TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 1, pp. 43-57, February 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

REMOTELY ALMOST PERIODIC SOLUTIONS TO PARABOLIC BOUNDARY VALUE INVERSE PROBLEMS

Fenglin Yang and Chuanyi Zhang

Abstract. Some properties of remotely almost periodic functions are studied. The existence and uniqueness of remotely almost periodic solution to Parabolic Inverse Problems for a type of boundary value problem are established. Stability of the solution is discussed.

1. Introduction

Sarason in [12] proposed the space $\mathcal{RAP}(\mathbb{R})$ of remotely almost periodic functions. This is a C^*subalgebra of $\mathcal{C}(\mathbb{R})$, the space of bounded, continuous, complex-valued functions f on $\mathbb{R}$ with the supremum norm $\|f\| = \sup\{|f(x)| : \ x \in \mathbb{R}\}$. Comparing with the space $\mathcal{AP}(\mathbb{R})$ of almost periodic functions, $\mathcal{RAP}(\mathbb{R})$ is a quite large space (see [12, 19]). $\mathcal{AP}(\mathbb{R})$ and some of its generalizations have many applications to the theory of differential equations (e.g., [10, 13, 14, 16-18] and references therein). It is reasonable to believe that $\mathcal{RAP}(\mathbb{R})$ has applications in this aspect too. The present paper is denoted to this. The central question is to investigate the remotely almost periodic solution of parabolic boundary value inverse problems.

To this end, we need first to extend the space $\mathcal{RAP}(\mathbb{R})$ to a more general setting. Let $J \in \{\mathbb{R}, \mathbb{R}^n\}$. Let $\mathcal{C}(J)$ (respectively, $\mathcal{C}(J\times\Omega)$, where $\Omega \subset \mathbb{R}^m$) denote the C^*-algebra of bounded continuous complex-valued functions on J (respectively $J\times\Omega$). For $f \in \mathcal{C}(J)$ (respectively, $\mathcal{C}(J \times \Omega)$) and $s \in J$, the translate of f by s is the function $R_s f(t) = f(t+s)$ (respectively, $R_s f(t, Z) = f(t+s, Z)$, $(t, Z) \in J\times\Omega$). Let

$$dist_\infty(f, g) = \lim \sup_{|t|\to\infty} |f(t) - g(t)|$$

Received May 24, 2006, accepted June 1, 2009.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: 35B15, 35R30.
Key words and phrases: Remotely almost periodic function, Inverse problems, Parabolic equations, Boundary value problems.
The research is supported by State Natural Science Funds of China (No. 10671046).

Definition 1.1.

(1) A function $f \in \mathcal{C}(J)$ is called remotely almost periodic if for every $\epsilon > 0$ the set

$$T(f, \epsilon) = \{\tau \in J : \ dist_\infty(R_\tau f, f) < \epsilon\}$$

is relatively dense in J. Denote by $\mathcal{RAP}(J)$ the set of all such functions. The number (vector) τ is called remote ϵ-translation number (vector) of f.

(2) A function $f \in \mathcal{C}(J \times \Omega)$ is said to be remotely almost periodic in $t \in J$ and uniform on compact subsets of Ω if $f(\cdot, Z) \in \mathcal{RAP}(J)$ for each $Z \in \Omega$ and is uniformly continuous on $J \times K$ for any compact subset $K \subset \Omega$. Denote by $\mathcal{RAP}(J \times \Omega)$ the set of all such functions. For convenience, such functions are also called uniformly remotely almost periodic.

(3) Let X be a Banach space and let $\mathcal{C}(J, X)$ be the space of bounded continuous functions from J to X. If we replace $\mathcal{C}(J)$ in (1) by $\mathcal{C}(J, X)$ then we get the definition of $\mathcal{RAP}(J, X)$.

As in [12], we always assume that $f \in \mathcal{RAP}(J)$ is uniformly continuous.

In the next sections, we will present some properties of remotely almost periodic functions. The main results are in Section 3, where a type of boundary value problem is investigated. The results in Section 3 are probably new even for the almost periodic functions.

2. Some Properties

The following proposition can be proved in the same way as Corollary 1.1.4 in [16], or Theorem 1.8 in [5] and Theorem 1.16 in [6].

Proposition 2.1. *Let $f \in \mathcal{RAP}(J)$ $(\mathcal{RAP}(J \times \Omega))$ be such that $\partial f / \partial x_i$ is uniformly continuous on J. Then $\partial f / \partial x_i \in \mathcal{RAP}(J)$ $(\mathcal{RAP}(J \times \Omega))$.*

Proposition 2.2. *Let $f_i \in \mathcal{RAP}(J)$, $i = 1, 2, \cdots, m$. Then for every $\epsilon > 0$ the set*

$$T(f_1, f_2, \cdots, f_m, \epsilon) = \{\tau \in J : \ dist_\infty(R_\tau f_i, f_i) < \epsilon, \ i = 1, 2, \cdots, m\}$$

is relatively dense in J.

Proof. The main result in [12] shows that $\mathcal{RAP}(\mathbb{R})$ is the closed subalgebra of $\mathcal{C}(\mathbb{R})$ generated by $\mathcal{AP}(\mathbb{R})$ and $\mathcal{SO}(\mathbb{R})$, the slowly oscillating function space consisting of the functions φ such that $R_a\varphi(x) - \varphi(x) \to 0$ as $|x| \to \infty$ for all $a \in \mathbb{R}$. Without difficulty this result can been generalized to $\mathbb{R}^n$. Thus, if $f \in \mathcal{RAP}(J)$ then for $\epsilon > 0$ there exist g_1, $g_2 \in \mathcal{AP}(J)$ and φ_1, $\varphi_2 \in \mathcal{SO}(J)$ such that $\|f - g_1 + \varphi_1 + g_2\varphi_2]\| < \epsilon/4$. If $\varphi_2 = 0$ then any ϵ-translation number of g_1 is a remote ϵ-translation number of f. In the case that $\varphi_2 \neq 0$. Let $\delta = \min\{\epsilon/4, \epsilon/(4\|\varphi_2\|)\}$ and let τ be a δ-translation number common to g_1 and g_2. We show that τ is a remote $\epsilon/4$-translation number of $g_1 + \varphi_1 + g_2\varphi_2$ and therefore is a remote ϵ-translation number of f. In fact,

$$\begin{aligned} & |g_2(t+\tau)\varphi_2(t+\tau) - g_2(t)\varphi_2(t)| \\ \leq\ & |g_2(t+\tau)\varphi_2(t+\tau) - g_2(t+\tau)\varphi_2(t)| + |g_2(t+\tau)\varphi_2(t) - g_2(t)\varphi_2(t)| \\ & \|g_2\||\varphi_2(t+\tau) - \varphi_2(t)| + \|\varphi_2\|\|R_\tau g_2 - g_2\| \end{aligned}$$

and so

$$dist_\infty(R_\tau g_2\varphi_2 - g_2\varphi_2) \leq \|\varphi_2\|\|R_\tau g_2 - g_2\| < \|\varphi_2\|\delta < \epsilon/4.$$

Now we show the proposition. By the fact we just showed, for $f_i \in \mathcal{RAP}(J)$ there exist $g_{i1},\ g_{i2} \in \mathcal{AP}(J),\ \varphi_{i1},\ \varphi_{i2} \in \mathcal{SO}(J)$, and $\delta_i > 0$ such that any δ_i-translation number τ common to g_{i1} and g_{i2} is a remote ϵ-translation number of f_i. Let $\delta = \min\{\delta_i :\ 1 \leq i \leq n\}$. Since the set of δ-translation number common to $g_{ij} :\ i = 1, 2, \cdots, n,\ j = 1, 2$ is relatively dense in J, so is the set of remote ϵ-translation number common to $f_i :\ i = 1, 2, \cdots, n$.

The proof is complete. ∎

For $H = (h_1, h_2, \cdots, h_n) \in \mathcal{C}(\mathbb{R})^n$, suppose that $H(t) \in \Omega$ for all $t \in \mathbb{R}$. Define $H \times \iota \to \Omega \times \mathbb{R}$ by

$$H \times \iota(t) = (h_1(t), h_2(t), \cdots, h_n(t), t) \qquad (t \in \mathbb{R}).$$

The following proposition shows the remote almost periodicity of the composite.

Proposition 2.3. *Let $f \in \mathcal{RAP}(\mathbb{R} \times \Omega)$. If $H \in \mathcal{RAP}(\mathbb{R})^n$ and $H(t) \in \Omega$ for all $t \in \mathbb{R}$ then $f \circ (H \times \iota) \in \mathcal{RAP}(\mathbb{R})$.*

Proof. Without loss of generality, we may assume that Ω is bounded and closed because the set $\{H(t) : t \in \mathbb{R}\}$ is bounded in $\mathbb{C}^n$.

Let $\epsilon > 0$. The uniform continuity of f implies that there exists $\epsilon > \delta > 0$ such that

$$|f(Z_1, t) - f(Z_2, t)| < \frac{\epsilon}{2} \qquad (t \in \mathbb{R},\ |Z_1 - Z_2| < \delta,\ Z_1, Z_2 \in \Omega).$$

Since $f \in \mathcal{RAP}(\mathbb{R} \times \Omega)$ and $H \in \mathcal{RAP}(\mathbb{R})^n$, we have, for any remote $\delta/2$-translation number τ common to f and H,

$$\begin{aligned} & \limsup_{|t|\to\infty} |f \circ H \times \iota(t+\tau) - f \circ H \times \iota(t)| \\ \leq\ & \limsup_{|t|\to\infty} |f(H(t+\tau), t+\tau) - f(H(t+\tau), t)| \\ & + \limsup_{|t|\to\infty} |f(H(t+\tau), t) - f(H(t), t)| < \epsilon. \end{aligned}$$

The proof is complete. ■

In the sequel we will use the notations: $\mathbb{R}^m_T = \mathbb{R}^m \times (0,T)$, $\|F\|_T = \sup\{ |F(x,t)| : x \in \mathbb{R}^n,\ 0 \le t \le T\}$. $F \in \mathcal{RAP}(\mathbb{R}^n \times \mathbb{R}^m_T)$ means that $F(x^{(1)}, x^{(2)}, t)$ is remotely almost periodic in $x^{(1)} \in \mathbb{R}^n$ and uniformly for $(x^{(2)}, t) \in \mathbb{R}^m_T$; $F \in \mathcal{RAP}(\mathbb{R}^n \times \mathbb{R}^m)$ means that $F(x^{(1)}, x^{(2)})$ is remotely almost periodic in $x^{(1)} \in \mathbb{R}^n$ and uniformly for $x^{(2)} \in \mathbb{R}^m$.

Let

$$Z(x,t;\xi,s) = \frac{1}{(2\sqrt{\pi(t-s)})^{n+m}} \exp\{-\frac{\sum(x_i - \xi_i)^2}{4(t-s)}\} \qquad (x, \xi \in \mathbb{R}^{n+m})$$

be the fundamental solution of heat equation [7].

Proposition 2.4. *Let $T > 0$. If $\varphi \in \mathcal{RAP}(\mathbb{R}^n \times \mathbb{R}^m)$ and*

$$u(x,t;s) = \int_{\mathbb{R}^{n+m}} \varphi(\xi) Z(x,t;\xi,s) d\xi$$

then for each fixed $s \in [0,T)$, $u \in \mathcal{RAP}(\mathbb{R}^n \times \mathbb{R}^m \times [s,T])$.

Proof. Let $\tau \in \mathbb{R}^n$ be a remote $\epsilon/3$-translation vector of φ.

$$\begin{aligned}
& u(x^{(1)}+\tau, x^{(2)}, t; s) - u(x^{(1)}, x^{(2)}, t; s) \\
&= \int_{\mathbb{R}^{n+m}} \varphi(\xi^{(1)}, \xi^{(2)})[Z(x^{(1)}+\tau, x^{(2)}, t; \xi^{(1)}, \xi^{(2)}, s) \\
&\quad - Z(x^{(1)}, x^{(2)}, t; \xi^{(1)}, \xi^{(2)}, s)] d\xi^{(1)} d\xi^{(2)} \\
&= \int_{\mathbb{R}^{n+m}} [\varphi(x^{(1)}+\tau+\xi^{(1)}, x^{(2)} \\
&\quad + \xi^{(2)}) - \varphi(x^{(1)}+\xi^{(1)}, x^{(2)}+\xi^{(2)})] Z(\theta, t; \xi, s) d\xi \\
&= \left(\int_{-\infty}^{-A} + \int_{-A}^{A} + \int_{A}^{\infty} \right) [\varphi(x^{(1)}+\tau+\xi^{(1)}, x^{(2)}+\xi^{(2)}) \\
&\quad - \varphi(x^{(1)}+\xi^{(1)}, x^{(2)}+\xi^{(2)})] Z(\theta, t; \xi, s) d\xi
\end{aligned} \tag{2.1}$$

where $\theta \in \mathbb{R}^{n+m}$ is the zero vector and by $\int_a^b F(\xi) d\xi$ we mean that

$$\int_a^b F(\xi) d\xi = \int_{[a,b]^{n+m}} F(\xi) d\xi = \int_a^b \cdots \int_a^b F(\xi_1, \xi_2, \cdots .\xi_{n+m}) d\xi_1 d\xi_2 \cdots d\xi_{n+m}.$$

Note that $\int_{\mathbb{R}^{n+m}} Z(\theta,t;\xi,s) d\xi = 1$, there exists an $A > 0$ such that for $(\xi^{(1)}, \xi^{(2)}) \in [-A, A]^{n+m}$,

$$\lim \sup_{|x^{(1)}|\to\infty} |\varphi(x^{(1)}+\tau+\xi^{(1)}, x^{(2)}+\xi^{(2)}) - \varphi(x^{(1)}+\xi^{(1)}, x^{(2)}+\xi^{(2)})| < \epsilon/3$$

and

$$2\|\varphi\| \int_A^\infty Z(\theta, t; \xi, s) d\xi < \epsilon/3.$$

It follows from (2.1) that

$$\lim \sup_{|x^{(1)}|\to\infty} |u(x^{(1)}+\tau, x^{(2)}, t; s) - u(x^{(1)}, x^{(2)}, t; s)| < \epsilon,$$

where $t \in [s, T]$ and $x^{(2)} \in B$ with B a bounded subset of $\mathbb{R}^m$. This shows that every remote $\epsilon/3$-translation vector of φ is a remote ϵ-translation vector of u and therefore, $u \in \mathcal{RAP}(\mathbb{R}^n \times \mathbb{R}^m \times [s, T])$. The proof is complete. ■

Proposition 2.5. *Let φ, $\partial\varphi/\partial x_i \in \mathcal{RAP}(\mathbb{R}^n \times \mathbb{R}^m)$ and let u be as in Proposition 2.4. Then $\partial u/\partial x_i \in \mathcal{RAP}(\mathbb{R}^n \times \mathbb{R}^m \times [s, T])$.*

Proof. Note that

$$\begin{aligned} \frac{\partial u(x,t;s)}{\partial x_i} &= \int_{\mathbb{R}^{n+m}} \varphi(\xi) \frac{\partial Z(x,t;\xi,s)}{\partial x_i} d\xi \\ &= -\int_{\mathbb{R}^{n+m}} \varphi(\xi) \frac{\partial Z(x,t;\xi,s)}{\partial \xi_i} d\xi = \int_{\mathbb{R}^{n+m}} \frac{\partial \varphi(\xi)}{\partial \xi_i} Z(x,t;\xi,s) d\xi. \end{aligned}$$

By Proposition 2.4 we get the conclusion. ■

Proposition 2.6. *If $f(x,t) \in \mathcal{RAP}(\mathbb{R}^n \times \overline{\mathbb{R}^m_T})$ and*

$$u(x,t) = \int_0^t ds \int_{\mathbb{R}^{n+m}} f(\xi, s) Z(x, t; \xi, s) d\xi$$

then u and $\partial u(x,t)/\partial x_i$ ($i = 1, 2, \cdots, n+m$) are all in $\mathcal{RAP}(\mathbb{R}^n \times \overline{\mathbb{R}^m_T})$. The proof is similar to that of Proposition 2.4, so we omit it.

3. A Type of Boundary Value Problem

We will keep the notations in the last section and at the same time, introduce the following new notations:

$$x = (x_1, x_2, \cdots, x_{n-1}) \qquad \xi = (\xi_1, \xi_2, \cdots, \xi_{n-1})$$

and

$$X = (x, x_n) \qquad \zeta = (\xi, \xi_n) \qquad D^n = \{X \in \mathbb{R}^n : x_n > 0\}.$$

In this section, we always assume: f, $f_{x_nx_n} \in \mathcal{RAP}(\mathbb{R}^{n-1} \times \overline{D_{T_0}})$, $h(x,t) \geq const > 0$, h, $(\Delta h - h_t) \in \mathcal{RAP}(\overline{\mathbb{R}^{n-1}_{T_0}})$, φ, $\varphi_{x_nx_n} \in \mathcal{RAP}(\mathbb{R}^{n-1} \times D)$, $\varphi \in C^3(\mathbb{R}^{n-1} \times D)$, and g, $(\Delta g - g_t) \in \mathcal{RAP}(\overline{\mathbb{R}^{n-1}_{T_0}})$.

Let

$$G(X,t;\zeta,\tau) = Z(X,t;\xi,\xi_n,\tau) + Z(X,t;\xi,-\xi_n,\tau)$$

be the Green's function for the boundary value problems [8, 15].

The following estimates are easily obtained:

$$\left\| \int_0^t ds \int_{D^n} G(X,t;\zeta,s)d\zeta \right\| \leq m_1(T)$$

$$\left\| \int_0^t ds \int_{\mathbb{R}^{n-1}} Z(X,t;\xi,0,s)d\xi \right\| \leq m_2(T)$$

$$\left\| \int_0^t ds \int_{\mathbb{R}^n} \frac{\partial Z(X,t;\zeta,s)}{\partial x_n} d\zeta \right\| \leq m_3(T),$$

where $m_i(T)$ $(i = 1,2,3)$ are positive and increasing for $T \geq 0$ and $m_i(T) \to 0$ as $T \to 0$.

To show the main results of this section, the following lemmas are needed. The first lemma is Lemma 3.1 on P15 in [9].

Lemma 3.1. *Let φ, ϕ and χ be real, continuous functions on $[0,T]$ with $\chi \geq 0$. If*

$$\varphi(t) \leq \phi(t) + \int_0^t \chi(s)\varphi(s)ds \qquad (t \in [0,T])$$

then

$$\varphi(t) \leq \phi(t) + \int_0^t \chi(s)\phi(s)\exp\{\int_s^t \chi(\rho)d\rho\}ds \qquad (t \in [0,T]).$$

Lemma 3.2. *Let φ be a continuous function on $[0,T]$. If ϕ, χ_1 and χ_2 are nondecreasing and nonnegative on $[0,T]$ and*

$$\varphi(t) \leq \phi(t) + \chi_1(t)\int_0^t \varphi(s)ds + \chi_2(t)\int_0^t \frac{\varphi(s)}{\sqrt{t-s}}ds \qquad (t \in [0.T]) \tag{3.0}$$

then

$$\varphi(t) \leq \phi(t)[1 + t\chi_1(t) + 2\sqrt{t}\chi_2(t)]e^{t\chi(t)}$$

where

$$\chi(t) = t\chi_1^2(t) + 4\sqrt{t}\chi_1(t)\chi_2(t) + \pi\chi_2^2(t).$$

Proof. Replacing $\varphi(s)$ in the two integrals of (3.0) by the expression of the right hand side in (3.0), changing the integral order of the resulting inequality, and making use of the monotonicity of ϕ, χ_1 and χ_2, one gets

$$\varphi(t) \leq \phi(t)[1+t\chi_1(t)+2\sqrt{t}\chi_2(t)]+[t\chi_1^2(t)+4\sqrt{t}\chi_1(t)\chi_2(t)+\pi\chi_2^2(t)]\int_0^t \varphi(s)ds.$$

Apply Lemma 3.1 to get the conclusion. ■

Lemma 3.3. *Let* $F(X,t) \in \mathcal{RAP}(\overline{D_T^n})$, $\phi(x,t), q(x,t) \in \mathcal{RAP}(\overline{\mathbb{R}_T^{n-1}})$ *and* $\varphi \in \mathcal{RAP}(D^n)$. *Then the equation*

$$\begin{cases} u_t - \Delta u + qu = F(X,t) & (X,t) \in D_T^n \\ u(X,0) = \varphi(X) & X \in D^n \\ u_{x_n}(x,0,t) = \phi(x,t) & (x,t) \in \mathbb{R}_T^{n-1} \end{cases}$$

has a unique solution u, u *is in* $\mathcal{RAP}(\overline{D_T^n})$ *and satisfies*

$$\|u\|_T \leq K(T)[T\|F\|_T + \|\varphi\| + \frac{\sqrt{T}}{2}\|\phi\|_T]$$

where $K(T) = 2(1 + T\|q\|_T e^{T\|q\|_T})$.

One sees that $K(T)$ depends on $\|q\|_T$ only and is bounded near zero.

Proof. The existence and uniqueness of the solution come from Theorem 5.3 on P320 in [11].

As [8, 15], the solution u can be written as

$$\begin{aligned} u(X,t) = &\int_{D^n} \varphi(\zeta)G(X,t;\zeta,0)d\zeta + \int_0^t ds \int_{D^n} F(\zeta,s)G(X,t;\zeta,s)d\zeta \\ &- \int_0^t ds \int_{D^n} q(\xi,s)u(\zeta,s)G(X,t;\zeta,s)d\zeta \\ &-2\int_0^t ds \int_{\mathbb{R}^{n-1}} \phi(\xi,s)Z(X,t;\xi,0,s)d\xi v(x,t) \\ &- \int_0^t ds \int_{D^n} q(\xi,s)u(\zeta,s)G(X,t;\zeta,s)d\zeta. \end{aligned}$$

So,

$$\|u\|_t \leq 2\|\varphi\| + 2\int_0^t \|F\|_s ds + 2\int_0^t \frac{\|\phi\|_s}{\sqrt{t-s}}ds + 2\int_0^t \|q\|_s\|u\|sds.$$

By Lemma 3.1 one gets the desired inequality.

Now we show that $u \in \mathcal{RAP}(\overline{D^n_T})$. As the proof of Propositions 2.4 and 2.6, one gets $v \in \mathcal{RAP}(\overline{D^n_T})$.

$$u(x+\tau, x_n, t) - u(x, x_n, t) = v(x+\tau, x_n, t) - v(x, x_n, t)$$
$$-\int_0^t ds \int_{D^n} q(\xi, s)u(\zeta, s)[G(x+\tau, x_n, t; \zeta, s) - G(x, x_n, t; \zeta, s)]d\zeta$$
$$= v(x+\tau, x_n, t) - v(x, x_n, t) - \int_0^t ds \int_{D^n} [q(x+\tau+\xi, s)u(x+\tau+\xi, x_n+\xi_n, s)$$
$$-q(x+\xi, s)u(x+\xi, x_n+\xi_n, s)]G(\theta, t; \zeta, s)d\zeta$$
$$= v(x+\tau, x_n, t) - v(x, x_n, t)$$
$$-\int_0^t ds \int_{D^n} [q(x+\tau+\xi, s) - q(x+\xi, s)]u(x+\tau+\xi, x_n+\xi_n, s)G(\theta, t; \zeta, s)d\zeta$$
$$-\int_0^t ds \int_{D^n} [u(x+\tau+\xi, x_n+\xi_n, s) - u(x+\xi, x_n+\xi_n, s)]q(x+\xi, s)G(\theta, t; \zeta, s)d\zeta.$$

Note that
$$\left| \int_{D^n} q(\xi, s)G(\theta, t; \zeta, s)d\zeta \right| \leq B\|q\|_s$$
where B is a constant.

As the proof of Proposition 2.4, for $\epsilon > 0$ there exists $A > 0$ such that

$$\left| \int_0^t ds \int_{D^n} [q(x+\tau+\xi, s) - q(x+\xi, s)]u(x+\tau+\xi, x_n+\xi_n, s)G(\theta, t; \zeta, s)d\zeta \right|$$
$$\leq \int_0^t ds \left(\int_{(\mathbb{R}^{n-1}\setminus[-A,A]^{n-1})\times\mathbb{R}_+} + \int_{[-A,A]^{n-1}\times\mathbb{R}_+} \right)$$
$$|[q(x+\tau+\xi, s). - q(x+\xi, s)]u(x+\tau+\xi, x_n+\xi_n, s)G(\theta, t; \zeta, s)|d\zeta$$
$$\leq \epsilon + \int_0^t ds \int_{[-A,A]^{n-1}\times\mathbb{R}_+} |q(x+\tau+\xi, s)$$
$$-q(x+\xi, s)]u(x+\tau+\xi, x_n+\xi_n, s)G(\theta, t; \zeta, s)|\, d\zeta.$$

Therefore,

$$dist_\infty(R_\tau u, u)_t \leq dist_\infty(R_\tau v, v)_t + B \cdot dist_\infty(R_\tau q, q)_t + \epsilon$$
$$+B \int_0^t dist_\infty(R_\tau u, u)_s \|q\|_s ds,$$

where
$$dist_\infty(R_\tau q, q)_t = \sup_{s\in[0,t]} dist_\infty(q(\cdot+\tau, s), q(\cdot, s)).$$

By Lemma 3.1, one has

$$dist_\infty(R_\tau u, u)_t \le m[dist_\infty(R_\tau v, v)_t + B \cdot dist_\infty(R_\tau q, q)_t + \epsilon],$$

where m is a constant. This means that $u \in \mathcal{RAP}(\overline{D_T^n})$. The proof is complete. ■

Consider the following problem:

Problem 3.1. Find functions $u \in \mathcal{RAP}(\mathbb{R}^{n-1} \times \overline{D_T})$ and $q \in \mathcal{RAP}(\overline{\mathbb{R}_T^{n-1}})$ such that

$$\begin{cases} u_t - \Delta u + q(x,t)u = f(X,t) & (X,t) \in D_T^n & (3.1) \\ u(X,0) = \varphi(X) & X \in D^n & (3.2) \\ u_{x_n}(x,0,t) = g(x,t) & (x,t) \in \mathbb{R}_T^{n-1} & (3.3) \\ u(x,a,t) = h(x,t) & (x,t) \in \mathbb{R}_T^{n-1} \quad a \in (0,\infty). & (3.4) \end{cases}$$

One sees that

$$h(x,0) = \varphi(x,a) \qquad \varphi_{x_n}(x,0) = g(x,0) \qquad x \in \mathbb{R}^{n-1}, \tag{3.5}$$

$$\begin{aligned} h_t(x,0) &= u_t|_{x_n=a,t=0} = [\Delta u - qu + f(X,t)]_{x_n=a,t=0} \\ &= \Delta\varphi(X)|_{x_n=a} - q(x,0)\varphi(x,a) + f(x,a,0) \end{aligned} \tag{3.6}$$

and

$$g_t(x,0) = u_{tx_n}|_{x_n=0,t=0} = \Delta\varphi_{x_n}(X)|_{x=0} - q(x,0)\varphi_{x_n}(x,0) + f_{x_n}(x,0,0) \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\begin{aligned} &\varphi_{x_n}(x,0)\Delta\varphi(X)|_{x_n=a} + f(x,a,0)\varphi_{x_n}(x,0) - h_t(x,0)\varphi_{x_n}(x,0) \\ &= \varphi(x,a)\Delta\varphi_{x_n}(X)|_{x_n=0} + f_{x_n}(x,0,0)\varphi(x,a) - g_t(x,0)\varphi(x,a) \end{aligned} \tag{3.8}$$

Let $V(X,t) = u_{x_n}(X,t)$ and $W(X,t) = V_{x_n}(X,t)$. We have the following two more problems for V and W respectively.

Problem 3.2. Find functions $V \in \mathcal{RAP}(\mathbb{R}^{n-1} \times \overline{D_T})$ and $q \in \mathcal{RAP}(\overline{\mathbb{R}_T^{n-1}})$ such that

$$\begin{cases} V_t - \Delta V + q(x,t)V = f_{x_n}(X,t) & (X,t) \in D_T^n & (3.9) \\ V(X,0) = \varphi_{x_n}(X) & X \in D^n & (3.10) \\ V(x,0,t) = g(x,t) & (x,t) \in \mathbb{R}_T^{n-1} & (3.11) \\ V_{x_n}(x,a,t) = h_t - \Delta h + qh - f(x,a,t) & (x,t) \in \mathbb{R}_T^{n-1} & (3.12) \end{cases}$$

Problem 3.3. Find functions $W \in \mathcal{RAP}(\mathbb{R}^{n-1} \times \overline{D_T})$ and $q \in \mathcal{RAP}(\overline{\mathbb{R}^{n-1}_T})$ such that

$$\begin{cases} W_t - \Delta W + q(x,t)W = f_{x_n x_n}(X,t) & (X,t) \in D^n_T & (3.13) \\ W(X,0) = \varphi_{x_n x_n}(X) & X \in D^n & (3.14) \\ W_{x_n}(x,0,t) = g_t - \Delta g + qg - f_{x_n}(x,0,t) & (x,t) \in \mathbb{R}^{n-1}_T & (3.15) \\ W(x,a,t) = h_t - \Delta h + hq - f(x,a,t) & (x,t) \in \mathbb{R}^{n-1}_T & (3.16) \end{cases}$$

Lemma 3.4. *Problems 3.1, 3.2 and 3.3 are equivalent each other.*

Proof. The existence and uniqueness of solution (V, q) of Problem 3.2 can be easily obtained from that of solution (u, q) of Problem 3.1. Conversely let (V, q) be solution of Problem 3.2, we show that Problem 3.1 has a unique solution (u, q). The uniqueness comes from the uniqueness of equations (3.1)-(3.3). For the existence, let

$$u(X,t) = \int_a^{x_n} V(x,y,t)dy + h(x,t). \tag{3.17}$$

Obviously, $u(X,t) \in \mathcal{RAP}(\mathbb{R}^{n-1} \times \overline{D_T})$ and satisfies (3.4). Also u satisfies (3.3) because $u_{x_n}(x,0,t) = V(x,0,t) = g(x,t)$. By (3.5) and (3.10) one sees that (3.2) is true. Finally we show that u satisfies (3.1) and therefore, along with q, constitutes solution of Problem 3.1. In fact,

$$\begin{aligned} & u_t - \Delta u + qu \\ &= h_t - \Delta h + qh + \int_a^{x_n} [V_t(x,y,t) - \Delta V(x,y,t) + qV(x,y,t)]dy \\ &\quad + \int_a^{x_n} \frac{\partial^2}{\partial y^2} V(x,y,t)dy - \frac{\partial^2}{\partial {x_n}^2} \int_a^{x_n} V(x,y,t)dy \\ &= h_t - \Delta h + qh + f(X,t) - f(x,a,t) + V_{x_n}(X,t) - V_{x_n}(x,a,t) - V_{x_n}(X,t) \\ &= f(X,t) \qquad (\text{by } (3.12)). \end{aligned}$$

Thus, we have shown the equivalence of Problems 3.1 and 3.2. Replacing (3.17) by the function

$$V(X,t) = \int_0^{x_n} W(x,y,t)dy + g(x,t)$$

the equivalence of Problems 3.2 and 3.3 can be proved similarly. The proof is complete. ∎

By Lemma 3.4, to solve Problem 3.1 we only need to solve Problem 3.3. By (3.13)-(3.15) we have the integral equation about W:

$$\begin{aligned} &W(X,t)\\ &=\int_{D^n}\varphi_{\xi_n\xi_n}(\zeta)G(X,t;\zeta,0)d\zeta+\int_0^t ds\int_{D^n} f_{\xi_n\xi_n}(\zeta,s)G(X,t;\zeta,s)d\zeta\\ &-\int_0^t ds\int_{D^n} q(\xi,s)W(\zeta,s)G(X,t;\zeta,s)d\zeta\\ &-2\int_0^t ds\int_{\mathbb{R}^{n-1}}[g_s-\Delta g+qg-f_{\xi_n}(\xi,0,s)]Z(X,t;\xi,0,s)d\xi. \end{aligned} \tag{3.18}$$

Rewrite (3.16) as

$$q=Lq=h^{-1}(x,t)[\Delta h-h_t+f(x,a,t)+W(x,a,t)], \tag{3.19}$$

where W is determined by (3.18).

One can directly test that Problem 3.3 is equivalent to (3.18)-(3.19).

Note that for a given $q(x,t)\in\mathcal{RAP}(\overline{\mathbb{R}_T^{n-1}})$, Lemma 3.3 shows that equations (3.13)-(3.15) (or equivalently, (3.18)) have a unique solution $W\in\mathcal{RAP}(\mathbb{R}^{n-1}\times\overline{D_T})$. Thus, (3.19) does define an operator L. Therefore, we only need to show that integral equation (3.19) has a unique solution q and $q\in\mathcal{RAP}(\overline{\mathbb{R}_T^{n-1}})$. That is, L has a fixed point in $\mathcal{RAP}(\overline{\mathbb{R}_T^{n-1}})$. Let

$$\begin{aligned} &\Big\{\|\Delta h-h_t+f(x,a,t)\|_{T_0}+2\|\varphi_{\xi_n\xi_n}\|+\Big\|\int_0^t ds\int_{D^n} f_{\xi_n\xi_n}(\zeta,s)G(x,a,t;\zeta,s)d\zeta\Big\|_{T_0}\\ &+2\Big\|\int_0^t ds\int_{\mathbb{R}^{n-1}}[\Delta g-g_s+f_{\xi_n}(\xi,0,s)]Z(x,a,t;\xi,0,s)d\xi\Big\|_{T_0}\Big\}\|h^{-1}\|_{T_0}=\frac{M}{2}. \end{aligned} \tag{3.20}$$

Set $B(M,T)=\{q\in\mathcal{RAP}(\overline{\mathbb{R}_T^{n-1}}):\|q\|_T\le M\}$ where $T\le T_0$. If $q\in B(M,t)$ then by Lemma 3.3, $W(X,t)$ is in $\mathcal{RAP}(\mathbb{R}^{n-1}\times\overline{D_T})$ and so, by (3.19) Lq is in $\mathcal{RAP}(\overline{\mathbb{R}_T^{n-1}})$ with

$$\|Lq\|_T\le\frac{M}{2}+\|h^{-1}\|_{T_0}[2m_2(T)\|g\|_{T_0}+m_1(T)\|W\|_T]M.$$

(3.18) gives the estimate

$$\begin{aligned} \|W\|_T\le{}&\|2\varphi_{\xi_n\xi_n}\|+2m_2(T_0)\|g_t-\Delta g-f_{x_n}(x,0,t)\|_{T_0}+2Mm_2(T_0)\|g\|_{T_0}\\ &+m_1(T_0)\|f_{x_nx_n}\|_{T_0}+Mm_1(T)\|W\|_T. \end{aligned}$$

Choose $t_0<T_0$ such that when $T\le t_0$ one has $1<2(1-Mm_1(T))$. It follows that

$$\begin{aligned} \|W\|_T\le 2\{&2\|\varphi_{x_nx_n}\|+2m_2(T_0)\|g_t-\Delta g-f_{x_n}(x,0,t)\|_{T_0}+2Mm_2(T_0)\|g\|_{T_0}\\ &+m_1(T_0)\|f_{x_nx_n}\|_{T_0}\}. \end{aligned}$$

Choose $T_1 \le t_0$ such that when $T \le T_1$ one has

$$2\|h^{-1}\|_{T_0}\{m_2(T)\|g\|_{T_0}+m_1(T)(2\|\varphi_{x_nx_n}\|+2m_2(T_0)\|g_t-\Delta g-f_{x_n}(x,0,t)\|_{T_0}$$
$$+2Mm_2(T_0)\|g\|_{T_0}+m_1(T_0)\|f_{x_nx_n}\|)\}<\tfrac{1}{2}$$

and therefore, $\|Lq\|_T \le M$.

Let q_1, $q_2 \in B(M,T)$. By (3.19), $\|Lq_1-Lq_2\|_T \le \|h^{-1}\|_T\|W_1-W_2\|_T$. Note that the function $W=W_1-W_2$ is the solution of the following problem

$$\begin{cases} W_t-\Delta W+qW=W_2(q_2-q_1) & (X,t)\in D_T^n \\ W(X,0)=0 & X\in D^n \\ W_{x_n}(x,0,t)=(q_2-q_1)g(x,t) & (x,t)\in \mathbb{R}_T^{n-1}. \end{cases}$$

So, by Lemma 3.3 one has

$$\|W\|_T \le K(T)(\frac{\sqrt{T}}{2}\|q_1-q_2\|_T\|g\|_T+T\|q_1-q_2\|_T\|W_2\|_T).$$

Choose $T_2 < t_0$ such that for $T \le T_2$, $\|h^{-1}\|_{T_0}\|W_1-W_2\|_T \le \frac{1}{2}\|q_1-q_2\|_T$. Now, set $T \le \min\{T_1,T_2\}$. Then L is a contraction from $B(M,T)$ into itself and therefore, has a unique fixed point. Thus, we have shown

Theorem 3.5. *Let functions f, g, h, and φ be as above. Then for small T Problem 3.3 has a unique solution (W,q) in $\mathbb{R}_T^n$ with $W\in\mathcal{RAP}(\mathbb{R}^{n-1}\times\overline{D_T})$ and $q\in\mathcal{RAP}(\overline{\mathbb{R}_T^{n-1}})$.*

Let (W^i,q_i) $(i=1.2)$ be the solutions of Problem 3.3 in D_T^n for the functions f^i, g^i, h^i and φ^i. Set $h^0=h^1-h^2$, $f^0=f^1-f^2$, $\varphi^0=\varphi^1-\varphi^2$ and $g^0=g^1-g^2$. For the stability of the solution, we have the following

Theorem 3.6. *For $0\le t\le T$, one has*

$$\begin{aligned}&\|q_1-q_2\|_t\\ &\le c_1\|h^0\|_t+c_2\|g^0\|_t+c_3\|f^0_{x_nx_n}\|_t+c_4\|\varphi^0_{x_nx_n}\|\\ &\quad +c_5\|h^0_t-\Delta h^0-f^0(x,a,t)\|_t+c_6\|g^0_t-\Delta g^0-f^0_{x_n}(x,0,t)\|_t,\end{aligned}$$

where c_i $(1\le i\le 6)$ depends on t, $\|h_1^{-1}\|_t$, $\|g^1\|_t$, $\|f^1_{x_nx_n}\|_t$, $\|\varphi^1_{x_nx_n}\|$, $\|q_1\|_t$, $\|q_2\|_t$ and $\|g^1_t-\Delta g^1-f^1_{x_n}(x,0,t)\|_t$.

Proof. By (3.16),

$$q_1-q_2=(h^1)^{-1}[\Delta h^0-h^0_t+f^0(x,a,t)-q_2h^0+W_1-W_2].$$

So,

$$\|q_1-q_2\|_t \le \|(h^1)^{-1}\|_t[\|\Delta h^0-h^0_t+f^0(x,a,t)\|_t+\|q_2\|_t\|h^0\|_t+\|W_1-W_2\|_t]. \tag{3.21}$$

Note that the function $W=W_1-W_2$ is the solution of the problem

$$\begin{cases} W_t-\Delta W+q_2W=f^0_{x_nx_n}-W_1(q_1-q_2) & (X,t)\in D^n_T \\ W(X,0)=\varphi^0_{x_nx_n}(X) & X\in D^n \\ W_{x_n}(x,0,t)=g^0_t-\Delta g^0+q_2g^0-f^0_{x_n}(x,0,t)+(q_1-q_2)g^1 & (x,t)\in \mathbb{R}^{n-1}_T. \end{cases}$$

Use a formula similar to (3.18) and Lemma 3.2 for function W, one gets

$$\begin{aligned} &\|W\|_t \\ &\le \Big\{t\|f^0_{x_nx_n}\|_t+\|\varphi^0_{x_nx_n}\|+2\sqrt{\frac{t}{\pi}}\|q_2\|_t\|g^0\|_t+2\sqrt{\frac{t}{\pi}}\|g^0_t-\Delta g^0-f^0_{x_n}(x,0,t)\|_t \\ &\quad +\|W_1\|_t\int_0^t\|q_1-q_2\|_sds+\frac{\|g^1\|_t}{\sqrt{\pi}}\int_0^t\frac{\|q_1-q_2\|_s}{\sqrt{t-s}}ds\Big\}\exp\{\int_0^t\|q_2\|_\rho d\rho\}\}. \end{aligned}$$

Apply Lemma 3.2 and (3.21), one gets the desired conclusion with

$$\begin{aligned} c_1 &= \phi(t)\|(h^1)^{-1}\|_t\|q_2\|_t \\ c_2 &= 2\phi(t)\sqrt{\frac{t}{\pi}}\|(h^1)^{-1}\|_t\|q_2\|_t\exp\{\int_0^t\|q_2\|_sds\} \\ c_3 &= t\phi(t)\|(h^1)^{-1}\|_t\exp\{\int_0^t\|q_2\|_sds \\ c_4 &= \phi(t)\|(h^1)^{-1}\|_t\exp\{\int_0^t\|q_2\|_sds \\ c_5 &= \phi(t)\|(h^1)^{-1}\|_t \\ c_6 &= 2\phi(t)\sqrt{\frac{t}{\pi}}\|(h^1)^{-1}\|_t\exp\{\int_0^t\|q_2\|_sds\}, \end{aligned}$$

where

$$\begin{aligned} \phi(t) &= (1+t\chi_1(t)+2\sqrt{t}\chi_2(t))e^{t\chi(t)} \\ \chi(t) &= t\chi_1^2(t)+4\sqrt{t}\chi_1(t)\chi_2(t)+\pi\chi_2^2(t) \\ \chi_1(t) &= \|(h^1)^{-1}\|_t\Phi(t)\exp\{\int_0^t\|q_2\|_sds\} \\ \chi_2(t) &= \pi^{-1/2}\|(h^1)^{-1}\|_t\|g^1\|_t\exp\{\int_0^t\|q_2\|_sds\} \end{aligned}$$

and $\Phi(t)$ is majorant of $\|W_1\|_t$. Specially, one can assume that

$$\Phi(t) = \left(\|\varphi^1_{x_n x_n}\| + t\|f^1_{x_n x_n}\|_t + \int_0^t \frac{\|g^1_s - \Delta g^1 - f^1_{x_n}(x,0,s)\|}{\sqrt{\pi(t-s)}} ds \right) \exp\{\int_s^t \|q_1\|_s ds\}.$$

The proof is complete. ■

Corollary 3.7. *Under the conditions in Theorem 3.6, the solution of Problem 3.3 is unique.*

References

1. E. Ait Dads, K. Ezzibi and O. Arino, Pseudo almost periodic solutions for some differential equations in a Banach space, *Nonlinear Analysis TMA*, **28** (1997), 1141-1155.
2. B. Basit and C. Zhang, New almost periodic type functions and solutions of differential equations, *Canadian J. Math.*, **48(6)** (1996), 1138-1153.
3. J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on Semigroups: Function Spaces*, Compactifications, Representations, Wiley, New York, 1989.
4. J. Bourgain, Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations, *Geometric and functional Analysis*, **6** (1996), 201-230.
5. C. Corduneanu, *Almost Periodic Functions*, Chelsea, New York, 1st Ed., 1968, 2nd Ed. 1989.
6. A. Fink, *Almost Periodic Differential Equations*, Lecture Notes in Mathematics 377, Springer-Verlag, Berlin Heidelber New York, 1974.
7. A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Englewood Cliffs, NJ, 1964.
8. B. Guo, *Inverse Problem of Parabolic Partial Differential Equations*, Science and Technology Press of Heilongjiang Province, Harbin, 1988, (in Chinese).
9. J. K. Hale and Sjoerd M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.
10. J. Hong, Rafael Obaya and Ana Sanz, Almost periodic type solutions of some differential equations with piecewise constant argument, *Nonlinear Analysis*, **45** (2001), 661-688.
11. O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, Linear and Quasi-linear Equations of Parabolic Type, *Amer. Math. Soc.*, Providence, Rhode Island, 1968.
12. D. Sarason, Remotely almost periodic functions, *Contemporary Mathematics*, **32** (1984), 237-242.

13. W. Shen, Traveling waves in time almost periodic structures governed by bistable nonlinearties,I. stability and uniqueness; II. existence, *J. Differential Equations*, **159** (1999), 1-101.

14. R. Yuan, Pseudo-almost periodic solutions of second-order neutral delay differential equations with piecewise constant argument, *Nonlinear Analysis, TMA*, **41** (2000), 871-890.

15. E. Zauderer, *Partial Differential Equation of Applied Mathematics*, John Wiley & Sons Inc., 1983.

16. C. Zhang, *Almost Periodic Type Functions and Ergodicity*, Science Press/Kluwer, Beijing New York Dordrecht Boston London, 2003.

17. C. Zhang, Pseudo almost periodic solutions of some differential equations, *J. Math. Anal. Appl.*, **181** (1994), 62-76.

18. C. Zhang, Integration of vector-valued pseudo almost periodic functions, *Proc. Amer. Math. Soc.*, **121** (1994), 167-174.

19. C. Zhang, New limit power function spaces, *IEEE Trans. Auto. Cont.*, **49** (2004), 763-766.

Fenglin Yang and Chuanyi Zhang
Department of Mathematics,
Harbin Institute of Technology,
Harbin 150001,
P. R. China
E-mail: czhang@hit.edu.cn