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COINCIDENCE AND MAXIMAL ELEMENT THEOREMS IN ABSTRACT CONVEX SPACES WITH APPLICATIONS

Ming-ge Yang, Nan-jing Huang* and Chin-San Lee

Abstract. In this paper, by using a nonempty intersection lemma due to the authors, we obtain two coincidence theorems involved $\mathfrak{RC}$-maps in abstract convex spaces, which are actually equivalent. We then derive some maximal element theorems for set-valued maps in abstract convex spaces. As an application, we study the existence of solutions for a system of generalized equilibrium problems in abstract convex spaces. We also give some examples to illustrate our results.

1. Introduction

Many problems in nonlinear analysis can be solved by showing the nonemptyness of the intersection of certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, or others of the corresponding equilibrium problems under consideration. The first remarkable result on the nonempty intersection was the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM principle) in 1929 [10], which concerns with certain types of maps called the KKM maps later.

At the beginning, the KKM theory was mainly devoted to the study of convex subsets of topological vector spaces mainly by Ky Fan [5-7]. Later it has been extended to convex spaces by Lassonde [11], to C-spaces (or H-spaces) by Horvath [8, 9] and others, and to generalized convex (G-convex) spaces by Park [21-24].

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*Corresponding author.

Recently, Park [16] introduced a new concept of abstract convex spaces which include convex subsets of topological vector spaces, convex spaces, C-spaces and G-convex spaces as special cases. Park [16] also introduced certain broad classes $\mathfrak{KO}$ and $\mathfrak{KC}$ of maps (having the KKM property). The class $\mathfrak{KC}(X,Y)$ includes the well-known class $KKM(X,Y)$ introduced by Chang and Yen [2] as a special case. With these new concepts, he obtained some coincidence theorems and fixed point theorems in abstract convex spaces. Very recently, Park [17-20] further studied KKM theory in abstract convex spaces with applications to fixed points, maximal elements, equilibria problems and other problems. It is worth mentioning that, in the KKM theory, there have appeared a number of coincidence theorems with many significant applications.

On the other hand, Ding and Park [3, 4] and Lin and Chen [14] studied the following more general equilibrium problems: Find $\bar{x} \in X$ such that one of the following situations occurs:

$$F(\bar{x}, z) \subset C(\bar{x}) \text{ for all } z \in Z;$$

$$F(\bar{x}, z) \cap C(\bar{x}) \neq \emptyset \text{ for all } z \in Z;$$

$$F(\bar{x}, z) \not\subset C(\bar{x}) \text{ for all } z \in Z;$$

$$F(\bar{x}, z) \cap C(\bar{x}) = \emptyset \text{ for all } z \in Z,$$

where X is a topological space, Z and V are nonempty sets, $F : X \times Z \multimap V$ and $C : X \multimap V$ are two maps.

Motivated and inspired by the works mentioned above, in this paper, by using a nonempty intersection theorem due to the authors, we obtain two coincidence theorems involved $\mathfrak{KC}$-maps in abstract convex spaces, which are actually equivalent. We then derive some maximal element theorems for set-valued maps in abstract convex spaces. As an application, we study the solvability of the following systems of generalized equilibrium problems: Find $\bar{x} \in X$ such that one of the following situations occurs:

$$F_i(\bar{x}, z_i) \subset C_i(\bar{x}) \text{ for each } i \in I \text{ and } z_i \in Z_i;$$

$$F_i(\bar{x}, z_i) \cap C_i(\bar{x}) \neq \emptyset \text{ for each } i \in I \text{ and } z_i \in Z_i;$$

$$F_i(\bar{x}, z_i) \not\subset C_i(\bar{x}) \text{ for each } i \in I \text{ and } z_i \in Z_i;$$

$$F_i(\bar{x}, z_i) \cap C_i(\bar{x}) = \emptyset \text{ for each } i \in I \text{ and } z_i \in Z_i,$$

where X is a topological space, I is an index set, $\{Z_i\}_{i \in I}$ and $\{V_i\}_{i \in I}$ are two families of nonempty sets, and $F_i : X \times Z_i \multimap V_i$ and $C_i : X \multimap V_i$ are two maps for all $i \in I$. We also give some examples to illustrate our results. The results presented in this paper improve and generalize some corresponding results due to Balaj [1] and Lin, Ansari and Wu [13].

2. Preliminaries

A multimap (or simply a map) $T : X \multimap Y$ is a function from a set X into the power set 2^Y of Y, that is a function with the values $T(x) \subset Y$ for $x \in X$. Given a map $T : X \multimap Y$, the map $T^- : Y \multimap X$ defined by $T^-(y) = \{x \in X : y \in T(x)\}$ for $y \in Y$, is called the (lower) inverse of T. For any $A \subset X$, $T(A) := \bigcup_{x\in A} T(x)$. For any $B \subset Y$, $T^-(B) := \{x \in X : T(x) \cap B \neq \emptyset\}$. For a set X, let $\langle X\rangle$ denote the family of all nonempty finite subsets of X.

If A is a subset of a topological space, we denote by int$A$ and $\overline{A}$ the interior and closure of A, respectively.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be compact if $T(X)$ is contained in a compact subset of Y. Let $\overline{T} : X \multimap Y$ be a map defined by $\overline{T}(x) = \overline{T(x)}$ for $x \in X$.

Definition 2.1. [25]. Suppose that X is a nonempty set and Y is a topological space. A map $T : X \multimap Y$ is said to be transfer open valued if, for any $(x, y) \in X \times Y$ with $y \in T(x)$, there exists an $x' \in X$ such that $y \in \text{int}\, T(x')$.

Obviously, if T has open values, then T is transfer open valued.

Lemma 2.1. [12]. *Let X be a topological space, Z be a nonempty set, and $P : X \multimap Z$ be a map. Then the following assertions are equivalent:*

(i) P^- is transfer open valued and P has nonempty values;

(ii) $X = \bigcup_{z\in Z}$ int $P^-(z)$.

Definition 2.2. [16]. An abstract convex space $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a map $\Gamma : \langle D\rangle \multimap E$ with nonempty values. Denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

In the case $E = D$, let $(E; \Gamma) = (E, E; \Gamma)$. It is obvious that any vector space E is an abstract convex space with $\Gamma =$ co, where co is the convex hull in vector spaces. In specially, $(\mathbb{R}; \text{co})$ is an abstract convex space. For more examples of abstract convex spaces, we refer to [16].

Let $(E, D; \Gamma)$ be an abstract convex space. For any $D' \subset D$, the Γ-convex hull of D' is denoted and defined by

$$\text{co}_\Gamma D' := \bigcup\{\Gamma_A | A \in \langle D'\rangle\} \subset E$$

(co is reserved for the convex hull in vector spaces). A subset X of E is called a Γ-convex subset of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$; that is, $\text{co}_\Gamma D' \subset X$. This means that $(X, D'; \Gamma|_{\langle D' \rangle})$ itself is an abstract convex space called a subspace of $(E, D; \Gamma)$. When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-convex if $\text{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' = X \cap D$. When $(E; \Gamma) = (\mathbb{R}; \text{co})$, the Γ-convex subset reduces to the ordinary convex subset.

Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a map $F : E \multimap Z$ with nonempty values, if a map $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A), \ \forall A \in \langle D \rangle,$$

then G is called a KKM map with respect to F. A classical KKM map $G : D \multimap E$ is a KKM map with respect to the identity map 1_E. A map $F : E \multimap Z$ is said to have the KKM property and called a $\mathfrak{K}$-map if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z | F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-map is defined for closed-valued maps G, and a $\mathfrak{KO}$-map is defined for open-valued maps G. Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$ and $\mathfrak{KO}$ are identical. Some authors use the notation KKM(E, Z) instead of $\mathfrak{KC}(E, Z)$.

For more details concerned with the abstract convex spaces with applications, we refer to [16, 17, 18, 19, 20] and the references therein.

Lemma 2.2. *Let X be a topological space and $(Y; \Gamma)$ be an abstract convex space. If $T \in \mathfrak{KC}(Y, X)$ is compact and $S : Y \multimap X$ is a KKM map with respect to T, then $\overline{T(Y)} \cap \bigcap_{y \in Y} \overline{S(y)} \neq \emptyset$.*

Proof. Since S is a KKM map with respect to T, we have $T(\Gamma_N) \subset S(N)$ for each $N \in \langle Y \rangle$. It is obvious that $T(\Gamma_N) \subset \overline{T(Y)}$ and $S(N) \subset \overline{S}(N)$. Hence, $T(\Gamma_N) \subset \overline{T(Y)} \cap \overline{S}(N)$. It follows that the map $F : Y \multimap X$ defined by

$$F(y) = \overline{T(Y)} \cap \overline{S}(y), \ \forall y \in Y$$

is a KKM map with respect to T. Note that F has closed values. By the definition of $T \in \mathfrak{KC}(Y, X)$, we have the family $\{F(y)\}_{y \in Y}$ has the finite intersection property. Moreover, $\overline{T(Y)}$ is compact. We have the family $\{F(y)\}_{y \in Y}$ has the nonempty intersection property, that is $\overline{T(Y)} \cap \bigcap_{y \in Y} \overline{S(y)} \neq \emptyset$. This completes the proof.

Remark 2.1. (a) If Y is a convex subset of a topological vector space and $\Gamma = \text{co}$, then Lemma 2.2 reduces to Lemma 3 in [1]; (b) If Y is a convex space and $\Gamma = \text{co}$, then by using Lemma 2.2, it is easy to derive Lemma 2.2 in [13].

Definition 2.3. Let $(X;\Gamma_1)$ and $(Y;\Gamma_2)$ be two abstract convex spaces. A map $T : X \multimap Y$ is called quasiconvex if, for each Γ_2-convex subset C of Y, $T^{-}(C)$ is a Γ_1-convex subset of X.

Remark 2.2. If X and Y are convex subsets of vector spaces, then Definition 2.3 reduces to the concept of quasiconvexity in [15].

Recall that if X and Y are convex subsets of vector spaces, a map $T : X \multimap Y$ is called convex if

$$\lambda T(x_1) + (1-\lambda)T(x_2) \subset T(\lambda x_1 + (1-\lambda)x_2)$$

for all $x_1, x_2 \in X$ and $\lambda \in [0,1]$. We try to provide evidence for the connection between the two concepts.

Proposition 2.1. *Let X and Y be two convex subsets of vector space. Then any convex map $T : X \multimap Y$ is quasiconvex.*

Proof. For any convex subset C of Y, we show that $T^{-}(C)$ is convex. For any $x_1, x_2 \in T^{-}(C)$ and $\lambda \in [0,1]$, there exists $y_1, y_2 \in C$ such that $y_1 \in T(x_1)$, $y_2 \in T(x_2)$. Let $y = \lambda y_1 + (1-\lambda)y_2$. Obviously, $y \in C$. Since T is convex, we have

$$y = \lambda y_1 + (1-\lambda)y_2 \in \lambda T(x_1) + (1-\lambda)T(x_2) \subset T(\lambda x_1 + (1-\lambda)x_2),$$

i.e.,

$$T(\lambda x_1 + (1-\lambda)x_2) \cap C \neq \emptyset.$$

Thus, T is quasiconvex. This completes the proof.

The following example shows that the converse of Proposition 2.1 is not true in general.

Example 2.1. Let $X = Y = \mathbb{R}$ and $T : X \multimap Y$ be defined by

$$T(x) = \begin{cases} [-1,0], & x \leq 0; \\ (0,1], & x > 0. \end{cases}$$

Then T is quasiconvex. In fact, for any subset C of $\mathbb{R}$, we have

$$T^-(C) = \{x \in \mathbb{R} : T(x) \cap C \neq \emptyset\}$$
$$= \begin{cases} \emptyset, & \text{if } C \subset \mathbb{R} \setminus [-1,1]; \\ (-\infty, 0], & \text{if } C \cap [-1,0] \neq \emptyset \text{ and } C \cap (0,1] = \emptyset; \\ (0, +\infty), & \text{if } C \cap [-1,0] = \emptyset \text{ and } C \cap (0,1] \neq \emptyset; \\ \mathbb{R}, & \text{if } C \cap [-1,0] \neq \emptyset \text{ and } C \cap (0,1] \neq \emptyset. \end{cases}$$

Hence, $T^-(C)$ is convex, and it follows that T is quasiconvex. However, we know that T is not convex. Indeed, let $x_1 = -1$, $x_2 = 1$ and $\lambda = \frac{1}{2}$. Then $T(x_1) = [-1,0]$, $T(x_2) = (0,1]$ and $\lambda x_1 + (1-\lambda)x_2 = 0$. For $y_1 = 0 \in T(x_1)$ and $y_2 = 1 \in T(x_2)$, we have

$$\lambda y_1 + (1-\lambda)y_2 = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2} \notin T(\lambda x_1 + (1-\lambda)x_2) = T(0) = [-1,0],$$

which shows that T is not convex.

3. Coincidence Theorems

Theorem 3.1. *Let X be a topological space and $(Y;\Gamma)$ be an abstract convex space. Let $S : X \multimap Y$ and $T : Y \multimap X$ be two maps satisfying the following conditions:*

(*i*) $X = \bigcup_{y \in Y}$ *int* $S^-(y)$;

(*ii*) *for each $x \in X$, $S(x)$ is Γ-convex;*

(*iii*) $T \in \mathfrak{RC}(Y,X)$ *is compact.*

Then there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \in T(y_0)$ and $y_0 \in S(x_0)$.

Proof. Suppose the conclusion would be false. Then for each $y \in Y$ and $x \in T(y)$, we have $y \notin S(x)$. Define a map $S^* : Y \multimap X$ by

$$S^*(y) = X \setminus S^-(y), \ \forall y \in Y.$$

We show that S^* is a KKM map with respect to T. Suppose to the contrary that there exist a finite subset $N = \{y_1, \cdots, y_n\}$ of Y and a point $x \in T(\Gamma_N) \setminus S^*(N)$. By $x \in T(\Gamma_N)$, there exists $y \in \Gamma_N$ such that $x \in T(y)$, and it follows that $y \notin S(x)$. By $x \notin S^*(N)$, we have $x \notin S^*(y_i)$ for each $y_i \in N$. It follows that $y_i \in S(x)$ for each $y_i \in N$ and so $N \in \langle S(x) \rangle$. By condition (ii), $\Gamma_N \subset S(x)$. Hence, $y \in S(x)$, which is a contradiction.

By Lemma 2.2, there exists $x_0 \in \bigcap_{y\in Y} \overline{S^*(y)}$. By condition (i), there exists $y_0 \in Y$ such that $x_0 \in \text{int } S^-(y_0)$. Hence, there exists a neighborhood V of x_0 such that $V \subset S^-(y_0)$. It follows that $y_0 \in S(x)$ for each $x \in V$. On the other hand, since $x_0 \in \overline{S^*(y_0)}$, we know that $V \cap S^*(y_0) \neq \emptyset$. Hence, there exists $x \in V$ such that $x \in S^*(y_0)$, and it follows $y_0 \notin S(x)$, which is a contradiction. This completes the proof.

Remark 3.1. (a) If Y is a convex space and $\Gamma = \text{co}$, then Theorem 3.1 reduces to Theorem 2.5 in [13]. (b) We would like to point out that the proof of Theorem 3.1 is quite different from the proof of Theorem 2.5 in [13].

Example 3.1. Let $X = [1, +\infty)$ be endowed with Euclidean topology, $Y = [0, +\infty)$ with $\Gamma = \text{co}$, $S : X \multimap Y$ and $T : Y \multimap X$ be two maps defined, respectively, by

$$S(x) = (x-1, x],\ \forall x \in X$$

and

$$T(y) = \{1\},\ \forall y \in Y.$$

Then we have the following conclusions.

(i) For each $y \in Y$, $S^- : Y \multimap X$ is defined by

$$S^-(y) = \begin{cases} [y, y+1), & y \geq 1; \\ [1, y+1), & 0 < y < 1; \\ \emptyset, & y = 0. \end{cases}$$

Since

$$\bigcup_{y\geq 1} \text{int } S^-(y) = \bigcup_{y\geq 1} \text{int } [y, y+1) = [1,2) \cup \bigcup_{y>1} (y, y+1) = [1, +\infty) = X,$$

we have $X = \bigcup_{y\in Y} \text{int } S^-(y)$ and so condition (i) of Theorem 3.1 is satisfied.

(ii) Obviously, for each $x \in X$, $S(x) = (x-1, x]$ is convex and so it is Γ-convex. Thus, condition (ii) of Theorem 3.1 is satisfied.

(iii) For any KKM map $F : Y \multimap X$ with regard to T, we have $T(\Gamma_A) \subset F(A)$ for each $A \in \langle Y \rangle$, i.e., $1 \in F(A)$ for each $A \in \langle Y \rangle$. Hence $1 \in \bigcap_{y\in Y} F(y)$, and it follows that $T \in \mathfrak{R}(Y, X) \subset \mathfrak{RC}(Y, X)$. Since $T(Y) = \{1\}$ and $\{1\}$ is compact, we know that T is compact. This shows that condition (iii) of Theorem 3.1 is satisfied.

From the discussions mentioned above, we know that all the conditions of Theorem 3.1 are satisfied.

Theorem 3.2. *Let X be a topological space, $(Y;\Gamma)$ be an abstract convex space and Z be a nonempty set. Let $P : X \multimap Z$, $Q : Y \multimap Z$ and $T : Y \multimap X$ be three maps satisfying the following conditions:*

(*i*) $X = \bigcup_{z\in Q(Y)}$ *int* $P^-(z)$;

(*ii*) *for each* $x \in X$, $\{y \in Y : P(x) \cap Q(y) \neq \emptyset\}$ *is* Γ*-convex;*

(*iii*) $T \in \mathfrak{RC}(Y, X)$ *is compact.*

Then there exists $(x_0, y_0) \in X \times Y$ *such that* $x_0 \in T(y_0)$ *and* $P(x_0) \cap Q(y_0) \neq \emptyset$.

Proof. Define a map $S : X \multimap Y$ by

$$S(x) = \{y \in Y : P(x) \cap Q(y) \neq \emptyset\},\ \forall x \in X.$$

It is easy to see that for a family $\{A_i\}_{i\in I}$ of subsets of a topological space, $\bigcup_{i\in I} \text{int}A_i \subset \text{int}(\bigcup_{i\in I} A_i)$. Having this fact in mind we obtain

$$\begin{aligned}
\bigcup_{y\in Y} \text{int } S^-(y) &= \bigcup_{y\in Y} \text{int } \{x \in X : P(x) \cap Q(y) \neq \emptyset\} \\
&= \bigcup_{y\in Y} \text{int} \left(\bigcup_{z\in Q(y)} P^-(z) \right) \\
&\supset \bigcup_{y\in Y} \bigcup_{z\in Q(y)} \text{int } P^-(z) \\
&= \bigcup_{z\in Q(Y)} \text{int } P^-(z).
\end{aligned}$$

Thus, by condition (i) it follows that $X = \bigcup_{y\in Y} \text{int } S^-(y)$. By condition (ii), $S(x)$ is Γ-convex for each $x \in X$. The conclusion follows from Theorem 3.1. This completes the proof.

When $Y = Z$ and $Q(y) = \{y\}$ for all $y \in Y$, Theorem 3.2 reduces to Theorem 3.1. Obviously, we know that Theorem 3.1 implies Theorem 3.2. Therefore, Theorems 3.1 and 3.2 are equivalent.

Remark 3.2. (a) If Y is a convex subset of a topological vector space and $\Gamma = \text{co}$, then Theorem 3.2 reduces to Theorem 4 in [1]. (b) We would like to point out that the proof of Theorem 3.2 is quite different from the proof of Theorem 4 in [1].

Example 3.2. Let $X = [1, +\infty)$ be endowed with Euclidean topology, $Y = [0, +\infty)$ with $\Gamma = \text{co}$, and $Z = [0, +\infty)$. Let $P : X \multimap Z$, $Q : Y \multimap Z$ and $T : Y \multimap X$ be three maps defined, respectively, by

$$P(x) = (x - 1, x],\ \forall x \in X,$$
$$Q(y) = [y, y + 1),\ \forall y \in Y$$

and

$$T(y) = \{1\},\ \forall y \in Y.$$

Then we have the following conclusions.

(i) By Example 3.1 (i), $X = \bigcup_{z \in Z} \text{int } P^-(z)$. Since $Q(Y) = Z$, we have $X = \bigcup_{z \in Q(Y)} \text{int } P^-(z)$. Thus, condition (i) of Theorem 3.2 is satisfied.

(ii) For each $x \in X$, it is easy to see that $\{y \in Y : P(x) \cap Q(y) \neq \emptyset\}$ is convex. For any $y_1, y_2 \in Y$ with $P(x) \cap Q(y_1) \neq \emptyset$ and $P(x) \cap Q(y_2) \neq \emptyset$, and any $\lambda \in [0, 1]$, we know that $P(x) \cap Q(\lambda y_1 + (1 - \lambda)y_2) \neq \emptyset$. By $P(x) \cap Q(y_1) \neq \emptyset$ and $P(x) \cap Q(y_2) \neq \emptyset$, there exist $z_1 \in P(x) \cap Q(y_1)$ and $z_2 \in P(x) \cap Q(y_2)$. Obviously, $P(x) = (x - 1, x]$ is convex for each $x \in X$. Hence, $\lambda z_1 + (1 - \lambda)z_2 \in P(x)$. By $z_1 \in Q(y_1) = [y_1, y_1 + 1)$ and $z_2 \in Q(y_2) = [y_2, y_2 + 1)$, we have $y_1 \leq z_1 < y_1 + 1$ and $y_2 \leq z_2 < y_2 + 1$ and so

$$\begin{aligned} &\lambda y_1 + (1 - \lambda)y_2 \leq \lambda z_1 + (1 - \lambda)z_2 < \lambda(y_1 + 1) + (1 - \lambda)(y_2 + 1) \\ = &\lambda y_1 + (1 - \lambda)y_2 + 1, \end{aligned}$$

i.e.,

$$\begin{aligned} &\lambda z_1 + (1 - \lambda)z_2 \in [\lambda y_1 + (1 - \lambda)y_2, \lambda y_1 + (1 - \lambda)y_2 + 1) \\ = &Q(\lambda y_1 + (1 - \lambda)y_2). \end{aligned}$$

Thus, there exists $z = \lambda z_1 + (1 - \lambda)z_2 \in Z$ such that $z \in P(x) \cap Q(\lambda y_1 + (1 - \lambda)y_2)$ and so $\lambda y_1 + (1 - \lambda)y_2 \in \{y \in Y : P(x) \cap Q(y) \neq \emptyset\}$. This shows that condition (ii) of Theorem 3.2 is satisfied.

(iii) By Example 3.1 (iii), $T \in \mathfrak{RC}(Y, X)$ is compact. Thus condition (iii) of Theorem 3.2 is satisfied.

From the discussions mentioned above, we know that all the conditions of Theorem 3.2 are satisfied.

Corollary 3.1. *Let X be a topological space, $(Y; \Gamma_1)$ and $(Z; \Gamma_2)$ be two abstract convex spaces. Let $P : X \multimap Z$, $Q : Y \multimap Z$ and $T : Y \multimap X$ be three maps satisfying conditions (i) and (iii) of Theorem 3.2. If P has Γ_2-convex values and Q is quasiconvex, then there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \in T(y_0)$ and $P(x_0) \cap Q(y_0) \neq \emptyset$.*

Proof. Since P has Γ_2-convex values and Q is quasiconvex, $Q^{-}(P(x)) = \{y \in Y : P(x) \cap Q(y) \neq \emptyset\}$ is Γ_1-convex for each $x \in X$. The conclusions follows from Theorem 3.2. This completes the proof.

4. Maximal Element Theorems

Theorem 4.1. *Let X be a topological space, $(Y;\Gamma)$ be an abstract convex space and Z be a nonempty set. Let $P : X \multimap Z$, $Q : Y \multimap Z$ and $T : Y \multimap X$ be three maps satisfying the following conditions:*

(i) *P^{-} is transfer open-valued;*

(ii) *for each $x \in X$, $\{y \in Y : P(x) \cap Q(y) \neq \emptyset\}$ is Γ-convex;*

(iii) *$T \in \mathfrak{RC}(Y, X)$ is compact;*

(iv) *for each $y \in Y$ and each $x \in T(y)$, $P(x) \cap Q(y) = \emptyset$;*

(v) *for each $z \in Z$, $Q^{-}(z) \neq \emptyset$.*

Then there exists $x_0 \in X$ such that $P(x_0) = \emptyset$.

Proof. Suppose that the conclusion would be false. Then P has nonempty values. By condition (v), $Q(Y) = Z$. Thus, by Lemma 2.1, we have

$$X = \bigcup_{z \in Z} \text{int } P^{-}(z) = \bigcup_{z \in Q(Y)} \text{int } P^{-}(z),$$

and hence condition (i) of Theorem 3.2 is fulfilled. According to Theorem 3.2, there exist $y_0 \in Y$ and $x_0 \in T(y_0)$ such that $P(x_0) \cap Q(y_0) \neq \emptyset$. This contradicts condition (iv). This completes the proof.

Remark 4.3. If Y is a convex subset of a topological vector space and $\Gamma = \text{co}$, then Theorem 4.1 reduces to Theorem 6 in [1].

Example 4.1. Let $X = [1, +\infty)$ be endowed with Euclidean topology, $Y = [0, +\infty)$ with $\Gamma = \text{co}$, and $Z = [0, +\infty)$. Let $P : X \multimap Z$, $Q : Y \multimap Z$ and $T : Y \multimap X$ be three maps defined, respectively, by

$$P(x) = \begin{cases} \emptyset, & \text{if } x = 1, \\ (x-1, x), & \text{if } x > 1 \end{cases}$$

$$Q(y) = [y, y+1), \ \forall y \in Y$$

and

$$T(y) = \{1\}, \ \forall y \in Y.$$

Then we have the following conclusions.

(i) For each $z \in Z$,

$$\begin{aligned} P^-(z) &= \{x \in X : z \in P(x)\} = \{x \in (1, +\infty) : z \in (x-1, x)\} \\ &= \{x \in (1, +\infty) : z < x < z+1\} = (1, +\infty) \cap (z, z+1) \\ &= \begin{cases} (z, z+1), & \text{if } z \geq 1, \\ (1, z+1), & \text{if } 0 < z < 1, \\ \emptyset, & \text{if } z = 0. \end{cases} \end{aligned}$$

Obviously, P^- has open values, and consequently is transfer open valued. Thus, condition (i) of Theorem 4.1 is satisfied.

(ii) Note that $P(x)$ is convex for each $x > 1$. Using the same argument as Example 3.2 (ii), we know that, for each $x \in X$, $\{y \in Y : P(x) \cap Q(y) \neq \emptyset\}$ is convex and so condition (ii) of Theorem 4.1 is satisfied.

(iii) By Example 3.1 (iii), $T \in \mathfrak{RC}(Y, X)$ is compact. Thus, condition (iii) of Theorem 4.1 is satisfied.

(iv) For each $y \in Y$ and each $x \in T(y)$, we have $P(x) = P(1) = \emptyset$ and so $P(x) \cap Q(y) = \emptyset$. This shows that condition (iv) of Theorem 4.1 is satisfied.

(v) Obviously, $Q(Y) = Z$, i.e., for each $z \in Z$, $Q^-(z) \neq \emptyset$. Thus condition (v) of Theorem 4.1 is satisfied.

From the discussions mentioned above, we know that all the conditions of Theorem 4.1 are satisfied.

Corollary 4.2. *Let X be a topological space, $(Y; \Gamma_1)$ and $(Z; \Gamma_2)$ be two abstract convex spaces. Let $P : X \multimap Z$, $Q : Y \multimap Z$ and $T : Y \multimap X$ be three maps satisfying conditions (i), (iii), (iv) and (v) of Theorem 4.1. If P has Γ_2-convex values and Q is quasiconvex, then there exists $x_0 \in X$ such that $P(x_0) = \emptyset$.*

Theorem 4.2. *Let X be a topological space, $(Y; \Gamma)$ be an abstract convex space, and $T \in \mathfrak{RC}(Y, X)$ be a compact map. Let I be any index set, $\{Z_i\}_{i \in I}$ be a family of nonempty sets. For each $i \in I$, if $P_i : X \multimap Z_i$ and $Q_i : Y \multimap Z_i$ are two maps satisfying the following conditions:*

(*i*) *P_i^- is transfer open valued;*

(*ii*) *for each $x \in X$, $\{y \in Y : P_i(x) \cap Q_i(y) \neq \emptyset\}$ is Γ-convex;*

(*iii*) *for each $y \in Y$ and each $x \in T(y)$, $P_i(x) \cap Q_i(y) = \emptyset$;*

(*iv*) *$Q_i(Y) = Z_i$.*

Then there exists $\bar{x} \in X$ *such that* $P_i(\bar{x}) = \emptyset$ *for each* $i \in I$.

Proof. Put $Z = \prod_{i \in I} Z_i$. Define two maps $P : X \multimap Z$ and $Q : Y \multimap Z$ by

$$P(x) = \prod_{i \in I} P_i(x),\ \forall x \in X$$

and

$$Q(y) = \prod_{i \in I} Q_i(y),\ \forall y \in Y,$$

respectively. We show that P and Q satisfy all the conditions of Theorem 4.1.

(a) $P^- : Z \multimap X$ is transfer open valued. For each $z \in Z$ and $x \in P^-(z)$, we have $z_i \in Z_i$ for each $i \in I$. Note that

$$\begin{aligned} P^-(z) &= \{x \in X : z \in P(x)\} \\ &= \{x \in X : z_i \in P_i(x) \text{ for all } i \in I\} \\ &= \{x \in X : x \in P_i^-(z_i) \text{ for all } i \in I\} \\ &= \bigcap_{i \in I} P_i^-(z_i). \end{aligned}$$

We have $x \in P_i^-(z_i)$ for each $i \in I$. For each $i \in I$, since P_i^- is transfer open valued, there exist a point $z_i^{'} \in Z_i$ and a neighborhood $N_i(x)$ of x such that $N_i(x) \subset P_i^-(z_i^{'})$. Let $z^{'} = (z_i^{'})_{i \in I}$ and $N(x) = \bigcap_{i \in I} N_i(x)$. It is obviously that $z^{'} \in Z$ and $N(x)$ is also a neighborhood of x satisfying $N(x) \subset \bigcap_{i \in I} P_i^-(z_i^{'}) = P^-(z^{'})$. It follows that $P^- : Z \multimap X$ is transfer open valued. Thus condition (i) of Theorem 4.1 is satisfied.

(b) For each $x \in X$, we have

$$\begin{aligned} \{y \in Y : P(x) \cap Q(y) \neq \emptyset\} &= \{y \in Y : \prod_{i \in I} P_i(x) \cap \prod_{i \in I} Q_i(y) \neq \emptyset\} \\ &= \{y \in Y : P_i(x) \cap Q_i(y) \neq \emptyset \text{ for all } i \in I\} \\ &= \bigcap_{i \in I} \{y \in Y : P_i(x) \cap Q_i(y) \neq \emptyset\}. \end{aligned}$$

By condition (ii), for each $i \in I$, $\{y \in Y : P_i(x) \cap Q_i(y) \neq \emptyset\}$ is Γ-convex. By the definition of Γ-convex subsets, we have the intersection of a family of Γ-convex subsets is also a Γ-convex subset. Hence, $\bigcap_{i \in I}\{y \in Y : P_i(x) \cap Q_i(y) \neq \emptyset\}$ is Γ-convex, that is $\{y \in Y : P(x) \cap Q(y) \neq \emptyset\}$ is Γ-convex. Thus condition (ii) of Theorem 4.1 is satisfied.

(c) For each $y \in Y$ and each $x \in T(y)$, we have

$$P(x) \cap Q(y) = \prod_{i \in I} P_i(x) \cap \prod_{i \in I} Q_i(y) = \prod_{i \in I} (P_i(x) \cap Q_i(y)) .$$

By condition (iii), we have $P(x) \cap Q(y) = \emptyset$. Thus condition (iv) of Theorem 4.1 is satisfied.

(d) By condition (iv), $Q(Y) = \prod_{i \in I} Q_i(Y) = \prod_{i \in I} Z_i = Z$. Thus condition (v) of Theorem 4.1 is satisfied.

Hence, P, Q satisfy all the conditions of Theorem 4.1. It follows from Theorem 4.1 that there exists $\bar{x} \in X$ such that $P(\bar{x}) = \emptyset$, i.e., $P_i(\bar{x}) = \emptyset$ for each $i \in I$. This completes the proof.

Remark 4.4. If I is a singleton, Theorem 4.2 reduces to Theorem 4.1.

Corollary 4.3. *Let X be a topological space, $(Y;\Gamma)$ be an abstract convex space, and $T \in \mathfrak{RC}(Y, X)$ be a compact map. Let I be any index set, $\{(Z_i;\Gamma_i)\}_{i \in I}$ be a family of abstract convex spaces. For each $i \in I$, let $P_i : X \multimap Z_i$ and $Q_i : Y \multimap Z_i$ be two maps satisfying conditions (i), (iii) and (iv) of Theorem 4.2. If P_i has Γ_i-convex values and Q_i is quasiconvex for each $i \in I$, then there exists $\bar{x} \in X$ such that $P_i(\bar{x}) = \emptyset$ for each $i \in I$.*

5. System of Generalized Equilibrium Problems

Let X be a topological space, I be any index set, $\{Z_i\}_{i \in I}$ and $\{V_i\}_{i \in I}$ be two families of nonempty sets. For each $i \in I$, let $F_i : X \times Z_i \multimap V_i$ and $C_i : X \multimap V_i$ be two maps. We consider the systems of generalized equilibrium problems: Find $\bar{x} \in X$ such that one of the following situations occurs:

$F_i(\bar{x}, z_i) \subset C_i(\bar{x})$ for each $i \in I$ and $z_i \in Z_i$;

$F_i(\bar{x}, z_i) \cap C_i(\bar{x}) \neq \emptyset$ for each $i \in I$ and $z_i \in Z_i$;

$F_i(\bar{x}, z_i) \not\subset C_i(\bar{x})$ for each $i \in I$ and $z_i \in Z_i$;

$F_i(\bar{x}, z_i) \cap C_i(\bar{x}) = \emptyset$ for each $i \in I$ and $z_i \in Z_i$.

Theorem 5.3. *Let X be a topological space, $(Y;\Gamma)$ be an abstract convex space, and $T \in \mathfrak{RC}(Y, X)$ be a compact map. Let I be any index set, $\{Z_i\}_{i \in I}$ and $\{V_i\}_{i \in I}$ be two families of nonempty sets. For each $i \in I$, let $F_i : X \times Z_i \multimap V_i$, $C_i : X \multimap V_i$ and $Q_i : Y \multimap Z_i$ be three maps satisfying the following conditions:*

(i) *for each* $x \in X$, $\{z_i \in Z_i : F_i(x, z_i) \not\subset C_i(x)\} \neq \emptyset$ *implies that there exist a neighborhood* $N_i(x)$ *of* x *and* $z_i' \in Z_i$ *such that* $F_i(x', z_i') \not\subset C_i(x')$ *for all* $x' \in N_i(x)$*;*

(ii) *for each* $x \in X$, $\{y \in Y : F_i(x, z_i) \not\subset C_i(x),$ *for some* $z_i \in Q_i(y)\}$ *is* Γ*-convex;*

(iii) *for each* $y \in Y$, $x \in T(y)$ *and* $z_i \in Q_i(y)$, *we have* $F_i(x, z_i) \subset C_i(x)$*;*

(iv) $Q_i(Y) = Z_i$.

Then there exists $\bar{x} \in X$ *such that* $F_i(\bar{x}, z_i) \subset C_i(\bar{x})$ *for each* $i \in I$ *and* $z_i \in Z_i$.

Proof. For each $i \in I$, define $P_i : X \multimap Z_i$ by

$$P_i(x) = \{z_i \in Z_i : F_i(x, z_i) \not\subset C_i(x)\}, \ \forall x \in X.$$

By condition (i), for each $x \in X$, $P_i(x) \neq \emptyset$ implies that there exist a neighborhood $N_i(x)$ of x and $z_i' \in Z_i$ such that $x' \in P_i^-(z_i')$ for all $x' \in N_i(x)$. Hence, P_i^- is transfer open valued. By condition (ii), for each $x \in X$, $\{y \in Y : P_i(x) \cap Q_i(y) \neq \emptyset\}$ is Γ-convex. By condition (iii), for each $y \in Y$ and each $x \in T(y)$, $P_i(x) \cap Q_i(y) = \emptyset$. Thus all the conditions of Theorem 4.2 are satisfied. Thus, Theorem 4.2 shows that there exists $\bar{x} \in X$ such that $P_i(\bar{x}) = \emptyset$ for each $i \in I$, i.e., $F_i(\bar{x}, z_i) \subset C_i(\bar{x})$ for each $i \in I$ and $z_i \in Z_i$. This completes the proof.

Theorem 5.4. *Let* X *be a topological space,* $(Y; \Gamma)$ *be an abstract convex space, and* $T \in \mathfrak{RC}(Y, X)$ *be a compact map. Let* I *be any index set,* $\{Z_i\}_{i \in I}$ *and* $\{V_i\}_{i \in I}$ *be two families of nonempty sets. For each* $i \in I$, *let* $F_i : X \times Z_i \multimap V_i$, $C_i : X \multimap V_i$ *and* $Q_i : Y \multimap Z_i$ *be three maps satisfying the following conditions:*

(i) *for each* $x \in X$, $\{z_i \in Z_i : F_i(x, z_i) \cap C_i(x) = \emptyset\} \neq \emptyset$ *implies that there exist a neighborhood* $N_i(x)$ *of* x *and* $z_i' \in Z_i$ *such that* $F_i(x', z_i') \cap C_i(x') = \emptyset$ *for all* $x' \in N_i(x)$*;*

(ii) *for each* $x \in X$, $\{y \in Y : F_i(x, z_i) \cap C_i(x) = \emptyset,$ *for some* $z_i \in Q_i(y)\}$ *is* Γ*-convex;*

(iii) *for each* $y \in Y$, $x \in T(y)$ *and* $z_i \in Q_i(y)$, *we have* $F_i(x, z_i) \cap C_i(x) \neq \emptyset$*;*

(iv) $Q_i(Y) = Z_i$.

Then there exists $\bar{x} \in X$ *such that* $F_i(\bar{x}, z_i) \cap C_i(\bar{x}) \neq \emptyset$ *for each* $i \in I$ *and* $z_i \in Z_i$.

Proof. For each $i \in I$, define $P_i : X \multimap Z_i$ by

$$P_i(x) = \{z_i \in Z_i : F_i(x, z_i) \cap C_i(x) = \emptyset\}, \ \forall x \in X.$$

Similar to the proof of Theorem 5.3, we can verify that all the conditions of Theorem 4.2 are satisfied. Thus, Theorem 4.2 shows that there exists $\bar{x} \in X$ such that $P_i(\bar{x}) = \emptyset$ for each $i \in I$, i.e., $F_i(\bar{x}, z_i) \cap C_i(\bar{x}) \neq \emptyset$ for each $i \in I$ and $z_i \in Z_i$. This completes the proof.

Theorem 5.5. *Let X be a topological space, $(Y;\Gamma)$ be an abstract convex space, and $T \in \mathfrak{RC}(Y,X)$ be a compact map. Let I be any index set, $\{Z_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ be two families of nonempty sets. For each $i \in I$, let $F_i : X \times Z_i \multimap V_i$, $C_i : X \multimap V_i$ and $Q_i : Y \multimap Z_i$ be three maps satisfying the following conditions:*

- (*i*) *for each $x \in X$, $\{z_i \in Z_i : F_i(x,z_i) \subset C_i(x)\} \neq \emptyset$ implies that there exist a neighborhood $N_i(x)$ of x and $z_i' \in Z_i$ such that $F_i(x',z_i') \subset C_i(x')$ for all $x' \in N_i(x)$;*
- (*ii*) *for each $x \in X$, $\{y \in Y : F_i(x,z_i) \subset C_i(x), \text{for some } z_i \in Q_i(y)\}$ is Γ-convex;*
- (*iii*) *for each $y \in Y$, $x \in T(y)$ and $z_i \in Q_i(y)$, we have $F_i(x,z_i) \not\subset C_i(x)$;*
- (*iv*) *$Q_i(Y) = Z_i$.*

Then there exists $\bar{x} \in X$ such that $F_i(\bar{x},z_i) \not\subset C_i(\bar{x})$ for each $i \in I$ and $z_i \in Z_i$.

Proof. For each $i \in I$, define $P_i : X \multimap Z_i$ by

$$P_i(x) = \{z_i \in Z_i : F_i(x,z_i) \subset C_i(x)\}, \ \forall x \in X.$$

Similar to the proof of Theorem 5.3, we can verify that all the conditions of Theorem 4.2 are satisfied. Thus, Theorem 4.2 shows that there exists $\bar{x} \in X$ such that $P_i(\bar{x}) = \emptyset$ for each $i \in I$, i.e., $F_i(\bar{x},z_i) \not\subset C_i(\bar{x})$ for each $i \in I$ and $z_i \in Z_i$. This completes the proof.

Theorem 5.6. *Let X be a topological space, $(Y;\Gamma)$ be an abstract convex space, and $T \in \mathfrak{RC}(Y,X)$ be a compact map. Let I be any index set, $\{Z_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ be two families of nonempty sets. For each $i \in I$, let $F_i : X \times Z_i \multimap V_i$, $C_i : X \multimap V_i$ and $Q_i : Y \multimap Z_i$ be three maps satisfying the following conditions:*

- (*i*) *for each $x \in X$, $\{z_i \in Z_i : F_i(x,z_i) \cap C_i(x) \neq \emptyset\} \neq \emptyset$ implies that there exist a neighborhood $N_i(x)$ of x and $z_i' \in Z_i$ such that $F_i(x',z_i') \cap C_i(x') \neq \emptyset$ for all $x' \in N_i(x)$;*
- (*ii*) *for each $x \in X$, $\{y \in Y : F_i(x,z_i) \cap C_i(x) \neq \emptyset, \text{for some } z_i \in Q_i(y)\}$ is Γ-convex;*
- (*iii*) *for each $y \in Y$, $x \in T(y)$ and $z_i \in Q_i(y)$, we have $F_i(x,z_i) \cap C_i(x) = \emptyset$;*
- (*iv*) *$Q_i(Y) = Z_i$.*

Then there exists $\bar{x} \in X$ such that $F_i(\bar{x},z_i) \cap C_i(\bar{x}) = \emptyset$ for each $i \in I$ and $z_i \in Z_i$.

Proof. For each $i \in I$, define $P_i : X \multimap Z_i$ by

$$P_i(x) = \{z_i \in Z_i : F_i(x,z_i) \cap C_i(x) \neq \emptyset\}, \ \forall x \in X.$$

Similar to the proof of Theorem 5.3, we can verify that all the conditions of Theorem 4.2 are satisfied. Thus, Theorem 4.2 shows that there exists $\bar{x} \in X$ such that $P_i(\bar{x}) = \emptyset$ for each $i \in I$, i.e., $F_i(\bar{x}, z_i) \cap C_i(\bar{x}) = \emptyset$ for each $i \in I$ and $z_i \in Z_i$. This completes the proof.

Remark 5.5. If Y is a convex subset of a topological vector space, $\Gamma = \text{co}$ and I is a singleton, then Theorems 5.3, 5.4, 5.5 and 5.6 reduce to Theorem 9 in [1].

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Ming-ge Yang[a,b], Nan-jing Huang[a] and Chin-San Lee[c]
[a]Department of Mathematics,
Sichuan University,
Chengdu, Sichuan 610064,
P. R. China

[b]Department of Mathematics,
Luoyang Normal University,
Luoyang, Henan 471022,
P. R. China

[c]Department of Leisure,
Recreation and Tourism Management,
Shu-Te University,
Kaohsiung, Taiwan