TAIWANESE JOURNAL OF MATHEMATICS
Vol. 1, No. 4, pp. 527-546, December 1997

ON CERTAIN OPERATOR FAMILIES RELATED TO COSINE OPERATOR FUNCTIONS*

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Abstract. This paper is concerned with two cosine-function-related functions which are called cosine step response and cosine cumulative output. We study some of their properties, such as measurability, continuity, infinitesimal operator, compactness, positivity, almost periodicity, and asymptotic behavior.

1. Introduction

Let X be a Banach space, and $B(X)$ denote the algebra of all bounded linear operators on X. Throughout this paper, $\{C(t); t \geq 0\}$ is a strongly continuous *cosine operator function* on X. By definition, it is a family of operators in $B(X)$ satisfying

(a) $C(0) = I$;
(b) $C(t+s) + C(t-s) = 2C(t)C(s)$ for t, $s \in (-\infty, \infty)$;
(c) the function $C(\cdot)x$ is continuous on $(-\infty, \infty)$ for every $x \in X$.

There exist some $M \geq 1$, $\omega \in R$ such that $\|C(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. The associated *sine operator function* $S(\cdot)$ is defined by the formula $S(t) = \int_0^t C(s)ds,\ t \in (-\infty, \infty)$. The *second infinitesimal generator* (or simply the generator) A of $C(\cdot)$ is defined as $Ax = \lim_{t\to 0+} 2t^{-2}(C(t) - I)x$, with natural domain. A cosine operator function gives the solution of a well-posed Cauchy problem

$$\text{(CP)} \qquad u''(t) = Au(t),\ u(0) = x, u'(0) = y,$$

Received July 10, 1996.
Communicated by the Editors.
1991 *Mathematics Subject Classification*: 47D09.
Key words and phrases: cosine operator function, cosine step response, cosine cumulative output.
* This research was partially supported by the NSC.

in the form $u(t) = C(t)x + S(t)y$, $-\infty < t < \infty$. For the theory of cosine operator function we refer to [5] and [16].

This paper is concerned with two cosine-function-related functions which we define as follows.

Definition 1.1. Let $C(\cdot)$ be a cosine operator function. A family $\{F(t);\ t \in (-\infty,\infty)\}$ of operators in $B(X)$ is called a *cosine step response* for $C(\cdot)$ if $F(0) = 0$ and

$$F(t+s) - 2F(t) + F(t-s) = 2C(t)F(s) \quad \text{for} \quad t,\ s \in (-\infty,\infty). \tag{1.1}$$

A family $\{G(t); -\infty < t < \infty\}$ in $B(X)$ is called a *cosine cumulative output* for $C(\cdot)$ if $G(0) = 0$ and

$$G(t+s) - 2G(t) + G(t-s) = 2G(s)C(t) \quad \text{for} \quad t,\ s \in (-\infty,\infty). \tag{1.2}$$

Clearly, $F(\cdot)$ and $G(\cdot)$ are even functions. If these families are strongly continuous at zero, we call them C_0-*cosine step response* and C_0-*cosine cumulative output*, respectively.

The above terminologies are chosen in view that the two functions $F(\cdot)$ and $G(\cdot)$ are related to the cosine function $C(\cdot)$ more or less the same way that a step response $U(\cdot)$ and a cumulative output $V(\cdot)$ are related to a C_0-semigroup $T(\cdot)$, and it turns out that they have similar properties. We recall that $U(\cdot)$ satisfies $U(0) = 0$ and $U(t+s) - U(t) = T(t)U(s)$, $t,\ s \geq 0$, and $V(\cdot)$ satisfies $V(0) = 0$ and $V(t+s) - V(t) = V(s)T(t)$, $t,\ s \geq 0$. Step responses and cumulative outputs for C_0-semigroups have been investigated in [11], [12], and [15].

C_0-cosine step responses and C_0-cosine cumulative outputs play interesting roles in the perturbation of cosine operator functions and the Cauchy problem (CP) (see [13]). For example, using C_0-cosine step responses we are able to consider the well-posedness of the perturbed Cauchy problem in the form

$$u''(t) = A(1 - \lambda\hat{F}(\lambda))u(t) + \lambda^3\hat{F}(\lambda)u(t),\ u(0) = x, u'(0) = y,\ t \geq 0.$$

Our purpose in this paper is to study some properties of cosine step responses and cosine cumulative outputs, such as, measurability and continuity, Laplace transform and infinitesimal operator, compactness, positiveness, almost periodicity, and asymptotic behavior; each subject will be discussed in a section.

2. Measurability and Continuity

It is known that a cosine operator function which is strongly (resp. uniformly) measurable on $(0,\infty)$ has to be strongly (resp. uniformly) continuous on $(-\infty,\infty)$ (see [4] and [7]). The next theorem shows that cosine step responses and cosine cumulative outputs share the same property.

Theorem 2.1. *If a cosine step response $F(\cdot)$ is strongly (resp. uniformly) measurable on $(0,\infty)$, then $F(\cdot)$ is strongly (resp. uniformly) continuous on $(-\infty,\infty)$. If a cosine cumulative output $G(\cdot)$ is uniformly measurable on $(0,\infty)$, then $G(\cdot)$ is uniformly continuous on $(-\infty,\infty)$.*

Proof. First of all the Lebesgue measurability of $F(\cdot)x$ on $(0,\infty)$ implies the Lebesgue measurability of $\|F(\cdot)x\|$ on $(0,\infty)$ (see [6]). Next, we show that $\|F(\cdot)x\|$ is bounded on any compact subinterval $[a,b]$ of $(0,\infty)$ for every $x \in X$. Suppose not. Then there are an $\tilde{x} \in X$, a number $\tau > 0$ and a sequence $\tau_n \in [a,b]$ such that $\tau_n \to \tau$ and

$$\|F(\tau_n)\tilde{x}\| \geq n \quad \text{as} \quad n \to \infty.$$

Because of the measurability of $\|F(\cdot)\tilde{x}\|$ there exist a constant c_1 and a Lebesgue measurable set $\Lambda \subset [0,\tau]$ with measure $m(\Lambda) > \frac{3}{4}\tau$ and

$$\sup_{t\in\Lambda} \|F(t)\tilde{x}\| \leq c_1. \tag{2.1}$$

Now following [2] we let

$$\mathcal{A}_k := \frac{\tau_k}{2} - \frac{\Lambda \cap [0,\tau_k]}{2}, \quad \mathcal{B}_k := \Lambda \cap [0,\tau_k/2]. \tag{2.2}$$

and

$$\mathcal{A} = \frac{\tau}{2} - \frac{\Lambda}{2}, \ \mathcal{B} = \Lambda \cap [0,\tau/2].$$

First we have $m(\mathcal{A}\cap\mathcal{B}) > 0$. To prove this, assume that $m(\mathcal{A}\cap\mathcal{B}) = 0$. Then $m(\mathcal{A}) + m(\mathcal{B}) \leq \tau/2$. But $m(\mathcal{A}) = m(\Lambda)/2$ by definition of set $\mathcal{A}$. So it means that $m(\Lambda) + 2m(\mathcal{B}) \leq \tau$. Hence $\frac{3}{4}\tau < m(\Lambda) \leq \tau - 2m(\mathcal{B})$, i.e.

$$m(\mathcal{B}) \leq \tau/8. \tag{2.3}$$

Now let us write

$$\Lambda = (\Lambda \cap [0,\tau/2]) \ \cup \ (\Lambda \cap [\tau/2,\tau]) = \mathcal{B} \cup \mathcal{D},$$

where $m(\Lambda) = m(\mathcal{B}) + m(\mathcal{D})$ with $m(\mathcal{D}) \leq \tau/2$. But

$$\frac{3}{4}\tau < m(\Lambda) = m(\mathcal{B}) + m(\mathcal{D}) \leq m(\mathcal{B}) + \tau/2$$

implies $m(\mathcal{B}) > \tau/4$, that is a contradiction to (2.3). We have proved that $m(\mathcal{A} \cap \mathcal{B}) \geq \delta > 0$.

Now we introduce the sets $E = \mathcal{A} \cap \mathcal{B}$, $E_n = \mathcal{A}_n \cap \mathcal{B}_n$ and $H_n = \{\tau_n - \eta\,;\ \eta \in E_n\}$. It is clear that $E_n \to E$ as $n \to \infty$, so that $m(H_n) > \delta/2$ for n large enough. For such n, if $\eta \in E_n$, then η and $\tau_n - 2\eta$ both belong to Λ because of (2.2). Using now (1.1) and (2.1) we get for $\eta \in E_n$

$$\begin{aligned} n &\leq \|F(\tau_n)\tilde{x}\| \\ &\leq 2\|F(\tau_n - \eta)\tilde{x}\| + \|F(\tau_n - 2\eta)\tilde{x}\| + 2\|C(\tau_n - \eta)\|\ \|F(\eta)\tilde{x}\| \\ &\leq 2\|F(\tau_n - \eta)\tilde{x}\| + c_1 + 2Me^{\omega b}c_1. \end{aligned} \tag{2.4}$$

Hence

$$\|F(t)\tilde{x}\| \geq \frac{n - c_1 - 2Mc_1e^{\omega\beta}}{2}$$

for $t \in H_n$, and denoting $\lim_{n\to\infty} H_n = H_\infty$ we have that $\|F(t)\tilde{x}\| = \infty$ for $t \in H_\infty$ with $m(H_\infty) \geq \delta/2 > 0$. This is a contradiction to the fact that $\|F(t)\tilde{x}\|$ is finite for all t.

Now we are going to prove that Lebesgue measurability together with boundedness implies the continuity of $F(\cdot)x$ for each $t > 0$ and each $x \in X$. For this purpose we choose four positive numbers α, β, ϵ and γ such that $\beta < t - \epsilon$ and $0 < \alpha < \gamma < \beta < t$. From (1.1) we have

$$F(t)x = 2F(t - \gamma/2)x - F(t - \gamma)x + 2C(t - \gamma/2)F(\gamma/2)x.$$

The left hand side being independent of γ is integrable with respect to γ and we have

$$\begin{aligned} (\beta - \alpha)(F(t \pm \epsilon)x - F(t)x) = &\int_\alpha^\beta 2(F(t \pm \epsilon - \gamma/2) - F(t - \gamma/2)x)d\gamma \\ &- \int_\alpha^\beta (F(t \pm \epsilon - \gamma) - F(t - \gamma))xd\gamma \\ &+ \int_\alpha^\beta 2(C(t \pm \epsilon - \gamma/2) \\ &- C(t - \gamma/2))F(\gamma/2)xd\gamma. \end{aligned}$$

Hence

$$\|(F(t\pm\epsilon)-F(t))x\| \leq \tfrac{1}{\beta-\alpha}\left[\int_{t-\beta/2}^{t-\alpha/2}\|(F(\zeta\pm\epsilon)-F(\zeta))x\|d\zeta \right.$$
$$+\int_{t-\beta}^{t-\alpha}\|(F(\zeta\pm\epsilon)-F(\zeta))x\|d\zeta$$
$$\left. +2\int_{\alpha}^{\beta}\|(C(t\pm\epsilon-\gamma/2)-C(t-\gamma/2))F(\gamma/2)x\|d\gamma\right].$$

By Theorem 3.8.3 of [6], $\int_{t-\beta/2}^{t-\alpha/2}\to 0$ and $\int_{t-\beta}^{t-\alpha}\to 0$ as $\epsilon\to 0$. The last term goes to zero because of the Lebesgue convergence theorem (see [6, Theorem 3.7.9]) . It follows now that $F(t)x$ is continuous for $t>0$. Replacing the t in (1.1) by $t+s$ we have for all $t,\ s>0$

$$F(t)x = 2C(t+s)F(s)x - F(t+2s)x + 2F(t+s)x,$$

which converges to $2C(s)F(s)x - F(2s)x + 2F(s)x = F(0)x = 0$ as $t\to 0^+$. Therefore $F(\cdot)$ is strongly continuous on $[0,\infty)$, and hence on $(-\infty,\infty)$, because $F(\cdot)$ is an even function. The proof for the case of uniform measurability is similar.

To prove the assertion for $G(\cdot)$ one can use the following form of (1.2):

$$G(\tau_n) = 2G(\tau_n-\eta) - G(\tau_n-2\eta) + 2G(\eta)C(\tau_n-\eta)$$

in an estimate analogous to (2.4). The proof then proceeds similarly.

Theorem 2.2. *C_0-cosine step responses and C_0-cosine cumulative outputs for a cosine operator function $C(\cdot)$ are strongly continuous on $[0,\infty)$. Moreover, uniform continuity at 0 implies uniform continuity on $[0,\infty)$.*

Proof. Following [3] we suppose in contrary that the C_0-cosine step response $F(\cdot)$ is not strongly continuous at some point $t_0>0$, i.e. there exists x_0 such that the nonincreasing sequence

$$K_n := \sup\left\{\|(F(t)-F(s))x_0\|\ ;\ |t-t_0|,\,|s-t_0|\leq\frac{t_0}{8n}\right\}$$

converges to some $K>0$ as $n\to\infty$. We can take sequences τ_n and σ_n such that

$$|\tau_n-t_0|\leq\frac{t_0}{8n}\ ,\ |\sigma_n-t_0|\leq\frac{t_0}{8n}$$

and

$$\|(F(\tau_n)-F(\sigma_n))x_0\|\geq K_n-\frac{1}{n},\ n\in N.$$

It is clear that $|\sigma_n - \tau_n| \leq \frac{t_0}{4n}$ and $|2\tau_{4n} - \sigma_{4n} - t_0| \leq \frac{t_0}{8n}, n \in N$. Therefore

$$\|(F(\sigma_{4n}) - F(2\tau_{4n} - \sigma_{4n}))x_0\| \leq K_n,\, n \in N.$$

Now, using identity (1.1) in the form

$$2(F(t+h) - F(t)) = (F(t+h) - F(t-h)) + 2C(t)F(h)$$

and putting $t_0 + h = \sigma_{4n}$ and $t_0 = \tau_{4n}$ we get

$$2\|(F(\sigma_{4n}) - F(\tau_{4n})x_0\| \leq K_n + 2Me^{\omega t_0}\|F(\sigma_{4n} - \tau_{4n})x_0\|.$$

Hence

$$2(K_{4n} - \frac{1}{4n}) \leq K_n + 2Me^{\omega t_0}\|F(\sigma_{4n} - \tau_{4n})x_0\|$$

and so

$$K_{4n} + (K_{4n} - K_n) \leq \frac{1}{2n} + 2Me^{\omega t_0}\|F(h)x_0\|. \tag{2.5}$$

Because $F(h)x_0 \to 0$ as $h \to 0$ (we recall that $h = \sigma_{4n} - \tau_{4n}$) and $K_{4n} - K_n \to 0$ as $n \to \infty$, we have that $K_n \to 0$ as $n \to \infty, n \in N$, which is a contradiction to our assumption that $K_n \to K,\ K > 0$.

To prove the same statement for $G(\cdot)$ by a similar argument one can use the identity

$$2(G(t+h) - G(t)) = (G(t+h) - G(t-h)) + 2G(t)(C(h) - I) + 2G(h),$$

which is obtained from (1.2) and Proposition 3.1 (i).

By a routine argument as in the proof of Proposition 2.6 of [11] one can prove the following proposition.

Proposition 2.3. *Suppose C_0-cosine step response $F(\cdot)$ and cosine operator function $C(\cdot)$ commute, i.e. $F(t)C(t) = C(t)F(t)$ for all $t \geq 0$. Then $F(\cdot)$ is a commutative family, i.e. $F(t)F(s) = F(s)F(t)$ for $s, t \geq 0$. The same is true for C_0-cosine cumulative outputs.*

3. Laplace Transform and Infinitesimal Operators

In this section we collect some basic properties of the Laplace transforms $\hat{F}(\cdot)$ and $\hat{G}(\cdot)$ of C_0-cosine step response $F(\cdot)$ and C_0-cosine cumulative output $G(\cdot)$.

Proposition 3.1. *Let $F(\cdot)$ be a C_0-cosine step response and $G(\cdot)$ be a C_0-cosine cumulative output for a cosine operator function $C(\cdot)$. The following properties are satisfied:*

(i) $(C(t)-I)F(s)=(C(s)-I)F(t)$ *and* $G(s)(C(t)-I)=G(t)(C(s)-I)$ *for* $t,s\geq 0$;

(ii) *The functions* $F(\cdot)$ *and* $G(\cdot)$ *are exponentially bounded*;

(iii) $\frac{d^2}{dt^2}[\lambda(\lambda^2-A)^{-1}F(t)x]=C(t)\lambda^2\hat{F}(\lambda)x$ *and*

$\frac{d^2}{dt^2}[G(t)\lambda(\lambda^2-A)^{-1}x]=\lambda^2\hat{G}(\lambda)C(t)x$ for $x\in X,\ \lambda>\omega,$ *and* $t>0$;

(iv) $F(t)x=(\lambda^2-A)\int_0^t S(s)\lambda\hat{F}(\lambda)x\,ds$

$=\int_0^t S(s)\lambda^3\hat{F}(\lambda)x\,ds-(C(t)-I)\lambda\hat{F}(\lambda)x$ *for* $x\in X,\ t\geq 0$;

(v) $G(t)x=\lambda\hat{G}(\lambda)(\lambda^2-A)\int_0^t S(s)x\,ds$

$=\lambda^3\hat{G}(\lambda)\int_0^t S(s)x\,ds-\lambda\hat{G}(\lambda)(C(t)-I)x$ *for* $x\in X,\ t\geq 0$.

Proof. Property (i) is an easy consequence of (1.1) and (1.2). To show that a C_0-cosine step response $F(\cdot)$ is exponentially bounded, let us choose $L\geq 1,\tau>0$ such that $\|C(s)\|\leq L,\|F(s)\|\leq L$ for $0\leq s\leq\tau$. Using

$$F(k\tau+s)=2F(k\tau)-F(k\tau-s)+2C(k\tau)F(s),$$

we have for $0\leq s\leq\tau$

$$\begin{aligned}\|F(\tau+s)\|&\leq\|2F(\tau)\|+\|F(\tau-s)\|+\ 2\ Me^{\tau\omega}\|F(s)\|\\&\leq 2L+L+2Me^{\tau\omega}L\leq Me^{\tau\omega}5L\leq Me^{2\tau\omega_1},\end{aligned}$$

where $5L\leq e^{\tau\omega_1}$ and $\omega\leq\omega_1$, and by induction

$$\begin{aligned}\|F(k\tau+s)\|&\leq 2\|F(k\tau)\|+\|F(k\tau-s)\|+2\|C(k\tau)\|\ \|F(s)\|\\&\leq 2Me^{k\tau\omega_1}+Me^{k\tau\omega_1}+2LMe^{k\tau\omega_1}\leq 5LMe^{k\tau\omega_1}\\&\leq Me^{(k+1)\tau\omega_1}\end{aligned}$$

for all $s\in[0,\tau]$. Hence we have $\|F(t)\|\leq M_1e^{\omega_1 t}$ for $M_1=Me^{\tau\omega_1}$ and all $t\geq 0$.

To show (iii), let $\Theta(t,\lambda)=\lambda(\lambda^2-A)^{-1}F(t)$ and $\Upsilon(t,\lambda)=G(t)\lambda(\lambda^2-A)^{-1},\lambda>\omega,t\geq 0$. From (1.1) and (1.2) one sees that

$$\Theta_t''(t,\lambda)=C(t)\lim_{s\to 0}2s^{-2}\lambda(\lambda^2-A)^{-1}F(s)=C(t)\Theta_t''(0,\lambda)$$

if $\Theta_t''(0,\lambda)$ exists, and

$$\Upsilon_t''(t,\lambda)=\lim_{s\to 0}2s^{-2}\lambda(\lambda^2-A)^{-1}G(s)C(t)=\Upsilon_t''(0,\lambda)C(t)$$

if $\Upsilon_t''(0,\lambda)$ exists. Hence it is enough to show $\Theta_t''(0,\lambda)=\lambda^2\hat{F}(\lambda)$ and $\Upsilon_t''(0,\lambda)=\lambda^2\hat{G}(\lambda)$.

Taking the Laplace transform of (1.1) with respect to t we have

$$(e^{\lambda s}-2+e^{-\lambda s})\hat{F}(\lambda)-e^{\lambda s}\int_0^s e^{-\lambda\tau}F(\tau)d\tau+e^{-\lambda s}\int_0^s e^{\lambda\tau}F(\tau)d\tau$$
$$=2\lambda(\lambda^2-A)^{-1}F(s)=2\Theta(s,\lambda).$$

Taking derivatives we obtain

$$2\Theta_s'(s,\lambda)=\lambda(e^{\lambda s}-e^{-\lambda s})\hat{F}(\lambda)-\lambda e^{\lambda s}\int_0^s e^{-\lambda\tau}F(\tau)d\tau-\lambda e^{-\lambda s}\int_0^s e^{\lambda\tau}F(\tau)d\tau$$

and

$$2\Theta_s''(s,\lambda)=\lambda^2(e^{\lambda s}+e^{-\lambda s})\hat{F}(\lambda)-\lambda^2e^{\lambda s}\textstyle\int_0^s e^{-\lambda\tau}F(\tau)d\tau$$
$$+\lambda^2e^{-\lambda s}\textstyle\int_0^s e^{\lambda\tau}F(\tau)d\tau-2\lambda F(s),$$

and so $\Theta_s'(0,\lambda)=0$ and $\Theta_s''(0,\lambda)=\lambda^2\hat{F}(\lambda)$. Similarly, one can show that $\Upsilon_s'(0,\lambda)=0$ and $\Upsilon_s''(0,\lambda)=\lambda^2\hat{G}(\lambda)$.

Integrating $\Theta_t''(t,\lambda)=C(t)\lambda^2\hat{F}(\lambda)$ from 0 to t twice and using the fact that $F(0)=0$ and $\Theta_t'(0,\lambda)=0$ we obtain

$$\lambda(\lambda^2-A)^{-1}F(t)x=\Theta(t,\lambda)x=\int_0^t S(s)\lambda^2\hat{F}(\lambda)xds,\ x\in X, \tag{3.1}$$

and hence (iv). Statement (v) is proved similarly.

Remark. If $C(\cdot)$ is uniformly continuous, then every C_0-cosine step response $F(\cdot)$ (resp. C_0-cosine cumulative output) for $C(\cdot)$ is also uniformly continuous. This is clear from formula (iv) (resp. (v)) of Proposition 3.1.

Definition 3.2. Let $F(\cdot)$ be a C_0-cosine step response for cosine operator function $C(\cdot)$. The *infinitesimal operator* W_s of $F(\cdot)$ is defined as $W_sx=\lim_{h\to0}\frac{2}{h^2}F(h)x$, with the natural domain. The *infinitesimal operator* A_s of the pair $(C(\cdot),F(\cdot))$ is defined as $A_sx:=\lim_{h\to0}\frac{2}{h^2}(C(h)+F(h)-I)x$, with the natural domain. The infinitesimal operator W_c of a C_0-cosine cumulative output $G(\cdot)$ and the infinitesimal operator A_c of the pair $(G(\cdot),C(\cdot))$ are defined in the same way as $W_cx=\lim_{h\to0}\frac{2}{h^2}G(h)x$ and $A_cx:=\lim_{h\to0}\frac{2}{h^2}(C(h)+G(h)-I)x$, respectively.

Theorem 3.3. *The above defined operators W_s and A_s are closed and*

(i) $W_s=\lambda(\lambda^2-A)\hat{F}(\lambda),\quad Re\lambda>\omega$;
(ii) $A_s=A(I-\lambda\hat{F}(\lambda))+\lambda^3\hat{F}(\lambda),\quad Re\lambda>\omega$;
(iii) $A_s=A(I-\frac{2}{t^2}\int_0^t\int_0^\tau F(s)dsd\tau)+\frac{2}{t^2}\Big(\lambda^2\int_0^t\int_0^\tau C(s)dsd\tau-(C(t)-I)\Big)\lambda\hat{F}(\lambda)$, $t>0$, $Re\lambda>\omega$.

Proof. Let $A_h = \frac{2}{h^2}(C(h)+F(h)-I)$. By (iv) of Proposition 3.1 one can write

$$\frac{2F(h)}{h^2}x = \frac{2}{h^2}\int_0^h S(s)\lambda^3\hat{F}(\lambda)xds - \frac{2}{h^2}(C(h)-I)\lambda\hat{F}(\lambda)x,$$

$$A_hx = 2h^{-2}\int_0^h S(s)\lambda^3\hat{F}(\lambda)xds + 2h^{-2}(C(h)-I)(I-\lambda\hat{F}(\lambda))x.$$

Since the first term on the right hand side of each equality converges to $\lambda^3\hat{F}(\lambda)x$ as $h\to 0$, we have

$$D(W_s) = D(A\hat{F}(\lambda)) \text{ and } W_sx = \lambda(\lambda^2-A)\hat{F}(\lambda)x \text{ for } x\in D(W_s),$$

and also $D(A_s) = D(A(I-\lambda\hat{F}(\lambda)))$ and

$$A_sx = \lambda^3\hat{F}(\lambda)x + A(I-\lambda\hat{F}(\lambda))x \ \text{ for } \ x\in D(A_s).$$

Since A is closed and $\hat{F}(\lambda)$ is bounded, it is easy to see that W_s and A_s are closed. This shows (i) and (ii).

To show (iii) we use (1.1) to write for all $x\in X$ and $s\geq 0$

$$\begin{aligned}&\frac{2}{h^2}(C(h)+F(h)-I)x\\ &= \frac{2}{h^2}(C(h)-I)x + \frac{2}{h^2}[F(s+h)-2C(h)F(s)+F(s-h)]x\\ &= \frac{2}{h^2}(C(h)-I)(I-F(s))x + \frac{1}{h^2}[F(s+h)-2F(s)+F(s-h)]x\\ &= \frac{2}{h^2}(C(h)-I)(I-F(s))x + \frac{2}{h^2}C(s)F(h)x.\end{aligned}$$

Now integration twice yields that for any $t>0$

$$\begin{aligned}\frac{2}{h^2}(C(h)+F(h)-I)x =& \frac{2}{h^2}(C(h)-I)\left(I-\frac{2}{t^2}\int_0^t\int_0^\tau F(s)dsd\tau\right)x\\ &+\tfrac{2}{t^2}(\lambda^2-A)\int_0^t\int_0^\tau C(s)dsd\tau(\lambda^2-A)^{-1}\frac{2}{t^2}F(h)x.\end{aligned}$$

Since the last term converges to $\frac{2}{t^2}(\lambda^2-A)\int_0^t\int_0^\tau C(s)dsd\tau\lambda\hat{F}(\lambda)x$ as $h\to 0^+$ for all $x\in X$ (Proposition 3.1 (iii)), we also have the representation of A_s as in (iii).

Remark. The definition of operator $\lim_{h\to 0+} h^{-1}F(h)$ does not make sense. Indeed, in this case using (3.1) we get it equals to 0 on the its domain.

In general, the domains of W_s and A_s are not necessarily dense. But, under suitable condition on $F(\cdot)$ (see [13]), A_s not only has dense domain, but also generates a cosine operator function $C_s(\cdot)$. In contrast, $D(W_c)$ and $D(A_c)$ always contain the dense set $D(A)$. This is shown in the next theorem.

Theorem 3.4. *The infinitesimal operators W_c and A_c have the following properties for $Re\lambda > \omega$:*

(i) $D(A) \subseteq D(W_c)$ *and* $W_c x = \lambda\hat{G}(\lambda)(\lambda^2 - A)x$ *for all* $x \in D(A)$;

(ii) $D(A) \subseteq D(A_c)$ *and for* $x \in D(A)$ $A_c x = Ax + W_c x = (I - \lambda\hat{G}(\lambda))Ax + \lambda^3\hat{G}(\lambda)x$;

(iii) $D(A) \subseteq D(A_c)$ and for $x \in D(A)$ and for $t > 0$

$$A_c x = \left(I - \frac{2}{t^2}\int_0^t\int_0^\tau G(s)dsd\tau\right)Ax$$
$$+\lambda\hat{G}(\lambda)\frac{2}{t^2}\left(\lambda^2\int_0^t\int_0^\tau C(s)dsd\tau - (C(t) - I)\right)x.$$

Moreover, if $G(t)$ is uniformly continuous in t, then A_c is closed, $D(A_c) = D(A)$, and $A_c = (I - \lambda\hat{G}(\lambda))A + \lambda^3\hat{G}(\lambda)$ for large λ. If $\hat{G}(\lambda)$ is invertible for some λ, then operator W_c is closed, $D(W_c) = D(A)$, and $W_c = A_c - A = \lambda\hat{G}(\lambda)(\lambda^2 - A)$.

Proof. Let $a_h = \frac{2}{h^2}(C(h) + G(h) - I)$. By (v) of Proposition 3.1 we have

$$\frac{2G(h)}{h^2}x = \lambda^3\hat{G}(\lambda)\frac{2}{h^2}\int_0^h S(s)xds - \lambda\hat{G}(\lambda)\frac{2}{h^2}(C(h) - I)x,$$

$$a_h x = 2\lambda^3\hat{G}(\lambda)h^{-2}\int_0^h S(s)xds + 2(I - \lambda\hat{G}(\lambda))h^{-2}(C(h) - I)x.$$

The first identity shows that $D(A) \subseteq D(W_c)$ and $W_c x = \lambda\hat{G}(\lambda)(\lambda^2 - A)x$ for $x \in D(A)$. The second identity shows that $D(A) \subseteq D(A_c)$ and $A_c x = Ax + W_c x = (I - \lambda\hat{G}(\lambda))Ax + \lambda^3\hat{G}(\lambda)x$ for $x \in D(A)$. The proof of (iii) is similar to that of assertion (iii) of Theorem 3.3.

If $\|G(t)\| \to 0$ as $t \to 0$, then $\|\lambda\hat{G}(\lambda)\| \to 0$ as $\lambda \to \infty$ (Proposition 3.5 (ii)). Thus $I - \lambda\hat{G}(\lambda)$ is invertible for large λ and we have $D(A_c) \subseteq D(A)$. If $\{x_n\}$ is a sequence in $D(A)$ such that $x_n \to x$ and $(I - \lambda\hat{G}(\lambda))Ax_n \to y$, then $Ax_n \to (I - \lambda\hat{G}(\lambda))^{-1}y$ so that $x \in D(A)$ and $Ax = (I - \lambda\hat{G}(\lambda))^{-1}y$. Hence $(I - \lambda\hat{G}(\lambda))A$ is closed. The same is A_c. The conclusion about operator W_c follows in the same way as for A_c.

It is clear from (1.1) that if $\|F(t)x\| = o(t^2)\,(t \to 0^+)$ for all $x \in X$, then $F''(t) = 0$ for all $t \geq 0$ so that $F'(\cdot) \equiv F'(0) = 0$, and then $F(\cdot) \equiv F(0) = 0$. Similarly, by (1.2), $\|G(t)x\| = o(t^2)\,(t \to 0^+)$ for all $x \in X$ implies $G(\cdot) \equiv 0$.

Hence the order of convergence at 0 of a non-trivial cosine step response or cosine cumulative output can not exceed $O(t^2)$.

Proposition 3.5. *The following statements about order of convergence hold:*

(i) *For $n = 0,1$, if $\|F(t)x\| = o(t^n)$ $(t \to 0^+)$ for all $x \in X$, then $\|\lambda^n \hat{F}(\lambda)\| = o(1)$ $(\lambda \to \infty)$ and $\|\lambda^{n+1}\hat{F}(\lambda)x\| = o(1)$ $(\lambda \to \infty)$ for all $x \in X$;*

(ii) *For $n = 0,1$, if $\|G(t)x\| = o(t^n)$ $(t \to 0^+)$ for all $x \in X$, then $\|\lambda^n \hat{G}(\lambda)\| = o(1)$ $(\lambda \to \infty)$, $\|\lambda^{n+1}\hat{G}(\lambda)x\| = o(1)$ $(\lambda \to \infty)$ for all $x \in X$, and $\|\lambda^3 \hat{G}(\lambda)x - (A_c - A)x\| = o(\lambda^{-n})$ for all $x \in D(A)$;*

(iii) *For $n = 0,1$, if $\|F(t)\| = o(t^n)$ (resp. $\|G(t)\| = o(t^n)$) $(t \to 0^+)$, then $\|\lambda^{n+1}\hat{F}(\lambda)\| = o(1)$ (resp. $\|\lambda^{n+1}\hat{G}(\lambda)\| = o(1)$) $(\lambda \to \infty)$;*

(iv) *For $n = 1,2$, if $\|F(t)\| = O(t^n)$ $(t \to 0^+)$, then $\|\lambda^{n+1}\hat{F}(\lambda)\| = O(1)$ $(\lambda \to \infty)$;*

(v) *For $n = 1,2$, if $\|G(t)\| = O(t^n)$ $(t \to 0^+)$, then $\|\lambda^{n+1}\hat{G}(\lambda)\| = O(1))$ $(\lambda \to \infty)$, and $\|\lambda^3 \hat{G}(\lambda)x - (A_c - A)x\| = O(\lambda^{-n})$ for all $x \in D(A)$.*

Proof. We only show (ii); the proofs of (i), (iii), (iv) and (v) are similar. For a given $\epsilon > 0$ let $\delta > 0$ be chosen so that $\|G(t)x\| \le \epsilon t^n$ for all $t \in [0,\delta]$. Then we have

$$\begin{aligned}
\|\lambda^{n+1}\hat{G}(\lambda)x\| &\le \lambda^{n+1}\Big(\int_0^\delta + \int_\delta^\infty\Big) e^{-\lambda t}\|G(t)x\| dt \\
&\le \epsilon\lambda^{n+1}\int_0^\infty e^{-\lambda t}t^n\, dt + \lambda^{n+1}\int_\delta^\infty e^{-\lambda t} M e^{wt} dt\|x\| \\
&\le \epsilon/n! + M\tfrac{\lambda^{n+1}}{\lambda - w} e^{-(\lambda-w)\delta}\|x\|.
\end{aligned}$$

This shows that $\|\lambda^{n+1}\hat{G}(\lambda)x\| = o(1)(\lambda \to \infty)$ for all $x \in X$. By the uniform boundedness principle we have $\|\lambda^n \hat{G}(\lambda)\| = o(1)(\lambda \to \infty)$. By (ii) of Theorem 3.4 we have $\|\lambda^3\hat{G}(\lambda)x - (A_c - A)x\| = o(\lambda^{-n})$ for all $x \in D(A)$.

Remark. The behavior of $\lambda^3 \hat{F}^*(\lambda)$ as $\lambda \to \infty$ is given in [13].

4. Compactness

A C_0-cosine step response $F(\cdot)$ (resp. C_0-cosine cumulative output $G(\cdot)$) is said to be *compact* if the operator $F(t)$ (resp. $G(t)$) is compact for every $t \ge 0$. In this section we discuss conditions for compactness of $F(\cdot)$ and $G(\cdot)$ and consequences.

Proposition 4.1. *If the sine operator function $S(\cdot)$ is compact and if the C_0-cosine step response $F(\cdot)$ is norm continuous at zero, then $F(\cdot)$ is compact. The same is true for C_0-cosine cumulative output.*

Proof. We integrate (1.1) with respect to t from 0 to τ

$$\int_{\tau}^{\tau+h} F(\eta)d\eta - \int_{\tau-h}^{\tau} F(\eta)d\eta = 2S(\tau)F(h). \tag{4.1}$$

The compactness of $S(\cdot)$ implies that the left hand side of (4.1) is compact for every $\tau, h \geq 0$. Because $S(\cdot)$ and $F(\cdot)$ are norm continuous, we can take in (4.1) the derivative with respect to τ without lost of compactness property, so the left hand side of (1.1) is a compact operator. Using condition $F(0) = 0$ and the uniform continuity of $F(\cdot)$ we get from (1.1) that $F(h)$ is compact for every $h \geq 0$.

Proposition 4.2. *If the C_0-cosine step response $F(\cdot)$ is compact, then $F(\cdot)$ is norm continuous on $[0, \infty)$.*

Proof. Since $F(\cdot)$ is compact, its Laplace transform $\hat{F}(\cdot)$ is also compact (see [17]). Then by the formula (iv) of Proposition 3.1 we are done because the strong convergence becomes uniform convergence after multiplication by a compact operator from the right.

Proposition 4.3. *If there is a cosine operator-function $C(\cdot)$ on a Banach space X such that each C_0-cosine step response $F(\cdot)$ (or C_0-cosine cumulative output $G(\cdot)$) for $C(\cdot)$ is compact, then X must be finite dimensional.*

Proof. By assumption, the two particular C_0-cosine step responses

$$F_1(t) = C(t) - I \text{ and } F_2(t) = \int_0^t S(s)ds$$

are compact. Then as we know (see [10]), $C(t) - I$ is compact for all $t > 0$ if and only if the generator A is compact. Hence $C(\cdot)$ is norm continuous on $[0, \infty)$. Thus the operator $C(0) = I$, being the limit in norm of the compact operators $2t^{-2}F_2(t)$ as $t \to 0$, is compact. This can happen only when X is finite dimensional.

5. Positivity

In case that X is a Banach lattice with positive cone X_+, we say a function $\mathcal{L}(\cdot)$ on X is *positive* if for every $t \geq 0$ the operator $\mathcal{L}(t)$ is positive (in notation,

$\mathcal{L}(t) \succeq 0$) in the sense that $\mathcal{L}(t)X_+ \subseteq X_+$. In case that X is a Hilbert space with inner product $(\cdot,\cdot)$, we say $\mathcal{L}(\cdot)$ is positive (in notation, $\mathcal{L}(t) \geq 0$) if for every $t \geq 0$ $\mathcal{L}(t)$ is positive in the sense that $(\mathcal{L}(t)x, x) \geq 0$ for all $x \in X$.

It is known [8] that a cosine operator function $C(\cdot)$ dominates I, i.e. $C(\cdot) - I$ is positive, either in the sense of Banach lattice or in the sense of Hilbert space, if and only if its generator A is bounded and positive.

The following propositions are just reformulations of this property in terms of C_0-cosine step response and C_0-cosine cumulative output. Let $F_B^\mu(\cdot)$ and $G_B^\mu(\cdot)$ be the functions defined by

$$F_B^\mu(t)x := (A-\mu)\int_0^t S(s)Bx\,ds,\ x \in X,\ t \geq 0, \tag{5.1}$$

$$G_B^\mu(t)x := B(A-\mu)\int_0^t S(s)x\,ds,\ x \in X,\ t \geq 0.$$

Then $F_B^\mu(\cdot)$ is a C_0-cosine step response and $G_B^\mu(\cdot)$ is a C_0-cosine cumulative output.

Proposition 5.1. *Let X be a Banach lattice. Each C_0-cosine step esponse $F_B^\mu(\cdot)$ for a cosine operator function $C(\cdot)$ on X defined by* (5.1) *with $\mu \leq 0$ and $B \succeq 0$ is positive if and only if the generator A of $C(\cdot)$ is positive. The same is true for C_0-cosine cumulative output.*

Proposition 5.2. *Let X be a Hilbert space. Each C_0-cosine step response $F_B^\mu(\cdot)$ for a cosine operator function $C(\cdot)$ on X defined by* (5.1) *with $\mu \leq 0$ and $B \geq 0$ and commuting with $C(\cdot)$ is positive if and only if the generator A of $C(\cdot)$ is positive. The same is true for C_0-cosine cumulative output.*

6. Almost Periodicity

A function $f(\cdot) : [0,\infty) \to X$ is said to be *almost periodic* if for every $\epsilon > 0$ the set $J(f,\epsilon) = \{\tau; \|f(t+\tau) - f(t)\| \leq \epsilon$ for all $t \geq 0\}$ is relatively dense in $[0,\infty)$, i.e., there exists an $l > 0$ such that every subinterval of $[0,\infty)$ of length l meets $J(f,\epsilon)$. An operator-valued function $Q(\cdot) : [0,\infty) \to B(X)$ is said to be *almost periodic* if for each $x \in X$ the function $Q(\cdot)x$ is almost periodic.

Lemma 6.1. *If a continuous function $f(\cdot) : [0,\infty) \to X$ converges to some $\varphi \in X$ as $t \to \infty$, then*

$$2t^{-2}\int_0^t sf(s)\,ds \ \to \varphi \text{ as } t \to \infty. \tag{6.1}$$

Proof. Clearly, it is enough to consider the case $\varphi = 0$. Put $t = \tau + \zeta$ and write

$$\frac{2}{t^2}\int_0^t sf(s)\ ds = \frac{2}{(\tau+\zeta)^2}\int_0^{\tau} sf(s)\ ds + \frac{2}{(\tau+\zeta)^2}\int_{\tau}^{\tau+\zeta} sf(s)\ ds.$$

Since $\|\frac{2}{(\tau+\zeta)^2}\int_{\tau}^{\tau+\zeta} sf(s)\ ds\| \le \sup_{t\ge\tau}\ \|f(t)\|$ for all ζ and τ and $f(t) \to 0$ as $t \to \infty$, we can choose τ so large that the second term becomes smaller than some given $\epsilon > 0$. Then we can take large ζ such that the first term is also smaller than ϵ. This shows (6.1) with $\varphi = 0$.

The next theorem gives a necessary and sufficient condition for every C_0-cosine step response (or every C_0-cosine cumulative output) to be almost periodic.

Theorem 6.2. *Every C_0-cosine step response $F(\cdot)$ for $C(\cdot)$ is almost periodic if and only if $C(\cdot)$ is almost periodic and $0 \in \rho(A)$. The same assertion is true for C_0-cosine cumulative outputs.*

Proof. Let $C(\cdot)$ be almost periodic. Then the condition $0 \in \rho(A)$ implies that $\int_0^t S(s)ds = \int_0^t S(s)AA^{-1}ds = (C(t) - I)A^{-1}$ is also almost periodic, so that from Proposition 3.1(iv) we see the almost periodicity of $F(\cdot)$.

Conversely, if each C_0-cosine step response is almost periodic, then the two particular C_0-cosine step responses $C(t) - I$ and $\int_0^t S(s)ds$ are almost periodic functions. If $x \in N(A)$, then $x = C(s)x - \int_0^s S(u)Axdu = C(s)x$ for all $s \ge 0$ and $x = 2t^{-2}\int_0^t S(s)xds \to 0$ as $t \to \infty$, because an almost periodic function is bounded. Hence A is injective. Next, since an almost periodic function is mean ergodic (see e.g. [1, p. 21]), the limit of $\frac{1}{s}\int_0^s\int_0^u S(v)x\ dv\ du$ exists as $s \to \infty$ for every $x \in X$. By Lemma 6.1, the limit $\frac{2}{t^2}\int_0^t s\frac{1}{s}\int_0^s\int_0^u\int_0^v C(\tau)x\ d\tau\ dv\ du\ ds$ exists as $t \to \infty$ for any $x \in X$. Since $C(\cdot)$ is uniformly bounded, it follows from Proposition 7.6 (ii) (i.e. [14, Theorem 3.7]) that $R(A) = X$. Therefore we have $0 \in \rho(A)$.

Remark. The assumption that $C(\cdot)$ is almost periodic and $0 \in \rho(A)$ is equivalent to the condition that (see [9]) each generalized solution of (CP) is almost periodic.

From the above theorem one can deduce the next theorem.

Theorem 6.3. *Each C_0-cosine step response $F(\cdot)$ for $C(\cdot)$ is periodic if and only if $C(\cdot)$ is periodic and $0 \in \rho(A)$. In this case, $F(\cdot)$ and $C(\cdot)$ have the same period. The same assertion is true for C_0-cosine cumulative outputs.*

7. Asymptotic Behavior

This section is concerned with asymptotic behavior of a C_0-cosine step response $F(t)$ and a C_0-cosine cumulative output $G(t)$ as $t \to \infty$. First we consider the problem under the assumption that there exist a real number λ_0 and a nonzero bounded operator $P \in B(X)$ such that

$$\lim_{t\to\infty} 2e^{-\lambda_0 t}\, C(t)x = Px \text{ for all } x \in X. \tag{7.1}$$

Clearly in this case there is a constant $M_1 \geq 1$ such that

$$\|C(t)\| \leq M_1 e^{\lambda_0 t} \text{ for } t \geq 0. \tag{7.2}$$

From the identity

$$2e^{-\lambda_0 2t}(C(2t)+I) = 2e^{-\lambda_0 t}C(t)2e^{-\lambda_0 t}C(t) \tag{7.3}$$

it is seen that in case (7.1) λ_0 cannot be negative. In case $\lambda_0 = 0$, we have from (7.3) $P + 2I = P^2$. On the other hand, by letting $t \to \infty$ and then $s \to \infty$ in the equation (b), one has $2P = P^2$. Hence we have $C(t) \to P/2 = I$ as $t \to \infty$. Letting $t \to \infty$ in (b) then leads to $C(s) = I$ for all $s \in R$. Then it follows from (iv) of Proposition 3.1 that $F(t)x = 2^{-1}t^2\lambda^3\hat{F}(\lambda)x$ (for any $\lambda > \omega$), which does not converge as $t \to \infty$ unless $x \in N(\hat{F}(\lambda))$. The same situation is for $G(\cdot)$. Thus the case $\lambda_0 = 0$ is not interesting, and we shall assume $\lambda_0 > 0$ from now on. From (7.3) it is clear that the operator P is now a projection.

It is known [5, p. 92] that the generator A of the cosine operator function $C(\cdot)$ generates also a C_0-semigroup $T(\cdot)$ defined by

$$T(t)x = \frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-\frac{s^2}{4t}}C(s)x\,ds,\ x \in X.$$

As will be seen in the next theorem, the convergence of $2e^{-\lambda_0 t}\, C(t)$ to P as $t \to \infty$ implies the convergence of $e^{-\lambda_0^2 t}T(t)$ to P in the same topology. When P has finite rank and $\|e^{-\lambda_0^2}T(t) - P\|$ as $t \to \infty$, Webb [18] says that the C_0-semigroup $T(\cdot)$ has asynchronous exponential growth with intrinsic growth constant λ_0^2. In this case, one has that $\lambda_0^2 > \omega_{ess}(A)$, where $\omega_{ess}(A)$ is the essential type or essential growth bound, the peripheral spectrum $\sigma_0(A) = \{\lambda_0^2\}$ and λ_0^2 is a simple pole of the resolvent $(\lambda - A)^{-1}$ (see [18]).

Theorem 7.1. *Suppose that* (7.1) *holds with* $\lambda_0 > 0$. *Then P is a projection with range* $R(P) = N(\lambda_0^2 - A)$ *and null space* $N(P) = \overline{R(\lambda_0^2 - A)}$.

If, in addition, P has finite rank and $\|2e^{-\lambda_0 t}C(t) - P\| \to 0$ as $t \to \infty$, then $\lambda_0^2 > \omega_{ess}(A)$, $\sigma_0(A) = \{\lambda_0^2\}$ and λ_0^2 is a simple pole of the resolvent $(\lambda - A)^{-1}$.

Proof. We shall prove that the semigroup $e^{-\lambda_0^2 t}T(t)$ converges to P strongly as $t \to \infty$. Then it follows from the mean ergodic theorem for semigroup that P is a projection with $R(P) = N(\lambda_0^2 - A)$ and $N(P) = \overline{R(\lambda_0^2 - A)}$.

We write

$$e^{-\lambda_0^2 t}T(t)x = \frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} e^{\lambda_0 s}[2e^{-\lambda_0 s}C(s) - P]x ds + \frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} e^{\lambda_0 s} ds Px.$$

It suffices to show that the first term $Q_1(t)$ on the right hand side converges to zero and the second term $Q_2(t)$ converges to Px as $t \to \infty$. We first see that

$$\frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} e^{\lambda_0 s} ds = \frac{1}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{(s-2\lambda_0 t)^2}{4t}} ds = \frac{1}{\sqrt{\pi}} \int_{-\lambda_0\sqrt{t}}^\infty e^{-u^2} du$$

converges to $\frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-u^2} du = 1$ as $t \to \infty$. Therefore $Q_2(t)$ converges to Px as $t \to \infty$. Let $\epsilon > 0$ be arbitrarily small and take $\tau > 0$ so large that $\|2e^{-\lambda_0 s}C(s)x - Px\| \le \epsilon$ for all $s \ge \tau$. Then

$$Q_1(t) \le \frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}} \int_0^\tau e^{-\frac{s^2}{4t}} e^{\lambda_0 s} \|(2e^{-\lambda_0 s}C(s) - P)x\| ds + \epsilon \frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} e^{\lambda_0 s} ds,$$

and hence is bounded by 2ϵ as $t \to \infty$. This shows that $Q_1(t) \to 0$ as $t \to \infty$.

When $\|2e^{-\lambda_0 t}C(t) - P\| \to 0$ as $t \to \infty$, a similar argument as above shows that $\|e^{-\lambda_0^2 t}T(t) - P\| \to 0$ as $t \to \infty$. When P has finite rank, the semigroup $T(\cdot)$ has asynchronous exponential growth with intrinsic growth constant λ_0^2. It follows from Theorem of [18] that $\lambda_0^2 > \omega_{ess}(A)$, $\sigma_0(A) = \{\lambda_0^2\}$, and λ_0^2 is a simple pole of resolvent $(\lambda - A)^{-1}$.

We shall need the following

Lemma 7.2. *If a strongly continuous function $f(\cdot) : [0, \infty) \to X$ is such that $lim_{t\to\infty} f(t) = \varphi$, $\varphi \in X$, then for any λ with $Re\lambda > 0$ we have*

$$e^{-\lambda t} \int_0^t e^{\lambda s} f(s)\, ds \to \varphi/\lambda \quad \text{as} \quad t \to \infty. \tag{7.4}$$

Proof. To prove (7.4) it is enough to consider the case $\varphi = 0$, because of

$$\lambda e^{-\lambda t}\int_0^t e^{\lambda s}(f(s)-\varphi)ds + \varphi - e^{-\lambda t}\ \varphi = \lambda e^{-\lambda t}\int_0^t e^{\lambda s} f(s)ds.$$

Now we put $t = \tau + \zeta$ and write

$$\begin{aligned} e^{-\lambda t}\int_0^t e^{\lambda s} f(s)ds &= e^{-\lambda(\tau+\zeta)}\int_\tau^{\tau+\zeta} e^{\lambda s} f(s)ds + e^{-\lambda(\tau+\zeta)}\int_0^\tau e^{\lambda s} f(s)ds \\ &= e^{-\lambda(\tau+\zeta)}\int_0^\zeta e^{\lambda\tau}e^{\lambda\eta} f(\tau+\eta)d\eta + e^{-\lambda\zeta}e^{-\lambda\tau}\int_0^\tau e^{\lambda s} f(s)ds. \end{aligned}$$

Since $\|e^{-\lambda\zeta}\int_0^\zeta e^{\lambda\eta} f(\tau+\eta)\ d\eta\|$ is less than $\lambda^{-1}(1-e^{-\lambda\zeta})\sup_{t\geq\tau}\|f(t)\|$ for all ζ and τ, we can choose τ so large that the first term becomes smaller than ϵ. Then we can take large ζ such that the second term is also smaller than ϵ. This shows (7.4) with $\varphi = 0$.

Proposition 7.3. *Suppose the cosine operator function $C(\cdot)$ satisfies* (7.1) *with $\lambda_0 > 0$. Then $2e^{-\lambda_0 t}S(t) \to P/\lambda_0$ and $2e^{-\lambda_0 t}\int_0^t S(s)ds \to P/\lambda_0^2$ strongly as $t \to \infty$.*

Proof. We can write

$$2e^{-\lambda_0 t}S(t) = e^{-\lambda_0 t}\int_0^t e^{\lambda_0 s} 2e^{-\lambda_0 s}C(s)ds$$

and

$$2e^{-\lambda_0 t}\int_0^t S(s)ds = e^{-\lambda_0 t}\int_0^t e^{\lambda_0 s}e^{-\lambda_0 s}\int_0^s e^{\lambda_0\eta}2e^{-\lambda_0\eta}C(\eta)\ d\eta\ ds.$$

The conclusion now follows by applying Lemma 7.2.

Theorem 7.4. *Suppose the cosine operator function $C(\cdot)$ satisfies* (7.1) *with $\lambda_0 > 0$. Then for every $\lambda > \lambda_0$ and $x \in X$ we have*

$$\lim_{t\to\infty} 2e^{-\lambda_0 t}F(t)x = \lambda(\lambda^2/\lambda_0^2 - 1)P\hat{F}(\lambda)x,$$

which is equal to $\frac{1}{\lambda_0^2}(A_s - A)x$ if $x \in D(A)$ and $\hat{F}(\lambda)x \in N(\lambda_0^2 - A)$ simultaneously, and is equal to zero if $\hat{F}(\lambda)x \in \overline{R(\lambda_0^2 - A)}$.

Proof. For $\lambda_0 > 0$ we write

$$2e^{-\lambda_0 t}F(t)x = 2e^{-\lambda_0 t}\int_0^t S(s)\lambda^3\hat{F}(\lambda)xds - 2e^{-\lambda_0 t}(C(t)-I)\lambda\hat{F}(\lambda)x.$$

Then applying Proposition 7.3, Theorem 3.3 and taking $\lambda \to \infty$ we obtain the asserted limit.

A similar argument using Proposition 7.3 and Theorem 3.4 shows the next theorem.

Theorem 7.5. *Suppose the cosine operator function* $C(\cdot)$ *satisfies* (7.1) *with* $\lambda_0 > 0$. *Then for every* $\lambda > \lambda_0$ *and* $x \in X$ *we have the limit*

$$\lim_{t\to\infty} 2e^{-\lambda_0 t}G(t)x = \hat{G}(\lambda)\lambda(\lambda^2/\lambda_0^2 - 1)Px,$$

which is equal to $\frac{1}{\lambda_0^2}(A_c - A)x$ *if* $x \in N(\lambda_0^2 - A)$, *and is equal to zero if* $x \in \overline{R(\lambda_0^2 - A)}$.

As mentioned previously, if $C(t)$ converges strongly as $t \to \infty$, then $C(\cdot) \equiv I$, and both $F(t)$ and $G(t)$ diverge. In the rest of this section we shall consider the behavior of $F(\cdot)$ and $G(\cdot)$ under the assumption:

$$\text{(7.5)}\quad \sup_{t>0} \|t^{-2}\int_0^t\int_0^s C(u)\,du\,ds\| < \infty \text{ and } t^{-2}C(t) \to 0 \text{ strongly as } t \to \infty.$$

We need the following Proposition (see [14, Theorems 3.5 and 3.7]).

Proposition 7.6. *Under the assumption* (7.5) *we have:*

(i) *The mapping* $P : x \to \lim_{t\to\infty} 2t^{-2}\int_0^t\int_0^s C(u)x\,du\,ds$ *is a linear projection with* $R(P) = N(A)$, $N(P) = \overline{R(A)}$ *and* $D(P) = N(A) \oplus \overline{R(A)}$;

(ii) $x := -\lim_{t\to\infty} 2t^{-2}\int_0^t\int_0^s\int_0^u\int_0^v C(\tau)y\ d\tau\ dv\ du\ ds$ *exists if and only if* $y \in A(D(A) \cap \overline{R(A)})$ (= $R(A)$ *in case* $C(\cdot)$ *is (C,2) mean ergodic, i.e.* $D(P) = X$). *Moreover, this element* x *is the unique solution of the equation* $Ax = y$ *in* $\overline{R(A)}$, *i.e.* $x = \tilde{A}^{-1}y$ *where* $\tilde{A} = A|_{\overline{R(A)}}$.

Using Proposition 3.1 (iv) and the above proposition we obtain

Theorem 7.7. *Under the assumption* (7.5) *the following assertions hold:*

(i) *The limit* $y = \lim_{t\to\infty} 2t^{-2}F(t)x$ *exists if and only if* $\hat{F}(\lambda)x \in N(A) \oplus \overline{R(A)}$ *for some (and all)* $\lambda > \omega$. *When the limit exists,* $y = \lambda^3 P\hat{F}(\lambda)x$, *which is independent of* λ;

(ii) *When* $\hat{F}(\lambda)x \in N(A) \oplus \overline{R(A)}$, $z = \lim_{t\to\infty} 2t^{-2}\int_0^t\int_0^s F(\tau)x\ d\tau ds$ *exists if and only if* $\hat{F}(\lambda)x \in A(D(A) \cap R(A))$ *for some (and all)* $\lambda > \omega$. *In this case,* $z = -\lambda(\lambda^2 - A)\tilde{A}^{-1}\hat{F}(\lambda)x$, *which is independent of* λ.

Proof. By (iv) of Proposition 3.1, we have

$$2t^{-2}F(t)x = 2t^{-2}(I - C(t))\lambda\hat{F}(\lambda)x + 2t^{-2}\int_0^t\int_0^s C(\tau)\lambda^3\hat{F}(\lambda)xd\tau ds$$

and

$$2t^{-2}\int_0^t\int_0^s F(\tau)xd\tau ds = 2t^{-2}\int_0^t\int_0^s\int_0^u\int_0^v C(\tau)\lambda^3\hat{F}(\lambda)xd\tau dvduds$$
$$-2t^{-2}\int_0^t\int_0^s C(\tau)\lambda\hat{F}(\lambda)xd\tau ds + \lambda\hat{F}(\lambda)x.$$

Then, as consequences of Proposition 7.6, assertions (i) and (ii) follow from (7.6) and (7.7), respectively.

Similarly, using Proposition 3.1 (v), Proposition 7.6, and Proposition 3.5 (ii) we have the next theorem.

Theorem 7.8. *Under the assumption* (7.5) *the following assertions hold:*
(i) *If* $x \in N(A) \oplus \overline{R(A)}$, *then* $\lim_{t\to\infty} 2t^{-2}G(t)x = A_cPx$;
(ii) *If* $x \in A(D(A) \cap R(A))$, *then* $\lim_{t\to\infty} 2t^{-2}\int_0^t\int_0^s G(\tau)x\ d\tau ds = -(A_c - A)\tilde{A}^{-1}x = x - A_c\tilde{A}^{-1}x$.

References

1. L. Amerio and G. Prouse, Almost-Periodic Functions and Functional Equations, Van Nostrand Reinhold Company, New York, 1971.
2. R. Chander and A. B. Buche, On some analytic properties of the modified sine operator, *Indian J. Pure Appl. Math.* **12** (1981), 585-593.
3. R. Chander and H. Singh, On the measurability and continuity properties of the modified exponential cosine operator, *J. Indian Math. Soc.* **44** (1980), 365-368.
4. R. Chander and H. Singh, On the measurability and continuity properties of the cosine operator, *Indian J. Pure Appl. Math.* **12** (1981), 81-83.
5. H. O. Fattorini, Ordinary differential equations in linear topological spaces, I., *J. Differential Eq.* **5** (1968), 72-105.
6. E. Hille and R. S. Phillips, Functional Analysis and Semi-groups, Amer. Math. Soc. Coll. Publ. Vol. 31, Providence, R. I., 1957.
7. S. Kurepa, Semigroups and cosine functions, *Funct. Anal. Lect. Notes Math.* **948** (1982), 47-72.
8. C. Lizama, On positive strongly continuous cosine functions, *Math. Balkanica* **4** (1990), 43-49.
9. S. Piskarev, Periodic and almost periodic cosine operator functions, *Math. USSR Sbornik* **46** (1983), 391-402.
10. S. Piskarev, Properties of compactness in the theory of cosine operator functions, *Mat. Zametki* (in Russian) **51** (1992), 151-153.

11. S. Piskarev and S.-Y. Shaw, On some properties of step responses and cumulative outputs, *Chinese J. Math.* **22** (1994), 321-336.

12. S. Piskarev and S.-Y. Shaw, Multiplicative perturbations of semigroups and applications to step responses and cumulative outputs, *J. Funct. Anal.* **128** (1995), 315-340.

13. S. Piskarev and S.-Y. Shaw, Perturbation and comparison of cosine operator functions, *Semigroup Forum* **51** (1995), 225-246.

14. S.-Y. Shaw, Mean ergodic theorems and linear functional equations, *J. Funct. Anal.* **87** (1989), 428-441.

15. S.-Y. Shaw and S. Piskarev, Asymtotic behavior of some semigroup-related functions, *Saitama Math. J.* **12** (1994), 7-12.

16. M. Sova, Cosine operator functions, *Rozprawy Mat.* **49** (1966), 1-47.

17. J. Voigt, On convex compactness property for the strong operator topology, preprint, 1990.

18. G. F. Webb, An operator-theoretic formulation of asynchronous exponential growth, *Trans. Amer. Math. Soc.* **303** (1987), 751-763.

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