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ANALYTIC SOLUTIONS OF A FUNCTIONAL DIFFERENTIAL EQUATION WITH STATE DEPENDENT ARGUMENT*

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Abstract. This paper is concerned with a functional differential equation $x'(z) = x(az + bx(z))$, where $a \neq 1$ and $b \neq 0$. By constructing a convergent power series solution $y(z)$ of a companion equation of the form $\beta y'(\beta z) = y'(z)[y(\beta^2 z) - ay(\beta z) + a]$, analytic solutions of the form $(y(\beta y^{-1}(z)) - az)/b$ for the original differential equation are obtained.

Functional differential equations of the form

$$x'(t) = x(t - \sigma(t))$$

have been studied to some extent by many authors. However, when the function $\sigma(t)$ is state dependent, say, $\sigma(t) = (1-a)t - bx(t)$, relatively little is known. Indeed, to the best of our knowledge, there are only a few reports (see [1, 3 - 10]) on functional differential equations with state dependent arguments. In this note, we will be concerned with a class of functional differential equation of the form

$$x'(z) = x(az + bx(z)). \tag{1}$$

When $a = 0$ and $b = 1$, equation (1) reduces to the iterative functional differential equation $x'(z) = x(x(z))$ which has been investigated by Eder [1] and analytic solutions are shown to exist by means of the Banach fixed point theorem. When $b = 0$ and $|a| \leq 1$, equation (1) reduces to the functional differential equation

$$x'(z) = x(az),$$

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which has an entire solution of the form (see Elbert [3])

$$x(z) = \sum_{n=0}^{\infty} \frac{a^{(n(n-1)/2)}}{n!} \eta z^n.$$

Indeed, if we seek a power series solution of the form

$$x(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then substituting it into the above equation leads to

$$(n+1)b_{n+1} = a^n b_n, \ n = 0, 1, 2, \ldots.$$

Taking $b_0 = \eta$, we see that

$$b_n = \frac{a^{(n(n-1)/2)}}{n!} \eta$$

and that

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{s\to\infty} \frac{a^n}{n+1} = 0,$$

as required.

When $a \neq 1$ and $b \neq 0$, by means of the reasoning just used and several more involved ideas, we will be able to construct analytic solutions for our equations in a neighborhood of the complex number $(\beta - a)/(1-a)$, where β satisfies either one of the following conditions:

(H1) $0 < |\beta| < 1$; or

(H2) $|\beta| = 1, \beta$ is not a root of unity, and

$$\log \frac{1}{|\beta^n - 1|} \leq \mu \log n, n = 2, 3, \ldots$$

for some positive constant μ.

The technique for obtaining such solutions is as follows. We first seek a formal power series solution for the following initial value problem

$$y'(\beta z) = \frac{1}{\beta} y'(z)\{y(\beta^2 z) - ay(\beta z) + a\}, \tag{2}$$

$$y(0) = \frac{\beta - a}{1 - a}. \tag{3}$$

Then we show that such a power series solution is majorized by a convergent power series. Then we show that

$$x(z) = \frac{1}{b} y(\beta y^{-1}(z)) - \frac{a}{b} z \tag{4}$$

is an analytic solution of (1) in a neighborhood of $(\beta - a)/(1 - a)$. Finally, we make use of a partial difference equation to show how to explicitly construct such a solution.

We begin with the following preparatory lemma, the proof of which can be found in [2, Chapter 6].

Lemma 1. *Assume that* (H2) *holds. Then there is a positive number* δ *such that* $|\beta^n - 1|^{-1} < (2n)^\delta$ *for* $n = 1, 2, \ldots$ *Furthermore, the sequence* $\{d_n\}_{n=1}^\infty$ *defined by* $d_1 = 1$ *and*

$$d_n = \frac{1}{|\beta^{n-1} - 1|} \max_{\substack{n=n_1+\cdots+n_t,\\ 0<n_1\le\cdots\le n_t, t\ge 2}} \{d_{n_1} \ldots, d_{n_t}\},\ n = 2, 3, \ldots,$$

will satisfy

$$d_n \le (2^{5\delta+1})^{n-1} n^{-2\delta},\ n = 1, 2, \ldots.$$

Lemma 2. *Suppose* (H1) *holds. Then for any nontrivial complex number* η*, equation* (2) *has an analytic solution of the form*

$$y(z) = \frac{\beta - a}{1 - a} + \eta z + \sum_{n=2}^\infty b_n z^n \tag{5}$$

in a neighborhood of the origin, and there exists a positive constant M *such that for* z *in this neighborhood,*

$$|y(z)| \le \left| \frac{\beta - a}{1 - a} \right| + \frac{1}{2M}.$$

Proof. We seek a solution of (2) in a power series of the form (5). By defining $b_0 = (\beta - a)/(1 - a)$ and $b_1 = \eta$ and then substituting (5) into (2), we see that the sequence $\{b_n\}_{n=2}^\infty$ is successively determined by the condition

$$\begin{aligned} &(\beta^{n+1} - \beta)\ (n+1) b_{n+1} \\ &= \sum_{k=0}^{n-1} (k+1)\ (\beta^{2(n-k)} - a\beta^{n-k}) b_{k+1} b_{n-k},\ n = 1, 2, \ldots \end{aligned} \tag{6}$$

in a unique manner. Furthermore, since $0 \le k \le n-1$,

$$\left|\frac{\beta^{2(n-k)} - a\beta^{n-k}}{\beta^{n+1} - \beta}\right| \le \frac{1+|a|}{|\beta^n - 1|} \le M, \ n \ge 2 \tag{7}$$

for some positive number M, thus if we define a sequence $\{B_n\}_{n=1}^{\infty}$ by $B_1 = |\eta|$ and

$$B_{n+1} = M \sum_{k=0}^{n-1} B_{k+1} B_{n-k}, \ n = 1, 2, \ldots,$$

then $|b_n| \le B_n$ for $n = 1, 2, \ldots$. Now if we define

$$G(z) = \sum_{n=1}^{\infty} B_n z^n,$$

then

$$\begin{aligned} G^2(z) &= \sum_{n=2}^{\infty} (B_1 B_{n-1} + B_2 B_{n-2} + \cdots + B_{n-1} B_1) z^n \\ &= \sum_{n=1}^{\infty} (B_1 B_n + B_2 B_{n-1} + \cdots + B_n B_1) z^{n+1} \\ &= \frac{1}{M} \sum_{n-1}^{\infty} B_{n+1} z^{n+1} = \frac{1}{M} G(z) - \frac{1}{M} |\eta| z. \end{aligned}$$

Hence

$$G(z) = \frac{1}{2M} \left\{ 1 \pm \sqrt{1 - 4M|\eta| z} \right\}.$$

But since $G(0) = 0$, only the negative sign of the square root is possible, so that

$$G(z) = \frac{1}{2M} \left\{ 1 - \sqrt{1 - 4M|\eta| z} \right\}.$$

It follows that the power series $G(z)$ converges for $|z| \le 1/(4M|\eta|)$, which implies that (5) is also convergent for $|z| \le 1/(4M|\eta|)$.

Next, note that for $|z| \le 1/(4M|\eta|)$,

$$\frac{1}{G(|z|)} = \frac{2M}{1 - \sqrt{1 - 4M|\eta|\ |z|}} = \frac{1 + \sqrt{1 - 4M|\eta|\ |z|}}{2|\eta|\ |z|} \ge \frac{1}{2|\eta|\ |z|},$$

or

$$G(|z|) \le 2|\eta|\ |z| \le 2|\eta| \frac{1}{4M|\eta|} = \frac{1}{2M}.$$

Thus

$$|y(z)| \le \left|\frac{\beta - a}{1 - a}\right| + \sum_{n=1}^{\infty} |b_n|\ |z|^n \le \left|\frac{\beta - a}{1 - a}\right| + \sum_{n=1}^{\infty} B_n |z|^n$$
$$= \left|\frac{\beta - a}{1 - a}\right| + G(|z|) \le \left|\frac{\beta - a}{1 - a}\right| + \frac{1}{2M}$$

as required. The proof is complete.

Next, we consider the case when (H2) holds.

Lemma 3. *Suppose* (H2) *holds. Then equation* (2) *has an analytic solution of the form*

$$y(z) = \frac{\beta - a}{1 - a} + z + \sum_{n=2}^{\infty} b_n z^n \tag{8}$$

in a neighborhood of the origin, and there exists a positive constant δ *such that*

$$|y(z)| \le \left|\frac{\beta - a}{1 - a}\right| + \frac{1}{2^{5\delta+1}} \sum_{n=1}^{\infty} \frac{1}{n^{2\delta}}.$$

Proof. As in the proof of Lemma 1, we seek a power series solution of the form (8). Then defining $b_0 = (\beta - a)/(1 - a)$ and $b_1 = 1$, (6) and (7) again hold so that

$$\begin{aligned} |b_{n+1}| &\le \frac{1 + |a|}{|\beta^n - 1|} \sum_{k=0}^{n-1} |b_{k+1}|\ |b_{n-k}| \\ &= \frac{1 + |a|}{|\beta^n - 1|} \sum_{\substack{n_1+n_2=n+1;\\ 1\le n_1, n_2 \le n}} |b_{n_1}|\ |b_{n_2}|,\ n = 1, 2, \ldots. \end{aligned} \tag{9}$$

Let us now consider the function

$$G(z) = \frac{1}{2(1 + |a|)} \left\{1 - \sqrt{1 - 4(1 + |a|)z}\right\}$$

which, in view of the binomial series expansion, can also be written as

$$G(z) = z + \sum_{n=2}^{\infty} C_n z^n$$

for $|z| < 1/4(1 + |a|)$. Since $G(z)$ satisfies the equation

$$(1 + |a|)G^2(z) + z = G(z),$$

thus, by the method of undetermined coefficients, it is not difficult to see that the coefficient sequence $\{C_n\}_{n=2}^{\infty}$ will satisfy $C_1 = 1$ and

$$\begin{aligned} C_{n+1} &= (1+|a|)\sum_{k=0}^{n-1} C_{k+1}C_{n-k} \\ &= (1+|a|)\cdot \sum_{\substack{n_1+n_2=n+1;\\ 1\le n_1,n_2\le n}} C_{n_1}C_{n_2}, n=1,2,\ldots. \end{aligned}$$

Hence by induction, we easily see from Lemma 1 that

$$|b_n| \le C_n d_n,\ n = 1,2,\ldots,$$

where the sequence $\{d_n\}_{n=1}^{\infty}$ is defined in Lemma 1.

Since $G(z)$ converges on the open disc $|z| < 1/4(1+|a|)$, there exists a positive constant T such that

$$C_n \le T^n$$

for $n = 1,2,\ldots$. In view of this and Lemma 1, we finally see that

$$|b_n| \le T^n Q^{n-1} n^{-2\delta},\ n = 1,2,\ldots,$$

where $Q = 2^{5\delta+1}$, which shows that the series (5) converges for $|z| < (TQ)^{-1}$.

Finally, when $|z| \le (TQ)^{-1}$, we have

$$\begin{aligned} |y(z)| &\le \left|\frac{\beta-a}{1-a}\right| + \sum_{n=1}^{\infty}|b_n|\ |z|^n \le \left|\frac{\beta-a}{1-a}\right| + \sum_{n=1}^{\infty} C_n d_n |z|^n \\ &\le \left|\frac{\beta-a}{1-a}\right| + \sum_{n=1}^{\infty} T^n Q^{n-1} n^{-2\delta}|z|^n \\ &\le \left|\frac{\beta-a}{1-a}\right| + \sum_{n=1}^{\infty} T^n Q^{n-1} n^{-2\delta}(TQ)^{-n} \\ &= \left|\frac{\beta-a}{1-a}\right| + \frac{1}{Q}\sum_{n=1}^{\infty}\frac{1}{n^{2\delta}}, \end{aligned}$$

as required. The proof is complete.

We now state and prove our main result in this note.

Theorem. *Suppose the complex number β satisfies either* (H1) *or* (H2). *Then equation* (1) *has an analytic solution $x(z)$ of the form* (4) *in a neighborhood of $(\beta-a)/(1-a)$, where $y(z)$ is an analytic solution of equation* (2). *Furthermore, when* (H1) *holds, there is a positive constant M such that*

$$|x(z)| \le \frac{1}{|b|}\left(\left|\frac{\beta-a}{1-a}\right| + \frac{1}{2M}\right) + \left|\frac{a}{b}\right|\ |z|$$

in a neighborhood of $(\beta - a)/(1-a)$*; and when* (H2) *holds, there is a positive number* δ *such that*

$$|x(z)| \le \frac{1}{|b|}\left(\left|\frac{\beta - a}{1-a}\right| + \frac{1}{Q}\sum_{n=1}^{\infty}\frac{1}{n^{2\delta}}\right) + \left|\frac{a}{b}\right| \ |z|, \ Q = 2^{5\delta+1},$$

in a neighborhood of $(\beta - a)/(1-a)$.

Proof. In view of Lemmas 2 and 3, we may find a sequence $\{b_n\}_{n=2}^{\infty}$ such that the function $y(z)$ of the form by (8) is an analytic solution of (2) in a neighborhood of the origin. Since $y'(0) = 1$, the function $y^{-1}(z)$ is analytic in a neighborhood of the point $y(0) = (\beta - a)/(1-a)$. If we now define $x(z)$ by means of (4), then

$$\begin{aligned} x'(z) &= \frac{1}{b}\cdot \beta y'(\beta y^{-1}(z))\cdot (y^{-1})'(z) - \frac{a}{b} = \frac{\beta}{b} y'(\beta y^{-1}(z))\cdot \frac{1}{y'(y^{-1}(z))} - \frac{a}{b} \\ &= \frac{1}{b}\{y(\beta^2 y^{-1}(z)) - a y(\beta y^{-1}(z)) + a\} - \frac{a}{b} \\ &= \frac{1}{b}\{y(\beta^2 y^{-1}(z)) - a y(\beta y^{-1}(z))\}, \end{aligned}$$

and

$$\begin{aligned} x(az + bx(z)) &= x\left(az + b\left[\frac{1}{b}y(\beta y^{-1}(z)) - \frac{a}{b}z\right]\right) = x(y(\beta y^{-1}(z))) \\ &= \frac{1}{b}y(\beta y^{-1}(y(\beta y^{-1}(z)))) - \frac{a}{b}y(\beta y^{-1}(z)) \\ &= \frac{1}{b}\{y(\beta^2 y^{-1}(z)) - a y(\beta y^{-1}(z))\} \end{aligned}$$

as required.

Next, if (H1) holds, then in view of Lemma 2,

$$\begin{aligned} |x(z)| &= \frac{1}{|b|}|y(\beta y^{-1}(z)) - az| \le \frac{1}{|b|}(|y(\beta y^{-1}(z))| + |a| \ |z|) \\ &\le \frac{1}{|b|}\left(\left|\frac{\beta - a}{1-a}\right| + \frac{1}{2M}\right) + \left|\frac{a}{b}\right| \ |z|; \end{aligned}$$

and if (H2) holds, then in view of Lemma 3,

$$\begin{aligned} |x(z)| &= \frac{1}{b}|y(\beta y^{-1}(z)) - az| \le \frac{1}{|b|}(|y(\beta y^{-1}(z))| + |a| \ |z|) \\ &\le \frac{1}{|b|}\left(\left|\frac{\beta - a}{1-a}\right| + \frac{1}{Q}\sum_{n=1}^{\infty}\frac{1}{n^{2\delta}}\right) + \left|\frac{a}{b}\right| \ |z|. \end{aligned}$$

The proof is complete.

We now show how to explicitly construct an analytic solution of (1) by means of (4). Since

$$x(z) = \frac{1}{b} y(\beta y^{-1}(z)) - \frac{a}{b} z,$$

thus

$$x\left(\frac{\beta - a}{1-a}\right) = \frac{1}{b} y(0) - \frac{a}{b}\frac{\beta - a}{1-a} = \frac{1}{b}\frac{\beta - a}{1-a} - \frac{a}{b}\frac{\beta - a}{1-a} = \frac{\beta - a}{b}.$$

Furthermore,

$$\begin{aligned} x'\left(\frac{\beta - a}{1-a}\right) &= x\left(a \cdot \frac{\beta - a}{1-a} + bx\left(\frac{\beta - a}{1-a}\right)\right) \\ &= x\left(a \cdot \frac{\beta - a}{1-a} + b \cdot \frac{\beta - a}{b}\right) = x\left(\frac{\beta - a}{1-a}\right) = \frac{\beta - a}{b}. \end{aligned}$$

By calculating the derivatives of both sides of (1), we obtain successively

$$x''(z) = x'(az + bx(z))\ (a + bx'(z)),$$

$$x'''(z) = x''(az + bx(z))(a + bx'(z))^2 + x'(az + bx(z))\ (bx''(z)),$$

so that

$$\begin{aligned} x''\left(\frac{\beta - a}{1-a}\right) &= x'\left(a \cdot \frac{\beta - a}{1-a} + bx\left(\frac{\beta - a}{1-a}\right)\right)\ \left(a + bx'\left(\frac{\beta - a}{1-a}\right)\right) \\ &= \beta x'\left(\frac{\beta - a}{1-a}\right) = \frac{\beta(\beta - a)}{b}, \end{aligned}$$

$$\begin{aligned} x'''\left(\frac{\beta - a}{1-a}\right) &= x''\left(\frac{\beta - a}{1-a}\right)\beta^2 + x'\left(\frac{\beta - a}{1-a}\right) \cdot bx''\left(\frac{\beta - a}{1-a}\right) \\ &= \frac{1}{b}[\beta(\beta - a)\ (\beta^2 + \beta - a)]. \end{aligned}$$

It seems from the above calculations that the higher derivatives $x^{(m)}(z)$ at $z = \xi \equiv (\beta - a)/(1-a)$ can be determined uniquely in similar manners. To see this, let us denote the derivative $(x^{(i)}(az + bx(z)))^{(j)}$ at $z = \xi$ by λ_{ij}, where $i, j \geq 0$. Note that the two derivatives $x^{(k)}(z)$ and $x^{(k)}(az + bx(z))$ are equal at the point $z = \xi$ since $a\xi + bx(\xi) = \xi$. In other words,

$$x^{(k)}(\xi) = \lambda_{k0}.$$

Furthermore, in view of (1), we see that $x^{(k+1)}(z) = (x(az + bx(z)))^{(k)}$ which implies

$$\lambda_{k+1,0} = \lambda_{0,k}.$$

Finally, since

$$\begin{aligned}(x^{(i)}(az + bx(z)))^{(j+1)} &= (x^{(i+1)}(az + bx(z)) \cdot (a + bx'(z)))^{(j)} \\ &= \sum_{k=0}^{j} \binom{j}{k} (a + bx'(z))^{(k)} \left(x^{(i+1)}(az + bx(z))\right)^{(j-k)},\end{aligned}$$

we see also that

$$\begin{aligned}\lambda_{i,j+1} &= \sum_{k=0}^{j} \binom{j}{k} \lambda_{i+1,j-k} \cdot (a + bx'(z))^{(k)}|_{z=\xi} \\ &= \beta\lambda_{i+1,j} + b\sum_{k=1}^{j} \binom{j}{k} \lambda_{i+1,j-k}\lambda_{0,k},\ i = 0, 1, \ldots;\ j = 0, 1, \ldots,\end{aligned}$$

where we have used the fact that $\lambda_{k+1,0} = \lambda_{0,k}$ in obtaining the last equality. Clearly, if we have obtained the derivatives $x^{(0)}(\xi) = \lambda_{00}, \ldots, x^{(m)}(\xi) = \lambda_{m0} = \lambda_{0,m-1}$, then by means of the above partial difference equation, we can successively calculate

$$\lambda_{m-1,1}, \lambda_{m-2,1}, \lambda_{m-2,2}, \ldots, \lambda_{11}, \lambda_{12}, \ldots, \lambda_{1,m-1}, \lambda_{0m}$$

in a unique manner. In particular, $\lambda_{0m} = \lambda_{m+1,0}$ is the desired derivative $x^{(m+1)}(\xi)$.

This shows that

$$\begin{aligned}x(z) &= \frac{\beta - a}{b} + \frac{1}{b}(\beta - a)\left(z - \frac{\beta - a}{1 - a}\right) + \frac{\beta(\beta - a)}{2!b}\left(z - \frac{\beta - a}{1 - a}\right)^2 \\ &\quad + \frac{\beta(\beta - a)\ (\beta^2 + \beta - a)}{3!b}\left(z - \frac{\beta - a}{1 - a}\right)^3 + \sum_{i=4}^{\infty} \frac{\lambda_{i,0}}{i!}\left(z - \frac{\beta - a}{1 - a}\right)^i.\end{aligned}$$

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