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BALL SEPARATION PROPERTIES IN BANACH SPACES AND EXTREMAL PROPERTIES OF UNIT BALL IN DUAL SPACES*

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Abstract. Various ball separation properties in Banach spaces and their relationships with the extremal structures of the unit ball of dual space are presented. The relationships of these properties with the ball convexity, ball topology in Banach spaces and Banach spaces with the Radon-Nikodyn property are discussed.

For a Banach space X, let S_X (resp. B_X) be the unit sphere (resp. ball) in X and let X^* be the dual space of X. We study Banach spaces with the following property:

($*$) For any two disjoint bounded closed convex sets $K_i, i = 1, 2$, in X, there exist balls $B_i, B_i \supset K_i, i = 1, 2$ and $B_1 \cap B_2 = \phi$.

Observe that if such balls B_1 and B_2 exist, then there exists a hyperplane H in X which separates B_1 and B_2, hence separates K_1 and K_2. This leads to the following problem:

(**) Determine $x^* \neq 0$ in X^* with the property that if $H = \ker\ x^* = \{x : x \in X, x^*(x) = 0\}$, then for any bounded closed convex set K in X such that $K \cap H = \phi$, there exists a ball B, $B \supset K$ and $B \cap H = \phi$.

In this survey paper, we shall show that these two properties and other related ball separation properties are related to the extremal structure of B_{X^*}. Applications to ball topology and ball convexity in Banach spaces will be discussed.

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For a set K in X, let $\bar{co}K$ denotes the closed convex hull of K. If K is a subset in X^*, let $\bar{co}^*K$ be the weak* closed convex hull of K and let $\bar{K}^{\omega*}$ be the weak* closure of K in X^*. A *weak* slice* of K is a set $S(K,x,\delta)=\{x^*: x^*\in K, x^*(x)>\sup_{y^*\in K} y^*(x)-\delta\}$ where $x\in X$ and $\delta>0$. x^* is called a *weak*-denting point* (w^*-denting point) of K if for every $\epsilon>0$, there exists a weak* slice $S(x,K,\delta)$ of K containing x^* and $\mathrm{diam}S(x,K,\delta)<\epsilon$. x^* is called a *weak** (resp. *weak*-weak*) *point of continuity* of K if the identity mapping $Id:(K,\text{weak}^*)\longrightarrow(K,\|\cdot\|)$ (resp. $(K,\text{ weak}^*)\rightarrow(K,\text{ weak})$) is continuous at x^*. For a set K in X, $x\in K$ is called an *extreme point* of K if for any $y,z\in K$, $x=\frac{1}{2}(y+z)$ implies that $x=y=z$. It is known [13] that for bounded closed convex set K in X^*, the following hold:

(1) weak*-denting points of $K\subset$ weak*-extreme points of K

$\subset$ extreme points of K;

(2) weak*-denting points of $K\subset$ weak*-points of continuity of K

$\subset$ weak*-weak points of continuity of K;

(3) $x^*\in K$ is a weak*-denting point of K if and only if x^* is a weak*-point of continuity of K and x^* is an extreme point of K.

All balls in this paper are closed balls, that is, set of the form $B(x,r)=\{y: y\in X, \|x-y\|\le r\}$. For balls in X^*, we use the notation $B^*(x^*,r)$ and for balls in X^{**}, we use the notation $B^{**}(x^{**},r)$.

1. Ball Separation Properties Related to Weak*-Denting Points of B_{X^*}

A Banach space X is said to have the *Mazur intersection property* (MIP) if every bounded closed convex set in X is an intersection of balls. Equivalently, for any bounded closed convex set K in X and for any $x\notin K$, there exists a ball B, $B\supset K$ and $x\notin B$. This is a special case of $(*)$ when one of the sets $K_i, i=1,2$ is a singleton. Since in this case, there is a hyperplane that separates K and x. Hence solutions of $(**)$ will yield solutions of Banach spaces with the (MIP). S. Mazur [17] was the first to study Banach spaces with (MIP). R. R. Phelps [18] showed that for a finite-dimensional Banach space X to have the (MIP), it is necessary and sufficient that the set of extreme points of B_{X^*} is dense in S_{X^*}. Since in finite-dimensional Banach spaces, the weak topology and norm topology coincide and so every point is a point of continuity. It follows that in finite-dimensional Banach spaces, extreme points, denting points and weak* denting points coincide. J. Giles, D. A. Gregory and B. Sims in 1978 proved the following theorem.

Theorem 1. [10] *A Banach space X has the (MIP) if and only if the set of weak*-denting points of B_{X^*} is dense in S_{X^*}.*

In the proof of Theorem 1, they have demonstrated that the (MIP), weak*-denting points of B_{X^*} are related to the smoothness of X. In [18], R. R. Phelps has established a basic result in determining the distance between elements in X^*. The following lemma is an easy consequence of results in [18].

Lemma 2. *For a normed space X, let x^* and y^* be elements in S_{X^*} and let $K = \{x : x \in B_X,\ x^*(x) > \frac{\epsilon}{2}\}$ where $0 < \epsilon < 1$. If $\inf y^*(K) > 0$ then $\|x^* - y^*\| < \epsilon$.*

Using Lemma 2 and the argument in [10], it is straightforward to prove the following result.

Lemma 3. (i) *Let $x \in B_X$, $\delta > 0$ and $\epsilon > 0$. If diam $S(B_{X^*}, x, \delta) \leq \epsilon$, then*

$$\sup_{y \in B_X} \frac{\|x + \frac{\delta}{2}y\| + \|x - \frac{\delta}{2}y\| - 2}{\frac{\delta}{2}} \leq \epsilon.$$

(ii) *Let $x \in S_X$, $\epsilon > 0$ and $n \in \mathbb{N}$. If*

$$\sup_{y \in B_X} \frac{\|x + \frac{1}{n}y\| + \|x - \frac{1}{n}y\| - 2}{\frac{1}{n}} \leq \epsilon,$$

then diam $S(B_{X^*}, x, \frac{\epsilon}{n}) \leq 3\epsilon$.

Using Lemma 3, we establish a solution for $(**)$.

Theorem 4. *Let X be a normed space and let $x_0^* \in S_X^*$. Then x_0^* is a weak*-denting point of B_X^* if and only if for any bounded closed convex set K in X such that inf $x_0^*(K) > 0$, there exists a ball B in X, $B \supset K$ and* inf *$x_0^*(B) > 0$.*

Corollary 5. *Let X be a Banach space. Then $x_0 \in S_X$ is a denting point of B_X if and only if for any bounded closed convex set K in X^* such that* $\inf x_0(K) > 0$*, there exists a ball B^* in X^*, $B^* \supset K$ and* $\inf x_0(B^*) > 0$.

Corollary 6. *Let X be a Banach space. Then $x_0^* \in S_{X^*}$ is a weak*-denting point of B_{X^*} if and only if for any bounded closed convex set K in X^{**} such that* $\inf x_0^*(K) > 0$*, there exists a ball B^{**} in X^{**} with center in X, $B^{**} \supset K$ and* $\inf x_0^*(B^{**}) > 0$.

Corollary 7. *A Banach space X has the (MIP) if and only if for any two disjoint bounded weak*-closed convex sets K_1, K_2 in X^{**}, there exist balls B_1^{**}, B_2^{**} in X^{**} with centers in X such that $B_i^{**} \supset K_i$, $i = 1, 2$, and $B_1^{**} \cap B_2^{**} = \phi$.*

In 1978, J. Giles, D. Gregory and B. Sims [10] raised the question whether every Banach space with the (MIP) is an Asplund space. Recently, M. J. Sevilla and J. P. Moreno [21] has exhibited a class of non-Asplund spaces that admit equivalent norms with the (MIP).

2. Ball Separation Properties and Weak*-Points of Continuity of B_{X^*}

We now consider some ball separation properties related to the weak*-points of continuity of B_{X^*}. The difference between weak*-denting points and weak*-points of continuity is that for weak*-denting points, we need to find weak*-slices with arbitrarily small diameter and for weak*-points of continuity, we need to find weak* open sets with arbitrarily small diameter. Since weak* open sets contain finite intersection of weak* slices this leads us to introduce the following generalized Mazur intersection property.

Definition. A Banach space X is said to have the property (II) if for every bounded closed convex set K in X, $K = \cap_{i \in I} K_i$ where for each $i \in I$, $K_i = \bar{co}\{\cup_{j=1}^n B_j\}$ and B_j, $j = 1, 2, \ldots, n$, are balls in X. Equivalently, for any bounded closed convex set K in X and for any $x \notin K$, there exist balls B_j, $j = 1, 2, \ldots, n$, in X such that $\bar{co}(\cup_{j=1}^n B_j) \supset K$ and $x \notin \bar{co}(\cup_{j=1}^n B_j)$.

The following theorem is the corresponding result of Theorem 4.

Theorem 8. [7] *Let X be a Banach space and let $x_0^* \in S_{X^*}$. Then x_0^* is a weak*-point of continuity of B_{X^*} if and only if for any bounded closed convex set K in X such that $\inf x_0^*(K) > 0$, there exist balls $B_1, B_2, \ldots, B_n$ in X such that $\bar{co}(\cup_{j=1}^n B_j) \supset K$ and $\inf x_0^*(\bar{co}(\cup_{j=1}^n B_j)) > 0$.*

Corollary 9. *A Banach space X has the property (II) if and only if the set of weak*-points of continuity of B_{X^*} is dense in S_{X^*}.*

There is a ball separation property in Banach spaces that is closely related to (MIP) and the property (II). A set K in a Banach space X is said to be *ball-generated* [12] if $K = \cap_{i \in I} K_i$ where each K_i is a finite union of balls in X. Equivalently, K is *ball-generated* if for every $x \notin K$, there exist balls

$B_1, \ldots, B_n$ in X such that $\cup_{i=1}^n B_i \supset K$ and $x \notin \cup_{i=1}^n B_i$. We say that a Banach space X has the *ball-generated property* (BGP) if all bounded closed convex sets in X are ball-generated. The (BGP) is related to the ball topology b_X of Banach space [12]. b_X is the weakest topology on X such that all balls in X are closed in b_X. A b_X-base of neighborhoods of a point x_0 in X is of the form $X \setminus \cup_{i=1}^n B_i$ where B_i are balls in X with $x_0 \notin B_i$, $i = 1, 2, \ldots, n$. A Banach space X has the (BGP) if and only if every bounded closed convex set in X is b_X-closed. The following theorem determines when an element in X^* is b_X-continuous.

Theorem 10. [8] *$x_0^* \in X^*$ is continuous on (B_X, b_X) if and only if for every $\epsilon > 0$, there exist weak* slices $S_1, S_2, \ldots, S_n$ of B_{X^*} and a function $F : \Pi_{i=1}^n S_i \to X^*$ such that $F(x_1^*, x_2^*, \ldots, x_n^*) = \sum_{i=1}^n a_i x_i^*$, where $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, are dependent on $(x_1^*, x_2^*, \ldots, x_n^*)$ and $\|x_0^* - F(x_1^*, x_2^*, \ldots, x_n^*)\| < \epsilon$ for all $(x_1^*, x_2^*, \ldots, x_n^*) \in \Pi_{i=1}^n S_i$.*

By Krein-Milman theorem and Theorem 10, it follows that every weak*-point of continuity of B_{X^*} is continuous on (B_X, b_X). By Theorem 10 again, every element in X^* is continuous on (B_X, b_X) if X^* is the closed linear span of weak* points of continuity of B_{X^*}. Thus we have the following result:

Theorem 11. *A Banach space X has the (BGP) if X^* is the closed linear span of weak* points of continuity of B_{X^*}.*

A Banach space X is called *nicely smooth* [11] if for all $x^{**} \in X^{**}$, $\cap_{x \in X} B^{**}(x, \|x - x^{**}\|) = \{x^{**}\}$. Equivalently, for all $x^{**} \neq y^{**}$ in X^{**}, there exists a ball B^{**} in X^{**}, center in X such that $x^{**} \in B^{**}$ and $y^{**} \notin B^{**}$. A closed linear subspace N in X^* is called a *norming subspace* for X if $\|x\| = \sup\{|x^*(x)| : x^* \in B_N\}$ for all $x \in X$. It is known that X is nicely smooth if and only if there is no proper norming subspace for X in X^*. Suppose that X has the (BGP) and let N be a norming subspace for X in X^*. Then B_X is closed under the weak topology $\sigma(X, N)$ induced by N. Since X has the (BGP), it follows that every bounded closed convex set in X is closed under $\sigma(X, N)$. Thus $N = X^*$. This proves that every Banach space with the (BGP) is nicely smooth. Now if X is an Asplund space, then B_X is Frechet differentiable on a dense subset F in S_X. Let D be the duality mapping of X. Then $D(F)$ norms X and every element in $D(F)$ is a weak*-strongly exposed point of B_{X^*}. If X is also nicely smooth, then X^* = the closed linear span of weak*-strongly exposed points of B_{X^*}. Since weak*-strongly exposed points are weak*-denting points, we obtain the following theorem.

Theorem 12. *Let X be a Banach space. Consider the following statements:*

(1) *X^* is the closed linear span of weak*-strongly exposed points of B_{X^*}.*

(2) *X^* is the closed linear span of weak*-denting points of B_{X^*}.*

(3) *X^* is the closed linear span of weak*-points of continuity of B_{X^*}.*

(4) *X has the (BGP).*

(5) *X is nicely smooth.*

Then (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(5). If X is Asplund, then all are equivalent.

For other properties of Banach spaces with (BGP), we refer to [6].

3. Ball Separation Property and Weak*-Weak Points of Continuity

The following ball separation property characterizes the weak*-weak points of continuity of B_{X^*}.

Theorem 13. [7] *Let X be a Banach space and let $x_0^* \in S_{X^*}$. Then x_0^* is a weak*-weak point of continuity of B_{X^*} if and only if for any x^{**} in X^{**} such that $x^{**}(x_0^*) \neq 0$, there exists a ball B^{**} in X^{**} with center in X, $x_0^{**} \in B^{**}$ and $B^{**} \cap ker\ x_0^* = \phi$, where $ker\ x_0^* = \{y^{**} \in X^{**} : y^{**}(x_0^*) = 0\}$.*

Corollary 14. *If the set of weak*-weak points of continuity is dense in S_{X^*}, then for any $x_1^{**} \neq x_2^{**}$ in X^{**} there exist balls B_i^{**} with center in X, $x_i^{**} \in B_i^{**}$, $i = 1, 2$, and $B_1^{**} \cap B_2^{**} = \phi$.*

4. Ball Separation Properties Related to Extreme Points and Weak*-Strongly Extreme Points of B_{X^*}

The ball separation properties of bounded closed convex sets in a Banach space is related to the weak*-denting points of B_{X^*}. The extreme points of B_{X^*} determine the ball separation properties for compact convex sets.

Theorem 15. [24 *and* 19] *Let X be a Banach space. Then every compact convex set in X is an intersection of balls if and only if the cone generated by the extreme points of B_{X^*} is τ_X dense in X^*, where τ_X is the topology of uniform convergence on compact subsets of X.*

Theorem 16. [19 *and* 20] *Let X be a Banach space. Then*

(i) *every compact convex set in X with dimension less than or equal to n is an intersection of balls if and only if for every $x^* \in X^*$, every $n+1$ points $x_1, x_2, \ldots, x_{n+1}$ in X and every $\epsilon > 0$, there exists $y^* \in ExtX^* \equiv \{\lambda z^* : \lambda > 0, z^* \in extB_{X^*}\}$ such that $|(x^* - y^*)(x_i)| < \epsilon, i = 1, 2, \ldots, n$;*

(ii) *every finite-dimensional compact convex set in X is an intersection of balls if and only if the cone Ext X^* generated by the extreme points of B_{X^*} is weak* dense in X^*.*

Using the argument similar to weak*-denting points discussed in section 1 by using the family of semi-norms p_M where M is compact convex set in X, the following result can be proved.

Theorem 17. [7] *Let X be a Banach space and let $x_0^* \in S_{X^*}$. Then the following are equivalent:*

(i) *x_0^* is an extreme point of B_{X^*};*

(ii) *for any compact convex set K in X, if $\inf x_0^*(K) > 0$, then there exists a ball B in X, $B \supset K$ and $\inf x_0^*(B) > 0$;*

(iii) *for any finite set K in X, if $\inf x_0^*(K) > 0$, then there exists a ball B in X, $B \supset K$ and $\inf x_0^*(B) > 0$.*

For a bounded closed convex set K in X^*, $x_0^* \in K$ is called a *weak* strongly extreme point* of K if the family of all weak* slices of K containing x_0^* forms a base for the weak* topology of x_0^* in K. It is clear that x_0^* is a weak* strongly extreme point of K if and only if x_0^* is an extreme point of K and x_0^* is a weak*-weak point of continuity of K. By combining Theorems 13 and 17, it is easy to prove the following result:

Theorem 18. [2] *Let X be a Banach space and let $x_0^* \in S_{X^*}$. Then the following are equivalent:*

(i) *x_0^* is a weak*-strongly extreme point of B_{X^*};*

(ii) *for any compact convex set K in X^{**}, if $\inf x_0^*(K) > 0$, then there exists a ball B^{**} in X^{**} with center in X such that $B^{**} \supset K$ and $\inf x_0^*(B^{**}) > 0$;*

(iii) *for any finite set K in X^{**}, if $\inf x_0^*(K) > 0$, then there exists a ball B^{**} in X^{**} with center in X such that $B^{**} \supset K$ and $\inf x_0^*(B^{**}) > 0$.*

Following the definition of usual convexity in linear spaces, M. Lassak [16] in 1977 introduced the concept of ball convexity in Minkowski-Banach spaces. A set K in a normed space is said to be *B-convex* if for every finite subset F in K, K contains the intersection of balls containing F. It is easy to see that a set K in a Banach space X is B-convex if and only if for every finite subset F in K and for any $x \notin K$, there exists a ball B in X with $B \supset F$ and $x \notin B$. It is known [16] that in finite-dimensional Banach spaces, the closure of every bounded B-convex set is an intersection of balls. However, there is an equivalent norm in Hilbert space ℓ_2 such that there is a bounded B-convex set K with the property that the closure of K is not an intersection of balls [9]. Let us call a set K a *Mazur set* if it is an intersection of balls. It is clear that every Mazur set is B-convex. We shall give a necessary condition for a Banach space X such that the closure of every bounded B-convex set is a Mazur set.

Let X be a Banach space. An element $x_0^* \in S_{X^*}$ is called a *semi-denting point* of B_{X^*} if for every $\epsilon > 0$, there exists a weak* slice $S(B_{X^*}, x, \delta)$ of B_{X^*} such that $\operatorname{diam}(x_0^* \cup S(B_{X^*}, x, \delta)) < \epsilon$. It is clear that every weak* denting point of B_{X^*} is a semi-denting point of B_{X^*}. Comparing to Theorem 4 in section I for weak*-denting points of B_{X^*}, the following ball separation property holds for semi-denting points of B_{X^*}.

Theorem 19. [9] *Let X be a Banach space. An element x_0^* in S_{X^*} is a semi-denting point of B_{X^*} if and only if for any bounded closed convex set K in X and for any x_0 in X, if x_0^* separates K and x_0, then there is a ball in X with $B \supset K$ and $x_0 \notin B$.*

Theorem 20. [9] *If X is a Banach space with the property that the closure of every bounded B-convex set in X is a Mazur set, then $S_{X^*} \cap \overline{Ext(X^*)}^{w^*}$ is the set of all semi-denting points of B_{X^*}, where $Ext(X^*) = \{\lambda x^* : \lambda > 0, x^*$ is an extreme point of $B_{X^*}\}$ is the cone generated by the set of extreme points of B_{X^*}.*

5. Weak* Asymptotic Norming Properties

The density of weak*-denting points (resp. weak* points of continuity etc.) of B_{X^*} in S_{X^*} plays an important role on the geometry of X. Let us consider the following four kinds of density in this respect.

(1) Every point of S_{X^*} is a weak*-denting point (resp. weak* point of continuity, etc.) of B_{X^*}.

(2) The set of weak*-denting points (resp. weak* points of continuity, etc.) of B_{X^*} is dense in S_{X^*} relative to certain topology on X^*.

(3) B_{X^*} is the closed convex hull of weak*-denting points (resp. weak* points of continuity, etc.) of B_{X^*}.

(4) X^* is the closed linear span of weak*-denting points (resp. weak* points of continuity, etc.) of B_{X^*}.

In this paper, we have demonstrated that (2) is necessary and sufficient for various ball separation properties. In [14], it is proved that if X is a Banach space such that the duality mapping $D : (S_X, \|\cdot\|) \to (S_{X^*}, w)$ is upper semi-continuous, then B_{X^*} is the closed convex hull of weak* denting points of B_{X^*}. In Section 2, we have showed that if X^* is the closed linear span of weak* points of continuity of B_{X^*} then X has the (BGP). To conclude this paper, we consider Banach spaces that have property (1). It is well-known that if every point of S_{X^*} is an extreme point of B_{X^*}, then X^* is said to be strictly convex and X is smooth. For the stronger kinds of extremal structures of B_{X^*}, there are related to the so-called *weak*-asymptotic norming properties* of X^*. The asymptotic norming properties I, II and III (ANP-I, II, III) were introduced by R. C. James and A. Ho in 1981 to show that the class of separable Banach spaces with Radon-Nikodym property (RNP) is larger than those Banach spaces which are isomorphic to subspaces of separable dual spaces. It was proved by N. Ghoussoub and B. Maurey that in separable Banach space, ANP and RNP are equivalent. However, it is still an open question whether the ANP and RNP are equivalent in every Banach spaces. Recently, it was proved in [13] tht the three ANP are equivalent in every Banach space that admits an equivalent locally uniformly convex norm. It is currently an active research topic to classify the class of Banach spaces that admit equivalent locally uniformly convex norm. Let us mention that a Banach space admits an equivalent uniformly convex norm if and only if the space is superreflexive.

A subset Φ of B_{X^*} is called a *norming set* for X if $\|x\| = \sup_{x^* \in \Phi} x^*(x)$ for all $x \in X$. A sequence $\{x_n\}$ in S_X is said to be *asymptotically normed by* Φ if for each $\epsilon > 0$, there exists $x^* \in \Phi$ and $N \in \mathbb{N}$ such that $x^*(x_n) > 1 - \epsilon$ for all $n \geq N$. For $\mathcal{K} = I, II, II' or III$, a sequence $\{x_n\}$ in X is said to have the *property* $\mathcal{K}$ if:

(I) $\{x_n\}$ is convergent.

(II) $\{x_n\}$ has a convergent subsequence.

(II') $\{x_n\}$ is weakly convergent.

(III) $\{x_n\}$ has a weakly convergent subsequence.

For $\mathcal{K} = I, II, II' or III, X$ is said to have *the asymptotic norming property* $\mathcal{K}$ *with respect to* $\Phi(\Phi - ANP - \mathcal{K})$ if every sequence in S_X that is asymptotically normed by Φ has property $\mathcal{K}$. A Banach space X is said to have *the asymptotic norming property* $\mathcal{K}(ANP - \mathcal{K})$, $\mathcal{K} = I, II, II'$ *or* III, if there exists an equivalent norm $\|\cdot\|$ on X and a norming set Φ for $(X, \|\cdot\|)$ such that X has $\Phi - ANP - \mathcal{K}$. The ANP-I, II and III were introduced by R. C. James and A. Ho. The ANP-II$'$ was recently introduced by P. Bandyopadhyay and S. Basu [2].

A dual space X^* is said to have *the weak* asymptotic norming property* $\mathcal{K}(w^* - ANP - \mathcal{K})$, $\mathcal{K}$ =I, II, II$'$ or III, if there exists an equivalent norm $\|\cdot\|$ on X and a norming set Φ for X^* in B_X such that X^* has $\Phi - ANP - \mathcal{K}$. The $w^* - ANP - \mathcal{K}$, $\mathcal{K} =$ I, II or III, were introduced by Zhibao Hu and Bor-Luh Lin [13]. The $w^* - ANP - II'$ was recently introduced by P. Bandyopadhyay and S. Basu [2]. It has been observed in [13] that X^* has $\Phi - ANP - \mathcal{K}$, $\mathcal{K} =$ I, II or III, where $\Phi \supset S_X$ is equivalent to $B_X - ANP - \mathcal{K}$ and similar arguments can be used to show $\Phi - ANP - II'$ is equivalent to $B_X - ANP - II'$. Thus for dual spaces X^*, we shall simply say that X^* has $w^* - ANP - \mathcal{K}$, $\mathcal{K} =$ I, II, II' or III, without going to equivalent norm or specifying the norming set Φ.

Theorem 21. [13] *Let X be a Banach space. Then*

(1) *X^* has $w^* - ANP - I$ if and only if every point in S_{X^*} is a weak*-denting point of B_{X^*}.*
(2) *X^* has $w^* - ANP - II$ if and only if every point in S_{X^*} is a weak* point of continuity of B_{X^*}.*
(3) *X^* has $w^* - ANP - III$ if and only if every point in S_{X^*} is a weak*-weak point of continuity of B_{X^*}.*

Theorem 22. [2] *Let X be a Banach space. Then X^* has $w^* - ANP - II'$ if and only if every point of S_{X^*} is a weak*-strongly extreme point of B_{X^*}.*

In [14], it is proved that X^* has $w^* - ANP - III$ if and only if X is Hahn-Banach smooth. Recall that a Banach space X is said to be Hahn- Banach smooth [22] if every element x^* in X^* has a unique Hahn-Banach extension in X^{***}. In [2], it is proved that X^* has the $w^* - ANP - II'$ if and only if X has the property V. A Banach space X is said to have the property (V) if there does not exist an increasing sequence of open balls $\{B_n\}$ with unbounded radii and norm-one functionals x^* and $\{y_k^*\}_{k\in\mathbb{N}}$ such that for some constant c,

(i) $x^*(x) > c$ for all $x \in \cup_{n=1}^{\infty} B_n$,
(ii) $y_k^*(x) > c$ for all $x \in B_n$, $n \leq k$, and
(iii) distance $(x^*, co(y_k)_{k\in\mathbb{N}}) > 0$.

The property (V) was introduced by F. Sullivan [23] based on a result of L. P. Vlasov which showed that if a Banach space X has the property V, then X^* is strictly convex. In fact, it is observed in [2] that X has the property V

if and only if X^* is strictly convex and X is Hahn-Banach smooth. It follows that $x^* \in S_X$ is a weak* strongly extreme point of B_{X^*} if and only if x^* is an extreme point of B_{X^*} and x^* is a weak*-weak point of continuity of B_{X^*}. It remains to find geometric properties in X which are equivalent to X^* having $w^* - ANP - \mathcal{K}$, $\mathcal{K} = \text{I}$ or II.

It is easy to see that

$$w^* - ANP - I \Rightarrow w^* - ANP - II \Rightarrow w^* - ANP - III$$

and

$$w^* - ANP - I \Rightarrow w^* - ANP - II' \Rightarrow w^* - ANP - III\,.$$

In [13], it is proved that X^* has $w^* - ANP - III$ is strictly stronger than X^* to have (RNP), hence X is an Asplund space. Thus if every point of S_{X^*} is a weak*-denting point of B_{X^*} then X is Asplund. However, if the set of weak*-denting points of B_{X^*} is dense in S_{X^*}, then X is not necessaryily an Asplund space [21].

References

1. P. Bandyopadhyay, The Mazur intersection property for families of closed bounded convex sets in Banach spaces, *Colloq. Math.* LX III (1992), 45-56.

2. P. Bandyopadhyay and S. Basu, On a new asymptotic norming property, preprint.

3. P. Bandyopadhyay and S. Basu, On nicely smooth Banach spaces, preprint.

4. P. Bandyopadhyay and A. K. Roy, Some stability results for Banach spaces with the Mazur intersection property, *Indagatione Math. New Series* **1**, No. 2 (1990), 137-154.

5. S. Basu and TSSRK Rao, Some stability results for asymptotic norming properties of Banach spaces, preprint.

6. D. Chen, Z. Hu, and B.-L. Lin, Ball intersection properties of Banach spaces, *Bull. Austral. Math. Soc.* **45** (1992), 333-342.

7. D. Chen and B.-L. Lin, Ball separation properties in Banach spaces, preprint.

8. D. Chen and B.-L. Lin, Ball topology on Banach spaces, *Houston J. Math.* **22** (1996), 821-833.

9. D. Chen and B.-L. Lin, On B-convex and Mazur sets of Banach spaces, *Bull. Acad. Polish Sci.* **43** (1995), 191-198.

10. J. Giles, D. A. Gregory, and B. Sims, Characterization of normed linear spaces with Mazur's intersection property, *Bull. Austral. Math. Soc.* **18** (1978), 105-123.

11. G. Godefroy, Nicely smooth Banach spaces, Longhorn Notes, The University of Texas at Austin (1984-85), 117-124.

12. G. Godefroy and N. Kalton, The ball topology and its applications, *Contemporary Math.* **85** (1989), 195-237.

13. Z. Hu and B.-L. Lin, On the asymptotic norming properties of Banach spaces, Proc. Conference on Function Spaces, vol. 136, Marcel and Dekker, SIUE, 1992, pp. 195-210.

14. Z. Hu and B.-L. Lin, Smoothness and asymptotic norming properties in Banach spaces, *Bull. Austral. Math. Soc.* **45** (1992), 285-296.

15. Z. Hu and B.-L. Lin, Asymptotic norming and Mazur's intersection property in Lebesque Bochner Function spaces, *Bull. Austral. Math. Soc.* **45** (1993), 177-186.

16. M. Lassak, Some properties of B-convexity in Minkowski-Banach spaces, *Bull. Acad. Pol. Math.* **27** (1979), 97-106.

17. S. Mazur, Uber Schwach Konvergenz in der Raume (L^p), *Studia Math.* **4** (1933), 128-133.

18. R. R. Phelps, A representation theorem for bounded closed convex sets, *Proc. Amer. Math. Soc.* **102** (1960), 976-983.

19. A. Sersouri, The Mazur's property for compact convex sets, *Pacific J. Math.* **113** (1988), 185-196.

20. A. Sersouri, Mazur's intersection property for finite dimensional sets, *Math. Annal.* **283** (1989), 165-170.

21. M. J. Sevilla and J. P. Moreno, The Mazur intersection property and Asplund spaces, *C. R. Acad. Sci. Paris* **321** (1995), 1219-1223.

22. M. Smith and F. Sullivan, Lecture Notes in Math. vol. 604, Springer-Verlag, 1977.

23. F. Sullivan, Geometrical properties determined by the higher duals of a Banach space, *Illinois J. Math.* **21** (1977), 315-331.

24. J. H. M. Whitfield and V. Zizler, Mazur's intersection property of balls for compact convex sets, *Bull. Austral. Math. Soc.* **35** (1987), 267-274.

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