s}$ for $w = b$ and $\varphi(w) = 0$ otherwise, where $b \in \mathbb{B}$ is arbitrary. What Theorem 1.3 now says is that $K_s(\cdot, b) \in \mathcal{B}_\alpha$ for every $b \in \mathbb{B}$. A quick estimate shows that $\|K_s(\cdot, b_1) - K_s(\cdot, b_2)\|_{\mathcal{B}_\alpha} \geq C\,|b_1 - b_2|$.

Remark 3.3. The operator P_s is not always a projection on a subspace, because $\mathcal{B}_\alpha$ need not be a subspace of L_α^∞. However, for t satisfying (1), $I_s^t(\mathcal{B}_\alpha)$ is an isometric copy of $\mathcal{B}_\alpha$ in L_α^∞ by (15), and hence a closed subspace of L_α^∞ by Corollary 5.5 below. Then by (3),

$$V_s^t := I_s^t P_s$$

is a true projection from L_α^∞ onto $I_s^t(\mathcal{B}_\alpha)$ for any s satisfying (2).

Corollary 3.4. *For a given s satisfying* (2)*, any two values of the order t of the differential operator D_s^t satisfying* (1) *generate the same space in Definition* 1.1 *for the same α.*

Proof. Suppose t_1, t_2 satisfy (1), $f \in H(\mathbb{B})$, and $\varphi = I_s^{t_1} f \in L_\alpha^\infty$ for a given α, where s satisfies (2). By Theorem 1.3, $P_s I_s^{t_1} f = C\,f$. Apply $I_s^{t_2}$ to both sides to get $V_s^{t_2}\varphi = C\,I_s^{t_2} f$, where

$$V_s^{t_2}\varphi(z) = (1-|z|^2)^{t_2} \int_{\mathbb{B}} \frac{(1-|w|^2)^{s-\alpha}}{(1-\langle z,w\rangle)^{N+1+s+t_2}}\,(1-|w|^2)^\alpha\,\varphi(w)\,d\nu(w)$$

when written explicitly. Since $\varphi(w) \in L^\infty_\alpha$, [16, Proposition 1.4.10] yields that $|V_s^{t_2}\varphi(z)| \leq C\,(1-|z|^2)^{-\alpha}$. In other words, $I_s^{t_2} f = C\,V_s^{t_2}\varphi \in L^\infty_\alpha$, which is the desired result.

The case for $\mathcal{B}_{\alpha 0}$ is entirely similar. ∎

Corollary 3.5. *For a given t satisfying* (1), *any two values of the parameter s of the differential operator D_s^t satisfying* (2) *generate the same space in Definition* 1.1 *for the same α if also $s > -1$ for $\alpha < 0$.*

Proof. Given α, suppose s_1, s_2 satisfy (2), $f \in H(\mathbb{B})$, and $I_{s_1}^t f \in L^\infty_\alpha$, that is, $g = D_{s_1}^t f \in L^\infty_{\alpha+t}$, where t satisfies (1). That is, there is a C such that

$$|g(z)| \leq \frac{C}{(1-|z|^2)^{\alpha+t}} \qquad (z \in \mathbb{B}). \tag{19}$$

Using (12) and (14), we write $D_{s_2}^t f = D_{s_2}^t D_{s_1+t}^{-t} D_{s_1}^t f = D_{s_2}^t D_{s_1+t}^{-t}\, g = Eg$, where

$$Eg(z) = C_{s_2} C_{s_1+t} \int_{\mathbb{B}} \frac{(1-|v|^2)^{s_2}}{(1-\langle z, v\rangle)^{N+1+s_2+t}} \int_{\mathbb{B}} \frac{(1-|w|^2)^{s_1+t}}{(1-\langle v, w\rangle)^{N+1+s_1}}\, g(w)\, d\nu(w)\, d\nu(v).$$

The required condition $s_1 + t > -1$ follows from (1) and (2). We would like to show that also $Eg = D_{s_2}^t f \in L^\infty_{\alpha+t}$. Again it is not necessary to check the $\mathcal{B}_{\alpha 0}$ spaces.

If $\alpha > 0$, using (19) and [16, Proposition 1.4.10], we obtain

$$\begin{aligned} |Eg(z)| &\leq C \int_{\mathbb{B}} \frac{(1-|v|^2)^{s_2}}{|1-\langle z, v\rangle|^{N+1+s_2+t}} \int_{\mathbb{B}} \frac{(1-|w|^2)^{s_1-\alpha}}{|1-\langle v, w\rangle|^{N+1+s_1}}\, d\nu(w)\, d\nu(v) \\ &\sim \int_{\mathbb{B}} \frac{(1-|v|^2)^{s_2-\alpha}}{|1-\langle z, v\rangle|^{N+1+s_2+t}}\, d\nu(v) \sim \frac{C}{(1-|z|^2)^{\alpha+t}}. \end{aligned}$$

If $\alpha = 0$, then the above computation and [16, Proposition 1.4.10] yield

$$\begin{aligned} |Eg(z)| &\leq C \int_{\mathbb{B}} \frac{(1-|v|^2)^{s_2}}{|1-\langle z, v\rangle|^{N+1+s_2+t}}\, \log \frac{1}{1-|v|^2}\, d\nu(v) \\ &\leq C \int_{\mathbb{B}} \frac{(1-|v|^2)^{s_2+\varepsilon}}{|1-\langle z, v\rangle|^{N+1+s_2+t}}\, d\nu(v) \sim \frac{C}{(1-|z|^2)^{t-\varepsilon}} \end{aligned}$$

for some $\varepsilon > 0$. We next let $\varepsilon \to 0$.

If $\alpha < 0$, (19) can be strengthened to $|g(z)| \leq C\,(1-|z|^2)^{-t}$. Using this estimate in $|Eg(z)|$ reduces this case to the case of $\alpha = 0$.

In the first two cases, from (1) and (2), $s + t > -1$ and $s > -1$ are automatic, but in the last case, $s > -1$ has to be additionally assumed. ∎

[26, Chapter 7] contains a collection of similar results dealing with various ranges of the parameters with further limitations on their values.

We have one more type of the Bergman projections that has already been utilized in [22]. Define the generalized Bergman projections P_{rs} on suitable φ by

$$P_{rs}\varphi(z) = \int_{\mathbb{B}} K_r(z,w)\,\varphi(w)\,d\nu_s(w).$$

Theorem 3.6. *$P_{rs} : L^\infty_\alpha \to \mathcal{B}_\alpha$ is bounded if and only if r, s satisfy* (2) *and*

$$r \leq s. \tag{20}$$

Given such r, s, if t satisfies (1)*, then for $f \in \mathcal{B}_\alpha$,*

$$P_{rs} I^t_s f = \frac{1}{C_{s+t}}\, D^{r-s}_s f. \tag{21}$$

Also either $P_{rs} : \mathcal{C}_\alpha \to \mathcal{B}_{\alpha 0}$ or $P_{rs} : \mathcal{C}_{\alpha 0} \to \mathcal{B}_{\alpha 0}$ is bounded if and only if (2) *and* (20) *hold.*

Proof. Take $\varphi \in L^\infty_\alpha$ and t so that (1) and $r + t > -(N+1)$ hold. Then

$$\begin{aligned}(1-|z|^2)^\alpha\, I^t_r(P_{rs}\varphi)(z) &= (1-|z|^2)^{\alpha+t} \int_{\mathbb{B}} \frac{(1-|w|^2)^{s-\alpha}\,\psi(w)}{(1-\langle z,w\rangle)^{N+1+(\alpha+t)+(s-\alpha)+(r-s)}}\,d\nu(w)\\ &= V\psi(z),\end{aligned}$$

where $\psi(z) = (1-|z|^2)^\alpha\,\varphi(z) \in L^\infty$. By [12, Theorem 7.2], $V\psi$ lies in L^∞ if and only if (2) and (20) hold since (1) is assumed anyway.

Equation (21) is obtained in exactly the same way as in the proof of Theorem 1.3, and so are the statements about $\mathcal{B}_{\alpha 0}$. ∎

Note that the only way to obtain f itself on the right side of (21) is to have $r = s$. Otherwise, the right side is a primitive of order $s - r > 0$ of f.

4. Gleason Problem

Let X be a space of functions on $\mathbb{B}$. Given $a \in \mathbb{B}$ and $f \in X$, the *Gleason problem* is to find $f_1, \dots, f_N \in X$ such that

$$f(z) - f(a) = \sum_{m=1}^{N} (z_m - a_m)\, f_m(z) \qquad (z \in \mathbb{B}).$$

This section is for explicit solutions to the Gleason problem in the spaces $\mathcal{B}_\alpha$.

Theorem 4.1. *For $a \in \mathbb{B}$, there are bounded linear operators $G_1, \ldots, G_N$ on $\mathcal{B}_\alpha$ satisfying*

$$f(z) - f(a) = \sum_{m=1}^{N} (z_m - a_m)\, G_m f(z) \qquad (f \in \mathcal{B}_\alpha,\ z \in \mathbb{B}). \tag{22}$$

The operators $G_1, \ldots, G_N$ are bounded on $\mathcal{B}_{\alpha 0}$ too.

Proof. We imitate the proof of [10, Theorem 6.1] and show little detail. Let $s > -(N+1)$ be an integer satisfying (2), let t satisfy (1), and define

$$G_m f(z) = C_{s+t} \int_{\mathbb{B}} \frac{K_s(z,w) - K_s(a,w)}{\langle z-a, w\rangle}\, \overline{w}_m\, I_s^t f(w)\, d\nu_s(w) \qquad (f \in \mathcal{B}_\alpha)$$

for $m = 1, \ldots, N$. It is easy to see that $G_1, \ldots, G_N$ satisfy (22).

The crucial part is to show that G_m is bounded. Proceeding as in the proof of [10, Theorem 6.1], by the fact that s is an integer, it is possible to write each G_m as a finite sum of operators T_j on $\mathcal{B}_\alpha$ such that

$$|I_{s-j}^u (T_j f)(z)| \le C\,(1-|z|^2)^u \int_{\mathbb{B}} \frac{(1-|w|^2)^{s-\alpha}\,(1-|w|^2)^\alpha\,|I_s^t f(w)|}{|1-\langle z,w\rangle|^{N+1+s+u}}\, d\nu(w), \tag{23}$$

where u satisfies (1) when substituted for t. Because $I_s^t f \in L_\alpha^\infty$ and s satisfies (2), [16, Proposition 1.4.10] yields that $(1-|z|^2)^\alpha\,|I_{s-j}^u (T_j f)(z)|$ is bounded on $\mathbb{B}$. Hence G_m is a bounded operator on $\mathcal{B}_\alpha$.

If $f \in \mathcal{B}_{\alpha 0}$, then given $\varepsilon > 0$, there is an $R < 1$ such that $(1-|w|^2)^\alpha\,|I_s^t f(w)| < \varepsilon$ for $|w| \ge R$. Split the integral in (23) into two parts, J_1 on $R\mathbb{B}$ and J_2 on $\mathbb{B} \setminus R\mathbb{B}$. Then $|J_1(z)| \le C$ and $|J_2(z)| \le C\,\varepsilon\,(1-|z|^2)^{-(\alpha+u)}$ for $z \in \mathbb{B}$. We multiply J_1 and J_2 by $(1-|z|^2)^{\alpha+u}$, add, and then let $|z| \to 1$. Because u satisfies (1), we obtain

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha\,|I_{s-j}^u (T_j f)(z)| \le C\,\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows that $T_j f$ and hence $G_m f$ belong to $\mathcal{B}_{\alpha 0}$. ■

5. Analytic Properties

In this section, we use Theorem 1.3 and in particular (4) to obtain some analytic properties of functions in $\mathcal{B}_\alpha$ spaces. Some of these properties are known, especially for $\alpha < 0$, the Lipschitz range. However the emphasis here is on how they are obtained so readily from Bergman projections and the ensuing integral representations, and on their uniformity for all real α.

But first, let's give a result that shows the versatility of the radial differential operators D_s^t. Given $N \ge 2$, $f \in H(\mathbb{B})$, and $\zeta \in \mathbb{C}^N$ with $|\zeta| = 1$, the holomorphic *slice functions* f_ζ are defined by $f_\zeta(x) = f(x\zeta)$ for $x \in \mathbb{D}$.

Theorem 5.1. *Suppose every slice function f_ζ of an $f \in H(\mathbb{B})$ belongs to $\mathcal{B}_\alpha$ of the disc with uniformly bounded norms. Then $f \in \mathcal{B}_\alpha$ of the ball.*

Proof. For $z = x\zeta \in \mathbb{B}$, by (11) we have

$$D_s^t f(z) = \sum_{k=0}^{\infty} d_k f_k(z) = \sum_{k=0}^{\infty} d_k f_k(\zeta)\, x^k = D_r^t \sum_{k=0}^{\infty} f_k(\zeta)\, x^k = D_r^t f_\zeta(x),$$

where $r = N - 1 + s$ by [10, Definition 3.1]. By assumption, there is a C such that $(1-|x|^2)^{\alpha+t}\,|D_r^t f_\zeta(x)| \leq C$ for all $x \in \mathbb{D}$ and $\zeta \in \partial\mathbb{B}$, and for some r, t satisfying (1). Then s, t satisfy (1) too, and obviously $(1-|z|^2)^{\alpha+t}\,|D_s^t f(z)| \leq C$ for all $z \in \mathbb{B}$. ■

The case $\alpha = 0$ of Theorem 5.1 is in [19, Theorem 4.10] with a much more roundabout proof.

Theorem 5.2. *Given α, there is a C such that for all $f \in \mathcal{B}_\alpha$ and $z \in \mathbb{B}$,*

$$|f(z)| \leq C\,\|f\|_{\mathcal{B}_\alpha} \begin{cases} (1-|z|^2)^{-\alpha}, & \text{if } \alpha > 0; \\ \log(1-|z|^2)^{-1}, & \text{if } \alpha = 0; \\ 1, & \text{if } \alpha < 0; \end{cases}$$

where $\|f\|_{\mathcal{B}_\alpha} = \|I_s^t f\|_{L_\alpha^\infty}$ with $s > -(N+1), t$ satisfying (2) and (1).

Proof. We use a simple estimate on (4) and obtain

$$|f(z)| \leq C\,\|f\|_{\mathcal{B}_\alpha} \int_{\mathbb{B}} \frac{(1-|w|^2)^{s-\alpha}}{|1-\langle z, w\rangle|^{N+1+s}}\, d\nu(w) \qquad (z \in \mathbb{B}).$$

We obtain all three cases by applying [16, Proposition 1.4.10]. ■

Corollary 5.3. *Under the same conditions as of Theorem 5.2 and for $r, u \in \mathbb{R}$, we have*

$$|D_r^u f(z)| \leq C\,\|f\|_{\mathcal{B}_\alpha} \begin{cases} (1-|z|^2)^{-(\alpha+u)}, & \text{if } \alpha + u > 0; \\ \log(1-|z|^2)^{-1}, & \text{if } \alpha + u = 0; \\ 1, & \text{if } \alpha + u < 0. \end{cases}$$

Proof. Just combine Theorem 5.2 with Proposition 2.1. ■

Theorem 5.2 says nothing new other than Definition 1.1 for $\alpha > 0$, and Corollary 5.3 is reminiscent of the classical definition of Lipschitz spaces for $u = -\alpha > 0$. Now we see how the two subfamilies for positive and negative values of α are combined in a uniform manner.

Theorem 5.4. $\Lambda_{-\alpha} \subset A(\mathbb{B}) \subset H^\infty \subset \mathcal{B}_\beta$ *for* $\alpha < 0$ *and* $\beta \geq 0$.

Proof. For $\alpha < 0$, let $f \in \Lambda_{-\alpha}$, and pick $s > -(N+1)$ and t so as to satisfy (2) and (1). Then by (4), we have

$$f(z) = C_{s+t} \int_{\mathbb{B}} \frac{(1-|w|^2)^{\alpha+t}\, D_s^t f(w)}{(1-\langle z,w\rangle)^{N+1+s}}\, (1-|w|^2)^{-\alpha+s}\, d\nu(w)$$

and

$$|f(z)| \leq C \int_{\mathbb{B}} \frac{(1-|w|^2)^{-\alpha+s}}{|1-\langle z,w\rangle|^{N+1+s}}\, d\nu(w)$$

Since $N+1+s-(N+1)-(-\alpha+s) = \alpha < 0$, by the proof of [16, Proposition 1.4.10], the last integral converges uniformly for $|z| \leq 1$. Thus also $f \in \mathcal{C}_0$.

The claim for $\beta \geq 0$ is the well-known fact proved via Schwarz lemma that the classical Bloch space contains H^∞ combined with (16). ∎

Corollary 5.5. *Given* $\alpha, r, u \in \mathbb{R}$ *and a compact subset* E *of* $\mathbb{B}$*, there is a* C *such that for all* $f \in \mathcal{B}_\alpha$*,*

$$\sup_{z\in E} |D_r^u f(z)| \leq C\, \|f\|_{\mathcal{B}_\alpha},$$

where $\|f\|_{\mathcal{B}_\alpha} = \|I_s^t f\|_{L_\alpha^\infty}$ *with* $s > -(N+1), t$ *satisfying* (2) *and* (1)*, Therefore point evaluations on* $\mathcal{B}_\alpha$ *are bounded linear functionals. Consequently, every* $\mathcal{B}_\alpha$ *is a Banach space.*

We now set $N = 1$ and look at the Taylor series coefficients of $f \in \mathcal{B}_\alpha$ on $\mathbb{D}$.

Theorem 5.6. *Given* α*, there is a* C *such that for all* $f \in \mathcal{B}_\alpha$*,*

$$|c_k| \leq C\, \|f\|_{\mathcal{B}_\alpha}\, k^\alpha,$$

where $c_k = f^{(k)}(0)/k!$ *and* $\|f\|_{\mathcal{B}_\alpha} = \|I_s^t f\|_{L_\alpha^\infty}$ *with* $s > -2, t$ *satisfying* (2) *and* (1).

Proof. Differentiation k times puts (4) into the form

$$f^{(k)}(z) = C\,(2+s)_k \int_{\mathbb{D}} \frac{\overline{w}^k\, I_s^t f(w)}{(1-\langle z,w\rangle)^{2+s+k}}\, d\nu_s(w) \qquad (z \in \mathbb{D}).$$

Then

$$|c_k| \leq C\, \|f\|_{\mathcal{B}_\alpha} \frac{\Gamma(2+s+k)}{\Gamma(1+k)} \int_{\mathbb{D}} |w|^k\, (1-|w|^2)^{s-\alpha}\, d\nu(w).$$

Evaluating the integral using [10, Proposition 2.1] yields

$$|c_k| \leq C\, \|f\|_{\mathcal{B}_\alpha} \frac{\Gamma(2+s+k)}{\Gamma(1+k)}\, \frac{\Gamma(1+k/2)}{\Gamma(2+s-\alpha+k/2)} \sim C\, \|f\|_{\mathcal{B}_\alpha}\, k^\alpha,$$

where the final estimate follows from (8). ∎

Remark 5.7. The statements of most results in this section could be guessed using the general principle stated in Remark 2.6. Theorem 5.2 and Corollary 5.3 are the $p = \infty$ versions of [13, Theorem 6.1 and Corollary 6.2], Theorem 5.6 is the $p = \infty$ version of [13, Theorem 7.1], all after setting $q = \alpha p$. Next we state without proof another result that we guess by employing the same principle. For it, we set $q = \alpha p$ first in [13, Theorem 4.2], and then replace the l^p condition with the l^∞ condition as well as p by ∞. For $\alpha > -1$, it is proved in [23, Theorem 1], while our result holds for all real α.

Theorem 5.8. *A Taylor series $f(z) = \sum_k c_k\, z^{n_k}$ with Hadamard gaps belongs to $\mathcal{B}_\alpha$ if and only if $\sup_k n_k^{-\alpha}\,|c_k| < \infty$, and belongs to $\mathcal{B}_{\alpha 0}$ if and only if $n_k^{-\alpha}\,|c_k| \to 0$ as $k \to \infty$.*

6. Duality and Interpolation

Duality results on Besov and Bloch spaces using different pairings dealing with various ranges of the parameters appear in several places; see, for example, [26, Sections 7.1 and 7.7]. Here we derive them for all real α using some general pairings directly from Theorems 1.3 and 2.5, (3), and some general results on Lebesgue classes.

Theorem 6.1. *The dual space of every $\mathcal{B}_{\alpha 0}$ can be identified with every B^1_q under each of the pairings*

$$\int_{\mathbb{B}} I^t_s f\, \overline{I^{-\alpha-q+s}_{\alpha+q+t} g}\, d\nu_{\alpha+q}, \tag{24}$$

where s, t are chosen to satisfy (2) *and* (1), *$f \in \mathcal{B}_{\alpha 0}$, and $g \in B^1_q$.*

Proof. It is clear that each pairing in (24) induces a bounded linear functional on $\mathcal{B}_{\alpha 0}$ via Hölder inequality. Note that $I^{-\alpha-q+s}_{\alpha+q+t} g \in L^1_q$ by (17).

Conversely, let T be a bounded linear functional on $\mathcal{B}_{\alpha 0}$ and M denote the operator of multiplication by $(1-|z|^2)^{-\alpha}$. Then $L = TP_sM$ is a bounded linear functional on $\mathcal{C}_{00}$ by Theorem 1.3. So there is a complex, hence finite, Borel measure μ on $\mathbb{B}$ such that $Lh = \int_{\mathbb{B}} h\, d\mu$ for all $h \in \mathcal{C}_{00}$. Pick $h = M^{-1} I^t_s f$ for $f \in \mathcal{B}_{\alpha 0}$. By (3), we have $Lh = TP_sMM^{-1}I^t_s f = TP_sI^t_s f = C^{-1}_{s+t} Tf$ so that $Tf = C_{s+t} \int_{\mathbb{B}} I^t_s f(z)\, (1-|z|^2)^\alpha\, d\mu(z)$.

Let $\varphi \in L^1_q$ be the Radon-Nikodym derivative of μ with respect to ν_q. We can also replace f by $C_{s+t} P_s I^t_s f$ by (3). Then $Tf = C^2_{s+t} \int_{\mathbb{B}} V^t_s(I^t_s f)\, \overline{\varphi}\, d\nu_{\alpha+q}$. Using the form of the adjoint of V^t_s computed in the proof of [26, Theorem 2.10], we obtain $Tf = C^2_{s+t} \int_{\mathbb{B}} I^t_s f\, \overline{(V^t_s)^* \varphi}\, d\nu_{\alpha+q} = C^2_{s+t} \int_{\mathbb{B}} I^t_s f\, \overline{V^{-\alpha-q+s}_{\alpha+q+t} \varphi}\, d\nu_{\alpha+q}$. Define $g = C^2_{s+t} P_{\alpha+q+t} \varphi$; then $g \in B^1_q$ by Theorem 2.5. This yields the desired form.

Under the conditions of the theorem, $g \in B^1_q$ obtained for a given T is unique. If there were two such g's, their difference, labeled g, would give 0 as the value of the integral in (24) for all $f \in \mathcal{B}_{\alpha 0}$. This forces $I^{-\alpha-q+s}_{\alpha+q+t} g = 0$. Applying $P_{\alpha+q+t}$ to this and using (3) show that $g = 0$. ■

Theorem 6.2. *The dual space of every B^1_q can be identified with every $\mathcal{B}_\alpha$ under each of the pairings in* (24), *where now s, t are chosen to satisfy* (18) *and* (17) *with $p = 1$, $f \in B^1_q$, and $g \in \mathcal{B}_\alpha$.*

Proof. The proof is almost identical to the proof of Theorem 6.1 with the roles of Bloch and Besov spaces interchanged. ■

Corollary 6.3. *There exist functions in every B^1_q and in every $\mathcal{B}_{\alpha 0}$, and hence in every $\mathcal{B}_\alpha$, whose Taylor series do not converge in norm.*

Proof. See [24], where this result is obtained in certain other settings. ■

Our next purpose is to establish interpolation relations among the $\mathcal{B}_\alpha$ family of spaces once again using Theorem 1.3. For basic definitions and notation regarding interpolation, we refer the reader to [26, Section 1.8]. We start with interpolation between Lebesgue classes, where $[X, Y]_\theta$ is the complex interpolation space between the Banach spaces X and Y.

Lemma 6.4. *Suppose $-\infty < \alpha < \sigma < \beta < \infty$ with $\sigma = (1-\theta)\alpha + \theta\beta$ for some $0 < \theta < 1$. Then $[L^\infty_\alpha, L^\infty_\beta]_\theta = L^\infty_\sigma$. Similar results hold for $\mathcal{C}_\sigma$ and $\mathcal{C}_{\sigma 0}$.*

Proof. To begin with, $L^\infty_\alpha \cap L^\infty_\beta = L^\infty_\alpha \subset L^\infty_\beta = L^\infty_\alpha + L^\infty_\beta$.

First suppose $\varphi \in L^\infty_\sigma$. For $\zeta \in S := \{\, \zeta : 0 \le \operatorname{Re}\zeta \le 1 \,\}$ and $z \in \mathbb{B}$, define $F_\zeta(z) = (1-|z|^2)^{\sigma-(1-\zeta)\alpha-\zeta\beta}\, \varphi(z)$, which, as a function of ζ, is continuous on S and holomorphic in its interior. Obviously, $F_\theta(z) = \varphi(z)$. We have

$$\|F_\zeta\|_{L^\infty_\alpha + L^\infty_\beta} \le \|\varphi\|_{L^\infty_\sigma} \sup_{z\in\mathbb{B}} (1-|z|^2)^{(1-\operatorname{Re}\zeta)(\beta-\alpha)} \le \|\varphi\|_{L^\infty_\sigma}$$

for all $\zeta \in S$. Similarly, $\|F_{iy}\|_{L^\infty_\alpha} \le \|\varphi\|_{L^\infty_\sigma}$ and $\|F_{1+iy}\|_{L^\infty_\beta} \le \|\varphi\|_{L^\infty_\sigma}$ for all $y \in \mathbb{R}$. Thus also $\|F\| \le \|\varphi\|_{L^\infty_\sigma}$ and $\varphi \in [L^\infty_\alpha, L^\infty_\beta]_\theta$.

Next suppose $\varphi \in [L^\infty_\alpha, L^\infty_\beta]_\theta$. Then there is an $F_\zeta(z)$ as above with $F_\theta(z) = \varphi(z)$. Put $M_0 = \sup_y \|F_{iy}\|_{L^\infty_\alpha}$, $M_1 = \sup_y \|F_{1+iy}\|_{L^\infty_\beta}$, and $M_\theta = \sup_y \|F_{\theta+iy}\|_{L^\infty_\sigma}$. Then

$$\|\varphi\|_{L^\infty_\sigma} \le \sup_{y\in\mathbb{R}, z\in\mathbb{B}} (1-|z|^2)^\sigma\, |F_{\theta+iy}(z)| = M_\theta \le M_0^{1-\theta} M_1^\theta$$

by Hadamard three lines theorem, and this shows $\varphi \in L^\infty_\sigma$. ■

Theorem 6.5. *Suppose* $-\infty < \alpha < \sigma < \beta < \infty$ *with* $\sigma = (1-\theta)\alpha + \theta\beta$ *for some* $0 < \theta < 1$. *Then* $[\mathcal{B}_\alpha, \mathcal{B}_\beta]_\theta = \mathcal{B}_\sigma$ *and* $[\mathcal{B}_{\alpha 0}, \mathcal{B}_{\beta 0}]_\theta = \mathcal{B}_{\sigma 0}$.

Proof. Let s satisfy (2) and t satisfy (1) both with β in place of α.

The operator P_s maps L^∞_α onto $\mathcal{B}_\alpha$, L^∞_β onto $\mathcal{B}_\beta$, and L^∞_σ onto $\mathcal{B}_\sigma$ boundedly by Theorem 1.3. By Lemma 6.4 and interpolation theory, P_s maps $L^\infty_\sigma = [L^\infty_\alpha, L^\infty_\beta]_\theta$ into $[\mathcal{B}_\alpha, \mathcal{B}_\beta]_\theta$. Thus $\mathcal{B}_\sigma \subset [\mathcal{B}_\alpha, \mathcal{B}_\beta]_\theta$.

On the other hand, the operator I^t_s maps $\mathcal{B}_\alpha$ into L^∞_α, $\mathcal{B}_\beta$ into L^∞_β, and $\mathcal{B}_\sigma$ into L^∞_σ boundedly by Definition 1.1. By Lemma 6.4 and interpolation theory, I^t_s maps $[\mathcal{B}_\alpha, \mathcal{B}_\beta]_\theta$ into $L^\infty_\sigma = [L^\infty_\alpha, L^\infty_\beta]_\theta$. By Definition 1.1, the last mapping just means that any f in $[\mathcal{B}_\alpha, \mathcal{B}_\beta]_\theta$ also belongs to $\mathcal{B}_\sigma$. ∎

7. Extremal Problem

Point evaluations already considered in Corollary 5.5 are important in function spaces for many reasons. Therefore it is of interest to know how large they can be as operators in any given space. In a weighted Bloch space, their size can be measured by the quantity $S_\alpha(b)$ of (5). In this section, we provide a solution of the related extremal problem.

Lemma 7.1. *Given* α, *pick* s, t *to satisfy* (1). *Then on* $\mathcal{B}_\alpha$ *considered with the norm* $\|f\|_{\mathcal{B}_\alpha} = \|I^t_s f\|_{L^\infty_\alpha}$, *the operator* W^α_ψ *of* (7) *is a linear surjective isometry with inverse* $(W^\alpha_\psi)^{-1} = D^{-t}_{s+t} T^\alpha_{\psi^{-1}} D^t_s$.

Proof. For $f \in \mathcal{B}_\alpha$, let $g = D^t_s f \in \mathcal{B}_{\alpha+t}$, note that $\alpha + t > 0$, and let $w = \psi(z)$. By (9) and Proposition 2.1, we have

$$\begin{aligned} \|W^\alpha_\psi f\|_{\mathcal{B}_\alpha} &= \|I^t_s W^\alpha_\psi f\|_{L^\infty_\alpha} = \|T^{\alpha+t}_\psi g\|_{L^\infty_{\alpha+t}} \\ &= \sup_{z\in\mathbb{B}} (1-|z|^2)^{\alpha+t} \frac{(1-|\psi(z)|^2)^{\alpha+t}}{(1-|z|^2)^{\alpha+t}} |g(\psi(z))| \\ &= \sup_{w\in\mathbb{B}} (1-|w|^2)^{\alpha+t} |g(w)| = \|g\|_{\mathcal{B}_{\alpha+t}} = \|f\|_{\mathcal{B}_\alpha}. \end{aligned}$$

The statement about the inverse is clear by (12) and $(J\psi^{-1})(\psi(z)) = 1/J\psi(z)$. ∎

Note that W^α_ψ is involutive when ψ is; this is so in particular with T^α_b. Because JU is constant, we see that T_U commutes with all the radial differential operators D^t_s and hence $W^\alpha_U f = T^\alpha_U f$. When $\alpha = 0$, Lemma 7.1 reduces to the Möbius-invariance of the classical Bloch space since then the derivaties cancel out and $W^0_\psi f = f \circ \psi$.

Proof of Theorem 1.4. Finding the extremal function at $b = 0$ is easy. First for $\alpha > 0$ and $t = 0$, clearly $f(0) \leq \sup_{z\in\mathbb{B}} (1-|z|^2)^\alpha |f(z)| = \|f\|_{\mathcal{B}_\alpha} = 1$, and equality holds if and only if f is identically 1 by the maximum modulus principle. So the unique extremal function is $f_0 \equiv 1$. Next for any α and $t \neq 0$ satisfying (1), we have

$$d_0\, f(0) = D_s^t f(0) \leq \sup_{z\in\mathbb{B}} (1-|z|^2)^{\alpha+t}\, |D_s^t f(z)| = \|f\|_{\mathcal{B}_\alpha} = 1.$$

Now the unique extremal function is $f_0 \equiv d_0^{-1}$.

To carry the result to other $b \in \mathbb{B}$, we use W_b^α of Lemma 7.1. Finding $S_\alpha(b)$ is equivalent to finding $\sup\big\{\, (W_b^\alpha f)(0) > 0 \,:\, f \in \mathcal{B}_\alpha,\ \|W_b^\alpha f\|_{\mathcal{B}_\alpha} = 1 \,\big\}$. The involutive property of W_b^α shows that the unique extremal function at b is $f_b = T_b^\alpha 1$ for $\alpha > 0$ and $t = 0$, and it is $f_b = d_0^{-1}\, W_b^\alpha 1$ for any α and $t \neq 0$ satisfying (1). To write the detailed form of f_b, we use for convenience $s_0 = 2\alpha + t - (N+1)$ when $t \neq 0$. Then by (10) and (13),

$$f_b(z) = \begin{cases} \dfrac{(1-|b|^2)^\alpha}{(1-\langle z,b\rangle)^{2\alpha}} & (\alpha > 0,\ t = 0); \\[2ex] (1-|b|^2)^{\alpha+t}\, K_{2\alpha+t-(N+1)}(z,b) & (\alpha + t > 0,\ s = s_0). \end{cases}$$

It can be checked that setting $b = 0$ in the second line above actually yields f_0 using the explicit forms of D_s^t in [10, Definition 3.1].

We are done with $\mathcal{B}_\alpha$. However, let's see that the extremal function lies in $\mathcal{B}_{\alpha 0}$ in each case using Definition 1.1. When $\alpha > 0$ and $t = 0$ in the norm, $(1-|z|^2)^\alpha\, f_b(z) \to 0$ as $|z| \to 1$ obviously. When $\alpha \leq 0$,

$$\begin{aligned}(1-|z|^2)^\alpha\, I_{s_0}^t f_b(z) &= (1-|z|^2)^{\alpha+t}\, (1-|b|^2)^{\alpha+t}\, K_{2(\alpha+t)-(N+1)}(z,b) \\ &= \frac{(1-|z|^2)^{\alpha+t}\, (1-|b|^2)^{\alpha+t}}{(1-\langle z,b\rangle)^{2(\alpha+t)}} \to 0\end{aligned}$$

as $|z| \to 1$, since (1) always holds. This completes the proof of the theorem. ■

Remark 7.2. As expected, Theorem 1.4 and Lemma 7.1 are the $p = \infty$ versions of [21, Theorem] and [10, Theorem 8.2] after setting $q = \alpha p$.

8. Maximality

The subject of this section is α-Möbius invariance and we prove Theorem 1.6 here. For each $\alpha > 0$, we give a nontrivial example of an α-Möbius-invariant space in Corollary 8.4. Similar results in the case $\alpha = 0$ are presented in [26, Lemma 3.18] and in [20, Theorem 0.3]. This case is different in its lack of α which causes its proofs to be more difficult and sometimes requiring stronger hypotheses. For example, the equivalent of Proposition 8.1 with $\alpha = 0$ requires the existence of a nonconstant function in the space. We concentrate on $\alpha \neq 0$ here. On the other hand, for any α, only diagonal unitary matrices need to be used in the proofs.

Proposition 8.1. *For $\alpha \neq 0$, an α-Möbius-invariant space X contains the polynomials.*

Proof. Fix $b \neq 0$ in $\mathbb{B}$. Since $1 \in X$, also $W_b^\alpha 1 \in X$. By Definition 1.2, (11), and (12),

$$\begin{aligned}(W_b^\alpha 1)(z) &= (D_{s+t}^{-t} T_b^{\alpha+t} D_s^t 1)(z) = d_0\,(D_{s+t}^{-t} T_b^{\alpha+t} 1)(z)\\ &= d_0\, D_{s+t}^{-t} \frac{(1-|b|^2)^{\alpha+t}}{(1-\langle z,b\rangle)^{2(\alpha+t)}} = d_0\,(1-|b|^2)^{\alpha+t} D_{s+t}^{-t} \sum_{k=0}^\infty c_k\,\langle z,b\rangle^k\\ &= d_0\,(1-|b|^2)^{\alpha+t} \sum_{k=0}^\infty d_k^{-1}\,c_k\,\langle z,b\rangle^k = \sum_\lambda c_\lambda\, z^\lambda,\end{aligned}$$

where $c_\lambda \neq 0$ for any $\lambda \in \mathbb{N}^N$. Fix any multi-index μ, let $U = \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_N})$ be unitary, $\theta = (\theta_1, \dots, \theta_N)$, and consider

$$f(z) = \frac{1}{(2\pi)^N} \int_{[-\pi,\pi]^N} (W_U^\alpha W_b^\alpha 1)(z)\, e^{-i\langle \mu,\theta\rangle}\, d\theta,$$

which belongs to X by the assumptions on the continuity of the W_U^α-action and the completenesss of X. Recalling the series for $W_b^\alpha 1$ and that $W_U^\alpha = T_U^\alpha$ shows that

$$f(z) = (\det U)^{2\alpha/(N+1)}\, c_\mu\, z^\mu.$$

Thus X contains any monomial z^μ. ■

Definition 8.2. A nonzero bounded linear functional on a normed space X of holomorphic functions on $\mathbb{B}$ is called *decent* if it extends to be continuous on $H(\mathbb{B})$.

Proof of Theorem 1.6. First let $\alpha > 0$. Let L be a decent functional on X, and let U and θ be as in the proof of Proposition 8.1. Define another linear functional by

$$L_U f = \frac{1}{(2\pi)^N} \int_{[-\pi,\pi]^N} L(T_U^\alpha f)\, (\det U)^{-2\alpha/(N+1)}\, d\theta \qquad (f \in X).$$

If we expand $T_U^\alpha f$ into series, the decency of L shows that $L_U f = f(0)\, L(1)$. On the other hand,

$$|f(0)|\,|L(1)| = |L_U f| \leq \frac{1}{(2\pi)^N} \int_{[-\pi,\pi]^N} \|L\|\, \|T_U^\alpha f\|_X\, d\theta \leq C\,\|L\|\,\|f\|_X.$$

Replacing f by $T_z^\alpha f$ gives

$$(1-|z|^2)^\alpha\, |f(z)|\,|L(1)| \leq C\,\|L\|\,\|T_z^\alpha f\|_X \leq C\,\|L\|\,\|f\|_X.$$

Hence if $L(1) \neq 0$, then $\|f\|_{\mathcal{B}_\alpha} \leq C\,\|L\|\,|L(1)|^{-1}\,\|f\|_X < \infty$, and $X \subset \mathcal{B}_\alpha$ with continuous inclusion.

Now we prove that a decent functional on X that is nonzero on 1 always exists. As in the proof of Proposition 8.1, for fixed $b \neq 0$ in $\mathbb{B}$ and by the decency of L,

$$L(T_b^\alpha 1) = \sum_\lambda c_\lambda \, L(z^\lambda)$$

with $c_\lambda \neq 0$ for any multi-index λ. If $L(T_b^\alpha 1) = 0$ for all $b \neq 0$ in $\mathbb{B}$, then $L(z^\lambda) = 0$ for all multi-indices λ and $L = 0$ because of its decency. So $L(T_b^\alpha 1) \neq 0$ for some $b \neq 0$ in $\mathbb{B}$. For such a b, define the linear functional $L_b = L T_b^\alpha$ on X. Then $L_b(1) \neq 0$ and $\|L_b\| \leq \|L\| \, \|T_b^\alpha\| \leq C \, \|L\|$. The functional L_b is also decent, because T_b^α is continuous on $H(\mathbb{B})$ since any Möbius transformation φ_b and $J\varphi_b$ take compact subsets of $\mathbb{B}$ to other compact subsets. This yields the desired result.

If $\alpha < 0$, pick s, t to satisfy (1), and consider the space $Y = \{ g = D_s^t f : f \in X \}$ with the norm $\|g\|_Y = \|f\|_X$. It is a matter of writing down the definitions and noting that T_U^α commutes with all D_s^t to check that Y is $(\alpha + t)$-Möbius-invariant. Then $Y \subset \mathcal{B}_{\alpha+t}$ and $X \subset \mathcal{B}_\alpha$ with continuous inclusions by Proposition 2.1. ■

Corollary 8.3. *There is no α-Möbius-invariant closed subspace of $H(\mathbb{B})$ other than $\{0\}$.*

Note that constants do not form an α-Möbius-invariant subspace. In the sense of Definition 1.5, $H(\mathbb{B})$ is not α-Möbius-invariant either, but this is a technicality.

Proof. Suppose Y is an invariant subspace that is properly contained in $H(\mathbb{B})$. By Hahn-Banach theorem, there is a continuous linear functional $L \neq 0$ on $H(\mathbb{B})$ whose restriction to Y is 0. For $f \in H(\mathbb{B})$, set $\chi(f) = \sup_\psi |L(T_\psi^\alpha f)|$ and define X as the completion of $\{ f \in H(\mathbb{B}) : \chi(f) < \infty \}$. It is easy to see that χ is a seminorm and that Theorem 1.6 is valid if X is given by a seminorm. Then $\chi(f) = 0$ for all $f \in Y$ and hence Y is contained in X on which L is a decent functional. By Theorem 1.6, X is contained in $\mathcal{B}_\alpha$ continuously, that is, $\|f\|_{\mathcal{B}_\alpha} \leq C \, \chi(f)$ for all $f \in X$. So if $f \in Y$, then $\|f\|_{\mathcal{B}_\alpha} = 0$ and thus $Y = \{0\}$. The conclusion also implies that χ is a true norm. ■

Corollary 8.4. *Suppose $p > 0$ and $q > -1$, or $p \geq 2$ and $-(N+1) \leq q \leq 1$. Suppose $\alpha > 0$. Suppose further α, q, and p are related by $N + 1 + q = \alpha p$. Then B_q^p is an α-Möbius-invariant space and thus $B_q^p \subset \mathcal{B}_\alpha$ with continuous inclusion.*

Proof. By [2, Theorem 3.3], [10, Theorem 8.2] and the given relation among the parameters, the Besov space B_q^p is α-Möbius-invariant. Then Theorem 1.6 applies. The set of (p, q) in the right half plane satisfying $N + 1 + q = \alpha p$ is a ray with slope α and q-intercept $-(N+1)$. ■

The inclusion part of this result appears in [2, Corollary 5.5] with a totally different proof, and is in fact a Sobolev-type imbedding.

9. Metrics and Lipschitz Property

In this section, we consider $\alpha > 0$ and develop the Hermitian metrics ρ_α with respect to which the weighted Bloch spaces $\mathcal{B}_\alpha$ have the Lipschitz property. We start with their infinitesimal forms. For a different point of view regarding Hermitian metrics and Bloch spaces, see [27].

For $z \in \mathbb{B}$, we define the matrix $g_\alpha(z)$ by

$$g_{\alpha_{ij}}(z) = \frac{1}{(1-|z|^2)^{2(1+\alpha)}} \left((1-|z|^2)\,\delta_{ij} + z_i \overline{z}_j\right) \qquad (1 \le i, j \le N),$$

where δ_{ij} is the Kronecker delta. The only difference of g_α from the infinitesimal Bergman metric g_0 is the presence of α in the power of the denominator. Clearly $g_{\alpha_{ji}} = \overline{g}_{\alpha_{ij}}$. Further, g_α is unitarily invariant in that $g_\alpha(Uz) = U\, g_\alpha(z)\, U^{-1}$ for a unitary transformation U of $\mathbb{C}^N$. We compute easily that

$$\det g_\alpha(z) = \frac{1}{(1-|z|^2)^{N+1+2N\alpha}} = K_{2N\alpha}(z,z) > 0 \qquad (z \in \mathbb{B}),$$

which also shows the form of g_α when $N = 1$. The leading principal minors of $g_\alpha(z)$ are just $\det g_\alpha(z)$ of all the dimensions from 1 through N, which are all positive. Thus $g_\alpha(z)$ is a positive definite matrix on $\mathbb{B}$. By the same reason, $g_\alpha(z)$ is invertible with its inverse given by

$$g_\alpha^{ij}(z) = (1-|z|^2)^{1+2\alpha}\,(\delta_{ij} - z_i \overline{z}_j) \qquad (1 \le i, j \le N).$$

Therefore g_α is an infinitesimal Hermitian metric on $\mathbb{B}$. It gives rise to a distance on $\mathbb{B}$ in the usual manner. If $\gamma = (\gamma_1, \ldots, \gamma_N)$ is a curve in $\mathbb{B}$ joining z and w in $\mathbb{B}$, its α-length is

$$l_\alpha(\gamma) = \int_0^1 \left\langle g_\alpha(\gamma(t))\gamma'(t), \gamma'(t) \right\rangle^{1/2} dt.$$

By taking the infimum of $l_\alpha(\gamma)$ over all curves joining z and w, we obtain a distance $\rho_\alpha(z,w)$ between z and w. If there exists a curve on which the infimum is attained, it is a geodesic of ρ_α.

Let $z = 0$ and $w = (r, 0, \ldots, 0)$ with $0 < r < 1$. The unitary invariance of g_α entails that the line segment γ_r joining z and w is a geodesic. Using the parametrization $\gamma_r(t) = (t, 0, \ldots, 0)$ for $0 \le t \le r$, we compute

$$l_\alpha(\gamma_r) = \int_0^r \frac{dt}{(1-t^2)^{1+\alpha}} \sim \int_0^r \frac{dt}{(1-t)^{1+\alpha}} = \frac{1}{\alpha}\left(\frac{1}{(1-r)^\alpha} - 1\right). \tag{25}$$

As $w \to 1$, $\lim_{r \to 1^-} l_\alpha(\gamma_r) = \infty$; that is, $\mathbb{B}$ is unbounded in the metric ρ_α. This simple result has important implications for ρ_α. By the Hopf-Rinow theorem (see

[5, Theorem 7.2.8]), $(\mathbb{B}, \rho_\alpha)$ is geodesically complete, complete as a metric space, its closed and bounded subsets are compact, and there exists a geodesic of g_α joining any two points in $\mathbb{B}$.

Associated to any Hermitian metric, there is defined a *Laplace-Beltrami operator* (see [18, Section 3.1]), which in our case is

$$\begin{aligned}
\widetilde{\Delta}_\alpha &= \frac{2}{\det g_\alpha} \sum_{i,j=1}^{N} \left[\frac{\partial}{\partial \overline{z}_j} \left((\det g_\alpha)\, g_\alpha^{ij} \frac{\partial}{\partial z_i} \right) + \frac{\partial}{\partial z_i} \left((\det g_\alpha)\, g_\alpha^{ij} \frac{\partial}{\partial \overline{z}_j} \right) \right] \\
&= 4\,(1-|z|^2)^{1+2\alpha} \left[\sum_{i,j=1}^{N} (\delta_{ij} - z_i \overline{z}_j)\, \frac{\partial^2}{\partial z_i\, \partial \overline{z}_j} + (N-1)\,\alpha\, \big(R + \overline{R}\big) \right] \\
&= 4 \sum_{i,j=1}^{N} g_\alpha^{ij}(z)\, \frac{\partial^2}{\partial z_i\, \partial \overline{z}_j} + 4\,(N-1)\,\alpha\,(1-|z|^2)^{1+2\alpha} \big(R + \overline{R}\big),
\end{aligned}$$

where $Rf(z) = \langle \nabla f(z), \overline{z} \rangle$ is the classical *radial derivative* of f, and ∇ denotes the complex gradient. The operators $\widetilde{\Delta}_\alpha$ are reminiscent of the variants of Laplace-Beltrami operators defined in a different context in [7, (1.11)], but there is a difference. Our $\widetilde{\Delta}_\alpha$ have no constant terms and annihilate constants. This can have interesting connections; see [11, Section 4]. Here also $(\widetilde{\Delta}_\alpha f)(0) = (\Delta f)(0)$, where Δ is the usual Laplacian. Further, $\widetilde{\Delta}_\alpha = (1-|z|^2)^{2(1+\alpha)} \Delta$ when $N = 1$, so $\widetilde{\Delta}_\alpha$ and Δ annihilate the same functions on $\mathbb{D}$.

The infinitesimal Bergman metric is obtained as

$$g_{0_{ij}} = \frac{1}{N+1}\, \frac{\partial^2 \log K_0(z,z)}{\partial z_i\, \partial \overline{z}_j}. \tag{26}$$

The presence of the first-order terms in $\widetilde{\Delta}_\alpha$ for $\alpha > 0$ and $N > 1$ imply that holomorphic functions are not annihilated by it, and more importantly, the corresponding g_α is not a Kähler metric; see [17, p. 26]. This is equivalent to the fact that g_α cannot be obtained by differentiation as in (26) for $\alpha > 0$ and $N > 1$. For $N = 1$ and small positive integer α, we can use integration by partial fractions to find formulas similar to (26). For example, let

$$L_1(z,w) = \frac{1}{3} \left(\log \frac{1}{1 - z\,\overline{w}} + \frac{1}{1 - z\,\overline{w}} + \frac{1}{2\,(1 - z\,\overline{w})^2} \right) \qquad (z, w \in \mathbb{D});$$

then

$$\frac{\partial^2 L_1(z,z)}{\partial z\, \partial \overline{z}} = \frac{1}{(1-|z|^2)^4} = g_1(z) \qquad (z \in \mathbb{D}).$$

Each Hermitian metric additionally gives rise to a gradient field; see [18, Section

3.4]. For $f \in H(\mathbb{B})$ and our g_α, it takes the form

$$|\widetilde{\nabla}_\alpha f(z)|^2 = 2 \sum_{i,j=1}^{N} g_\alpha^{ij} \frac{\partial f}{\partial z_i} \overline{\frac{\partial f}{\partial z_j}} = 2\,(1-|z|^2)^{1+2\alpha} \big(|\nabla f(z)|^2 - |Rf(z)|^2\big)$$
$$= 2\,(1-|z|^2)^{2\alpha}\,|\widetilde{\nabla} f(z)|^2,$$

where $\widetilde{\nabla} = \widetilde{\nabla}_0/\sqrt{2}$ and is called the *invariant gradient* since $\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$.

Before relating $\mathcal{B}_\alpha$ to ρ_α, we find an equivalent definition of $\mathcal{B}_\alpha$ much the same way as the early definition of $\mathcal{B}_0$ in [19, Definition 3.1]. To this end, set

$$Q_\alpha f(z) = \sup_{w \neq 0} \frac{|\langle \nabla f(z), \overline{w} \rangle|}{\sqrt{\langle g_\alpha(z) w, w\rangle}} \qquad (z \in \mathbb{B}),$$

and $f \in H(\mathbb{B})$.

Lemma 9.1. *If $\alpha > 0$ and $f \in H(\mathbb{B})$, then $f \in \mathcal{B}_\alpha$ if and only if $Q_\alpha f \in L^\infty$ and if and only if $|\widetilde{\nabla}_\alpha f| \in L^\infty$.*

Proof. The proof of [26, Theorem 3.1] with straightforward modifications for the presence of α yields that $Q_\alpha f(z) = (1-|z|^2)^\alpha\,|\widetilde{\nabla} f(z)| = |\widetilde{\nabla}_\alpha f(z)|/\sqrt{2}$. Then [26, Theorem 7.2 (a)] gives us what we want. ∎

When $N=1$, $Rf(z) = zf'(z)$, $\nabla f(z) = f'(z)$, $\widetilde{\nabla} f(z) = (1-|z|^2)\,f'(-z)$, and $Q_\alpha f(z) = (1-|z|^2)^{1+\alpha}\,|f'(z)|$.

For $f \in H(\mathbb{B})$, a direct computation shows that

$$\widetilde{\Delta}_\alpha(|f|^2)(z) = 2\,|\widetilde{\nabla}_\alpha f(z)|^2 + 2\,(N-1)\,\alpha\,\mathrm{Re}(\overline{f}(z)\,Rf(z)).$$

Thus $\widetilde{\Delta}_\alpha(|f|^2) = C\,|\widetilde{\nabla}_\alpha f|^2$ for $f \in H(\mathbb{B})$ if and only if $\alpha = 0$ or $N = 1$.

Proof of Theorem 1.7. Let $z, w \in \mathbb{B}$ and γ_α be a geodesic of ρ_α joining them, which we know exists. Then

$$|f(z) - f(w)| \leq \int_{\gamma_\alpha} \big|\langle \nabla f(u), \overline{du}\rangle\big| \leq \int_{\gamma_\alpha} Q_\alpha f(u)\,\sqrt{\langle g_\alpha(u) du, du\rangle}$$
$$\leq C\,\|f\|_{\mathcal{B}_\alpha} \int_0^1 \sqrt{\big\langle g_\alpha(\gamma_\alpha(t))\gamma_\alpha'(t), \gamma_\alpha'(t)\big\rangle}\,dt = C\,\|f\|_{\mathcal{B}_\alpha}\,\rho_\alpha(z,w).$$

In the converse direction $N = 1$. Consider the curve $\gamma(t) = t$ for $0 \leq t \leq r < 1$ between 0 and r. Let $z \in \mathbb{D}$ and $\xi(t) = \varphi_z(\gamma(t))$. Then $\xi(t)$ is a curve between z and $w = \varphi_z(r)$, $\xi'(t) = \varphi_z'(t)$, and

$$l_\alpha(\xi) = \int_0^r \frac{|\varphi_z'(t)|}{(1-|\varphi_z(t)|^2)^{1+\alpha}}\,dt = \int_0^r \frac{1}{(1-t^2)^{1+\alpha}} \frac{|1-\overline{z}t|^{1+\alpha}}{(1-|z|^2)^\alpha}\,dt$$
$$\leq \frac{(1+|z|)^{2\alpha}}{(1-|z|^2)^\alpha} \int_0^r \frac{dt}{(1-t)^{1+\alpha}} = \frac{(1+|z|)^{2\alpha}}{(1-|z|^2)^\alpha}\,\frac{1}{\alpha}\bigg(\frac{1}{(1-r)^\alpha} - 1\bigg).$$

by [16, Theorem 2.2.2 (iv)] and (25). The same is true if also φ_z is composed with rotations in obtaining ξ from γ. In all cases $r = |\varphi_z(w)|$. Then

$$\rho_\alpha(z,w) \leq \frac{2^{2\alpha}}{\alpha} \frac{1}{(1-|z|^2)^\alpha} \left(\frac{1}{(1-|\varphi_z(w)|)^\alpha} - 1 \right).$$

Now let $w = z + h$ with $|h|$ small. Then

$$\varphi_z(w) \sim \frac{|h|}{1-|z|^2} \qquad \text{and} \qquad (1-|\varphi_z(w)|)^\alpha \sim 1 - \frac{\alpha|h|}{1-|z|^2},$$

and thus

$$\rho_\alpha(z,w) \leq \frac{C}{(1-|z|^2)^\alpha} \frac{|h|}{1-|z|^2-\alpha|h|} \,.$$

Using the assumption,

$$C \geq \frac{|f(z)-f(w)|}{\rho_\alpha(z,w)} \geq C \, \frac{|f(z)-f(z+h)|}{|h|} \, (1-|z|^2)^\alpha \, (1-|z|^2-\alpha|h|).$$

Letting $h \to 0$, we obtain $(1-|z|^2)^{\alpha+1} |f'(z)| \leq C$ for all $z \in \mathbb{D}$. Therefore $f \in \mathcal{B}_\alpha$. ■

The case of the classical Bloch space has some extra properties which we do not know for $\alpha > 0$. The metric ρ_0 is invariant under Mőbius transformations, but we do not know of any isometries of ρ_α other than unitary transformations. The invariance gives rise to the well-known explicit logarithmic formula for $\rho_0(z,w)$ which we do not have for $\alpha > 0$. This lack of explicit formula for ρ_α is the main obstacle to obtaining the converse in Theorem 1.7 for $N > 1$. The computations involved for the style of proof presented above for $N > 1$ are prohibitively complicated. For $N = 1$ and small positive integer α, it is possible to obtain explicit expressions for $\rho_\alpha(z,w)$ using integration by partial fractions and the unitary invariance of g_α. For example, with $\alpha = 1$, a tedious computation yields that

$$\rho_1(z,w) = \frac{1}{4} \log \frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|} + \frac{1}{2} \, \frac{|\varphi_z(w)| \, (1-2\,\mathrm{Re}(\overline{z}\,\varphi_z(w))+|z|^2)}{1-|\varphi_z(w)|^2} \quad (z,w \in \mathbb{D})\,.$$

As a final note, let's compute the *holomorphic sectional curvatures* (see [14, Section 2.1]) of the new metrics. When $N = 1$, they are

$$\kappa_\alpha(z) = -\frac{\Delta \log g_\alpha(z)}{g_\alpha(z)} = -4\,(1+\alpha)\,(1-|z|^2)^{2\alpha} \qquad (z \in \mathbb{D}).$$

Clearly $\kappa_\alpha(z) \leq 0$, and $\kappa_\alpha(z) \to 0$ as $z \to \partial\mathbb{D}$. This curvature is more difficult to compute exactly for higher N, but it is clear that the factor $1-|z|^2$ will persist with a positive power. So the new metrics for $\alpha > 0$ have curvatures that are neither constant nor bounded away from 0 unlike the Bergman metric.

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H. Turgay Kaptanoğlu
Department of Mathematics,
Bilkent University,
Ankara 06800,
Turkey
E-mail: kaptan@fen.bilkent.edu.tr

Serdar Tülü
Department of Mathematics,
Bilkent University,
Ankara 06800,
Turkey

(Current address)
Department of Mathematics,
İzmir Institute of Technology (İYTE),
Urla, İzmir 35437,
Turkey
E-mail: serdartulu@iyte.edu.tr