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ON THE SOLVABILITY OF SOLUTIONS TO SOME QUASILINEAR ELLIPTIC PROBLEMS

Tsang-Hai Kuo and Chiung-Chiou Tsai

Abstract. Let Ω be a bounded open set in $\mathbb{R}^N$ and $1 < p < \infty$. We study the following quasilinear elliptic problem:

$$\begin{cases} Lu = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where L is a Leray-Lions type operator from $W_0^{1,p}(\Omega)$ into its dual space. It is shown that there exists a solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to the problem provided that $|f(x, r, \xi)| \le C(1 + |r|^\delta + |\xi|^\eta)$ where C is a nonnegative constant and $0 \le \delta, \eta < p - 1$.

1. Introduction

In this paper, Ω shall be a bounded open set in $\mathbb{R}^N$ and $1 < p < \infty$. $W^{m,p}(\Omega) = \{u \in L^p(\Omega) |$ weak derivatives $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| \le m\}$, $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ and $W^{-m,p'}(\Omega)$ is the dual space of $W_0^{m,p}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$. ∇u denotes the gradient of u.

Consider the following nonlinear elliptic problem:

$$(1.1) \qquad \begin{cases} Lu = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) = f(x, u, \nabla u) \quad \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \end{cases}$$

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where L is a Leray-Lions type operator from $W_0^{1,p}(\Omega)$ into its dual space $W^{-1,p'}(\Omega)$. An operator $L : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined in (1.1) is called a Leray-Lions type operator if a_i is a Carathédory function, that is,

$$\begin{cases} x \to a_i(x,r,\xi) \text{ is measurable } \forall (r,\xi) \in \mathbb{R} \times \mathbb{R}^N, \\ (r,\xi) \to a_i(x,r,\xi) \text{ is continuous for a.e. } x \in \Omega; \end{cases} \tag{1.2}$$

and satisfies the following hypotheses:

$$\begin{cases} \text{There is } \alpha > 0 \text{ such that, for a.e. } x \in \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,r,\xi) \cdot \xi_i \geq \alpha |\xi|^p \quad \forall r \in \mathbb{R}, \forall \xi \in \mathbb{R}^N; \end{cases} \tag{1.3}$$

$$\begin{cases} \text{There exist } \beta > 0,\ k \in L^{p'}(\Omega) \text{ such that, for a.e. } x \in \Omega, \\ |a_i(x,r,\xi)| \leq \beta(|r|^{p-1} + |\xi|^{p-1} + k(x)) \quad \forall r \in \mathbb{R}, \forall \xi \in \mathbb{R}^N; \end{cases} \tag{1.4}$$

$$\begin{cases} \displaystyle\sum_{i=1}^{N} (a_i(x,r,\xi) - a_i(x,r,\hat{\xi})) \cdot (\xi_i - \hat{\xi}_i) > 0 \\ \text{for a.e. } x \in \Omega, \forall r \in \mathbb{R}, \forall \xi, \hat{\xi} \in \mathbb{R}^N,\ \xi \neq \hat{\xi}. \end{cases} \tag{1.5}$$

Suppose that f is a Carathéodory function satisfying

$$|f(x,r,\xi)| \leq h(|r|)(1 + |\xi|^p),$$

where h is an increasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$, and that there exist a subsolution φ and a supersolution ψ with $\varphi,\ \psi \in W^{1,\infty}(\Omega)$ and $\varphi \leq \psi$ a.e. in Ω. Suppose further that there exists $\epsilon > 0$ such that $k \in L^{p'+\epsilon}(\Omega)$ in hypothesis (1.4). Then it has been shown that in [1] there exists a solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \leq u \leq \psi$ a.e. in Ω. However, the existence of a subsolution and a supersolution is a structure hypothesis.

In this paper, instead of this hypothesis, we impose the growth condition on f with respect to r, ξ by

$$|f(x,r,\xi)| \leq C(1 + |r|^\delta + |\xi|^\eta), \tag{1.6}$$

where C is a nonnegative constant and $0 \leq \delta, \eta < p - 1$. We then prove the following main result.

Theorem 1.1. *Under hypotheses* (1.2)-(1.6), *there exists a solution to problem* (1.1).

As an example, when $p > 1$ and $0 \le \delta, \eta < p-1$, consider the following problem:

$$(1.7)\quad \begin{cases} Lu = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right) + u^{\delta} + |\nabla u|^{\eta} = h \quad \text{in } \mathcal{D}'(\Omega), \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $h \in L^{\infty}(\Omega)$. Theorem 1.1 then ensures the existence of solutions to problem (1.7) without knowing apriori the existence of subsolutions and supersolutions.

2. Lemmas and Proof of Main Results

We first give some lemmas which will be employed in the proof of Theorem 1.1.

Lemma 2.1. ([3, Lemma 1.3]) *Let $g \in L^q(\Omega)$, $g_\mu \in L^q(\Omega)$ and $\|g_\mu\|_{L^q(\Omega)} \le C$, $1 < q < \infty$. If $g_\mu \to g$ a.e., then $g_\mu \rightharpoonup g$ weakly in $L^q(\Omega)$.*

Lemma 2.2. ([3, Lemma 2.1]) *If $u_\mu \to u$ in $L^p(\Omega)$ and $v \in W^{1,p}(\Omega)$, then*

$$a_i(x, u_\mu, \nabla v) \to a_i(x, u, \nabla v) \text{ in } L^{p'}(\Omega).$$

Lemma 2.3. *If $u_\mu \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and if*

$$(2.1)\qquad \left\langle -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x, u_\mu, \nabla u_\mu) - a_i(x, u_\mu, \nabla u)\right), u_\mu - u \right\rangle \to 0,$$

then $u_\mu \to u$ in $W_0^{1,p}(\Omega)$.

Proof. By the compact imbedding theorem, we have $u_\mu \to u$ in $L^p(\Omega)$. It follows from Lemma 2.2 that, for $u \in W^{1,p}(\Omega)$,

$$(2.2)\qquad a_i(x, u_\mu, \nabla u) \to a_i(x, u, \nabla u) \quad \text{in } L^{p'}(\Omega).$$

Combining (2.1) with (2.2), we have

$$\left\langle -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(a_i(x, u_\mu, \nabla u_\mu) - a_i(x, u, \nabla u)), u_\mu - u \right\rangle \to 0.$$

By a well-known result on mappings of type (S) [2, Lemma 3], it follows that $u_\mu \to u$ in $W_0^{1,p}(\Omega)$.

For $v \in W^{1,p}(\Omega)$, we associate the Nemytskii operator F with respect to f, defined by

$$F(v, \nabla v)(x) = f(x, v, \nabla v) \quad \text{for a.e. } x \in \Omega.$$

Lemma 2.4. *The operator $v \to F(v, \nabla v)$ is continuous from $W^{1,p}(\Omega)$ into $L^{p'}(\Omega)$.*

Proof. By hypothesis (1.6), we have

$$|f(x, r, \xi)| \le C(3 + |r|^{p-1} + |\xi|^{p-1}) \tag{2.3}$$

which implies $F(v, \nabla v) \in L^{p'}(\Omega)$. Since f is a Carathédory function, the lemma follows immediately by applying [4, Theorem 2.1].

Proof of Theorem 1.1. We will show that the operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$A(v) = Lv - f(x, v, \nabla v)$$

is a variational operator ([3, p. 180]) and satisfies the coercive condition

$$\lim_{\|v\|_{W^{1,p}} \to \infty} \frac{\langle A(v), v\rangle}{\|v\|_{W^{1,p}}} = \infty. \tag{2.4}$$

The detailed proof is achieved as follows.

(1) Let

$$A(u, v) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla v) - f(x, u, \nabla u).$$

Then $A(v, v) = A(v)$ for all $v \in W_0^{1,p}(\Omega)$. It is easy to see that A and the operator $v \to A(u, v)$ is bounded for all $u \in W_0^{1,p}(\Omega)$. Now we claim that the operator $v \to A(u, v)$ is hemicontinuous for all $u \in W_0^{1,p}(\Omega)$, i.e., the operator

$$\lambda \ \to \ \langle A(u, v_1 + \lambda v_2), w\rangle$$

is continuous for all $v_1, v_2, w \in W_0^{1,p}(\Omega)$. Since a_i is a Carathédory function,

$$a_i(x, u, \nabla(v_1 + \lambda v_2)) \to a_i(x, u, \nabla v_1) \quad \text{a.e. as } \lambda \to 0.$$

Further, we know from (1.4) that $a_i(x, u, \nabla(v_1 + \lambda v_2))$ is bounded in $L^{p'}(\Omega)$. Thus, by Lemma 2.1,

$$a_i(x, u, \nabla(v_1 + \lambda v_2)) \rightharpoonup a_i(x, u, \nabla v_1) \quad \text{weakly in } L^{p'}(\Omega).$$

Hence, as $\lambda \to 0$,

$$
\begin{aligned}
&\langle A(u, v_1 + \lambda v_2), w\rangle \\
&\quad = \int_\Omega a_i(x, u, \nabla(v_1 + \lambda v_2))\frac{\partial w}{\partial x_i}dx - \int_\Omega f(x, u, \nabla u)wdx \\
&\quad \to \int_\Omega a_i(x, u, \nabla v_1)\frac{\partial w}{\partial x_i}dx - \int_\Omega f(x, u, \nabla u)wdx \\
&\quad = \langle A(u, v_1), w\rangle
\end{aligned}
$$

for all $v_1, v_2, w \in W_0^{1,p}(\Omega)$. Similarly, it follows from the proof as stated above that the operator $u \to A(u, v)$ is bounded and hemicontinuous for all $v \in W_0^{1,p}(\Omega)$.

(2) By (1.5), we have, for all $u, v \in W_0^{1,p}(\Omega)$,

$$
\begin{aligned}
&\langle A(u, u) - A(u, v), u - v\rangle \\
&\quad = \sum_{i=1}^N \int_\Omega (a_i(x, u, \nabla u) - a_i(x, u, \nabla v))\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right)dx \geq 0.
\end{aligned}
$$

(3) Let $u_\mu \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $\langle A(u_\mu, u_\mu) - A(u_\mu, u), u_\mu - u\rangle \to 0$. We claim that $A(u_\mu, v) \rightharpoonup A(u, v)$ weakly in $W^{-1,p'}(\Omega)$ for all $v \in W_0^{1,p}(\Omega)$. Since $u_\mu \to u$ in $L^p(\Omega)$ by the compact imbedding theorem, we can obtain from Lemma 2.2 that

$$
a_i(x, u_\mu, \nabla v) \to a_i(x, u, \nabla v) \quad \text{in } L^{p'}(\Omega). \tag{2.5}
$$

By Lemma 2.3, we have $u_\mu \to u$ in $W_0^{1,p}(\Omega)$ and it follows from Lemma 2.4 that

$$
f(x, u_\mu, \nabla u_\mu) \to f(x, u, \nabla u) \quad \text{in } L^{p'}(\Omega). \tag{2.6}
$$

Hence, by (2.5) and (2.6), we have

$$
\begin{aligned}
\langle A(u_\mu, v), w\rangle &= \sum_{i=1}^N \int_\Omega a_i(x, u_\mu, \nabla v)\frac{\partial w}{\partial x_i}dx - \int_\Omega f(x, u_\mu, \nabla u_\mu)wdx \\
&\to \sum_{i=1}^N \int_\Omega a_i(x, u, \nabla v)\frac{\partial w}{\partial x_i}dx - \int_\Omega f(x, u, \nabla u)wdx \\
&= \langle A(u, v), w\rangle \quad \text{for all } w \in W_0^{1,p}(\Omega).
\end{aligned}
$$

(4) If $u_\mu \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $A(u_\mu, v) \rightharpoonup \phi$ weakly in $W^{-1,p'}(\Omega)$. We claim that $\langle A(u_\mu, v), u_\mu \rangle \to \langle \phi, u \rangle$. As stated in (3), we have

$$a_i(x, u_\mu, \nabla v) \to a_i(x, u, \nabla v) \quad \text{in } L^{p'}(\Omega)$$

and so

$$\int_\Omega a_i(x, u_\mu, \nabla v) \frac{\partial u_\mu}{\partial x_i} dx \to \int_\Omega a_i(x, u, \nabla v) \frac{\partial u}{\partial x_i} dx.$$

Hence, together with

$$\sum_{i=1}^N \int_\Omega a_i(x, u_\mu, \nabla v) \frac{\partial u}{\partial x_i} dx - \int_\Omega f(x, u_\mu, \nabla u_\mu) u dx \ \to \ \langle \phi, u \rangle,$$

we have

$$\begin{aligned} & \langle A(u_\mu, v), u_\mu \rangle \\ = \ & \sum_{i=1}^N \int_\Omega a_i(x, u_\mu, \nabla v) \frac{\partial u_\mu}{\partial x_i} dx - \int_\Omega f(x, u_\mu, \nabla u_\mu) u_\mu dx \\ = \ & \sum_{i=1}^N \int_\Omega a_i(x, u_\mu, \nabla v) \left(\frac{\partial u_\mu}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) dx + \sum_{i=1}^N \int_\Omega a_i(x, u_\mu, \nabla v) \frac{\partial u}{\partial x_i} dx \\ & - \int_\Omega f(x, u_\mu, \nabla u_\mu) u dx - \int_\Omega f(x, u_\mu, \nabla u_\mu)(u_\mu - u) dx \ \to \ \langle \phi, u \rangle. \end{aligned}$$

(5) Now we claim (2.4). By (1.3), we have

$$\begin{aligned} \langle Av, v \rangle \ &= \ \sum_{i=1}^N \int_\Omega a_i(x, v, \nabla v) \frac{\partial v}{\partial x_i} dx - \int_\Omega f(x, v, \nabla v) v dx \\ &\geq \ \alpha \|\nabla v\|_p^p - \int_\Omega f(x, v, \nabla v) v dx. \end{aligned}$$

It follows from the Poincaré inequality that

$$\frac{\langle Av, v \rangle}{\|v\|_{W^{1,p}}} \geq C_0 \|v\|_{W^{1,p}}^{p-1} - \frac{\displaystyle\int_\Omega f(x, v, \nabla v) v dx}{\|v\|_{W^{1,p}}}$$

for some constant $C_0 > 0$. By (1.6), there exist nonnegative constants C_1, C_2 and C_3 such that

$$\int_\Omega f(x, v, \nabla v) v dx \leq C_1 + C_2 \|v\|_{W^{1,p}}^{\delta+1} + C_3 \|v\|_{W^{1,p}}^{\eta+1}.$$

Since $0 \leq \delta, \eta < p - 1$, we can conclude that

$$\frac{\langle Av, v \rangle}{\|v\|_{W^{1,p}}} \to \infty.$$

Therefore, by (1)-(5), there exists a solution $u \in W_0^{1,p}(\Omega)$ to problem (1.1) by applying Corollary 2.1 of [3]. Furthermore, by (1.3) and (2.3), we can obtain from [5, Theorem 10.9] that $u \in L^\infty(\Omega)$.

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Department of Applied Mathematics, National Chiao Tung University
Hsinchu, Taiwan