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INFINITESIMAL GENERATORS OF RANDOM POSITIVE SEMIGROUPS*

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Abstract. We first show the existence of a random infinitesimal generator of a given random positive semigroup under some conditions. Then we represent it as a random second order integro-differential operator.

1. Introduction

In the previous paper Kifer-Kunita [1], we studied the random infinitesimal generator of a random positive semigroup, or a positive semigroup in random environments. There, the random semigroup with independent increments is studied. Its random infinitesimal generator is represented as a second order stochastic partial differential operator, where the first order coefficients and the potential part are Brownian motions with spatial parameter but the second order (highest order) coefficients are deterministic ones. In this paper, we will study a more general random positive semigroup, not necessarily with independent increments. Its random infiniesimal generator will be a semimartingale with values in intego-differential operators.

In the next section, we will show the existence and the uniqueness of the random infinitesimal generator of a given random positive semigroup. Assumptions needed for the random positive semigroup are relaxed considerably from those in [1]. See Theorem 2.1 and Theorem 2.3. Then we will represent it as a random integro-differential operator: It will be represented as a sum of a second order stochastic partial diffrential operator and a random integral operator involving a Lévy measure and a counting measure. See Theorem 3.1. In section 4, we discuss the asymptotic properties of coefficients of the random infinitesimal generator.

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2. Ito's Infinitesimal Generators of Random Positive Semigroups

Let $\mathbf{C} = C(\mathbb{R}^d)$ be the totality of real continuous functions on $\mathbb{R}^d$ such that $\lim_{x\to\infty} f(x)$ exists and equals 0. It is a real separable Banach space with the supremum norm $\| \quad \|$. Let $\{T_{s,t}, 0 \le s \le t < \infty\}$ be a family of stochastic processes with values in linear operators on $\mathbf{C}$, cadlag with respect to $t(\ge s)$, defined on a probability space $(\Omega, \mathcal{F}, P)$. It is called a *random positive semigroup* if (i) $f \ge 0$ implies $T_{s,t} f \ge 0$ a.s. for any s, t, (ii) for each s $\lim_{t\to s} T_{s,t} f = f$ holds a.s. for $f \in \mathbf{C}$, and (iii) for each $s < t < u$ $T_{s,t}T_{t,u} f = T_{s,u} f$ holds a.s. for $f \in \mathbf{C}$. Further, if $T_{t_i,t_{i+1}}, i = 0, \ldots, n-1$ are independent for any $0 \le t_0 < t_1 < \ldots < t_n < \infty$, it is said to have *independent increments*.

We set $\mathcal{F}_t = \sigma(T_{s,r}; s, r \le t)$. Then $\{\mathcal{F}_t\}_{t>0}$ is an increasing family of sub σ-fields of $\mathcal{F}$.

We shall obtain the infinitesimal generator of a given random positive semigroup under two different conditions. Results will be stated in Theorems 2.1 and 2.3. To make statements precisely, we introduce some function spaces. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_d)^{\alpha_d}$. Let m be a positive integer. Set $\|f\|_m = \sum_{|\alpha|\le m} \|D^\alpha f\|$. Denote by $\mathbf{C}^m$ the totality of $f \in \mathbf{C}$ such that $\|f\|_m < \infty$. We set $\mathbf{C}^\infty = \cap_m \mathbf{C}^m$. A function f is said to belong to $\mathbf{C}^m_{loc}$ etc. if $f\psi$ belongs to $\mathbf{C}^m$ for any $\mathbf{C}^\infty$-function ψ with compact support.

Let $A(t), t \ge 0$ be a family of random linear maps from $\mathbf{C}^\infty$ to $\mathbf{C}_{loc}$ such that $A(t) f(x)$ is a semimartingale with respect to $\{\mathcal{F}_t\}$ for any $f \in \mathbf{C}^\infty$ and $x \in \mathbb{R}^d$. It is called the *random infinitesimal generator* of $\{T_{s,t}\}$, if it satisfies

$$T_{s,t}(x) f(x) = f(x) + \int_s^t T_{s,r-} A(dr) f(x) \quad \text{a.s.} \quad \forall s < t, \tag{2.1}$$

for any $f \in \mathbf{C}^\infty$ and $x \in \mathbb{R}^d$, where the right hand side is Itô's stochastic integral.

We introduce two assumptions for $\{T_{s,t}\}$. Set

$$U_{s,t} f(x) = E\,[T_{s,t} f(x) | \mathcal{F}_s]. \tag{2.2}$$

Then $\{U_{s,t}\}$ is a family of random popsitive linear operators on $\mathbf{C}$. In general, $\{U_{s,t}\}$ might not satisfy the semigroup property. However, if $\{T_{s,t}\}$ has independent increments, $\{U_{s,t}\}$ defines a deterministic positive semigroup.

(**A.1**). For each $f \in \mathbf{C}^\infty$, the limit

$$L(t) f(x) = \lim_{h\to 0} \frac{U_{t,t+h} f(x) - f(x)}{h} \tag{2.3}$$

exists uniformly in x a.s. and it is a continuous function of (t, x) a.s.

The family of operators $\{L(t)\}$ is called the *(random) infinitesimal generator* of $\{U_{s,t}\}$. Note that if $\{T_{s,t}\}$ has independent increments, its infinitesimal generator is deterministic.

$(\mathbf{A.2})_m$. For any $s<t$, $T_{s,t}$ maps $\mathbf{C}^{m+2}$ to $\mathbf{C}^m_{loc}$ a.s. Furher, for any $N>0$ there exists a positive constant c_N such that

$$\sup_{|x|\le N} E[|D^\alpha T_{s,t}f(x)-D^\alpha f(x)|^2]\le c_N|t-s|\ \|f\|^2_{|\alpha|+2},\quad \forall f\in \mathbb{C}^{|\alpha|+2} \tag{2.4}$$

hold for any α with $|\alpha|\le m$.

Theorem 2.1. (c.f. [1]) *Let $\{T_{s,t}\}$ be a random positive semigroup satisfying $(A.1)$, $(A.2)_m$ for some $m\ge[d/2]+1$. Then it admits a unique random infinitesimal generator $A(t)$, which maps $\mathbf{C}^{m+2}$ into $\mathbf{C}^{m-[d/2]-1}_{loc}$.*

Let $\Delta_n=\{0<t_1^{(n)}<\cdots<t_l^{(n)}<\cdots\}$ be a partition such that $t_l^{(n)}=(l-1)/2^n$. Set for $f\in\mathbf{C}^{m+2}$,

$$A^{(n)}(t)f(x)=\sum_{i;t_{i+1}^{(n)}\le t}\{T_{t_i^{(n)},t_{i+1}^{(n)}}f(x)-f(x)\}. \tag{2.5}$$

Then it holds

$$A(t)f(x)=\lim_{n\to\infty}A^{(n)}(t)f(x) \tag{2.6}$$

locally uniformly in (t,x) in probability.

In particular, if the random positive semigroup has independent increments, then its random infinitesimal generator has also independent increments, i.e., $A(t_{i+1})-A(t_i),\ i=0,1,\dots,n-1$ are independent for any $0<t_0<t_1<\ldots<t_n$.

For the proof of the theorem we need a lemma.

Lemma 2.2. *For $f\in\mathbf{C}^{m+2}$, set*

$$B^{(n)}(t)f(x)=\sum_{i;t_{i+1}^{(n)}\le t}\left\{T_{t_i^{(n)},t_{i+1}^{(n)}}f(x)-U_{t_i^{(n)},t_{i+1}^{(n)}}f(x)\right\}. \tag{2.7}$$

Then for any $N>0$ there exists a positive constant c'_N not depending on $t,\ f,\ n$ such that

$$\sup_{|x|\le N}E[|D^\alpha B^{(n)}(t)f(x)|^2]\le c'_N t\|f\|^2_{|\alpha|+2}, \tag{2.8}$$

holds for any α with $|\alpha|\le m$.

Proof. Set $M_t^{(n)} := D^\alpha B^{(n)}(t)f(x)$. Then $M_t^{(n)}, t \geq 0$ is a real valued martingale. Set $t_{j+1} = j/2^n$. Then we have $E[|M_t^{(n)}|^2] = \sum_{j=1}^{[2^n t]} E[|M_{t_{j+1}}^{(n)} - M_{t_j}^{(n)}|^2]$. Note that

$$E[|D^\alpha U_{s,t}f(x) - D^\alpha f(x)|^2] \leq E[|D^\alpha T_{s,t}f(x) - D^\alpha f(x)|^2].$$

Then we have

$$E\left[\left|M_{t_{j+1}}^{(n)} - M_{t_j}^{(n)}\right|^2\right] \leq c_N' 2^{-n} \|f\|_{|\alpha|+2}^2 \tag{2.9}$$

by $(A.2)_m$. Therefor we have $E[|M_t^{(n)}|^2] \leq c_N' t \|f\|_{|\alpha|+2}^2$. The proof is complete.

Proof of Theorem 2.1. We shall apply Sobolev's imbedding theorem. For a positive integer m and positive number N, set

$$\|f\|_{m,2,N} = \left(\sum_{\alpha;|\alpha|\leq m} \int_{|x|\leq N} |D^\alpha f(x)|^2 dx \right)^{1/2},$$

where $D^\alpha f(x)$ is the distributional derivative of f. Denote by $H_{m,2,N}$ the set of all f such that $\|f\|_{m,2,N} < \infty$. Let $X = X(x)$ be a $H_{m,2,N}$-valued random variable. We set

$$\|X\|_{m,2,N}' = \left(\sum_{\alpha;|\alpha|\leq m} \int_{|x|\leq N} E[|D^\alpha X(x)|^2] dx \right)^{1/2}.$$

We denote by $W_{m,2,N}$ the set of all $H_{m,2,N}$-valued random variables X such that the above norm is finite. It is a real separable Hilbert space.

In view of Lemma 2.2, we can regard that $B^{(n)}(t)f$ of (2.7) are $W_{m,2,N}$-valued random variables for any N. For a fixed $T > 0$, they satisfy

$$\|B^{(n)}(T)f\|_{m,2,N}' \leq c_N'' \|f\|_{m+2}, \qquad \forall n = 1, 2, \dots \tag{2.10}$$

for any $f \in \mathbf{C}^{m+2}$ in view of (2.8). Then $\{B^n(T)f, n = 1, 2, \dots,\}$ is compact with respect to the weak topology of $W_{m,2,N}$. Therefore for a countable dense linear subspace D of $\mathbf{C}^{m+2}$, we can choose a subsequence $B^{(n_i)}(T)f,\ f \in D$ which converge with respect to the weak topology of $W_{m,2,N}$ for any N. Let $B(T)f,\ f \in D$ be their weak limits. They admit the linear property $B(T)(\alpha_1 f_+ \alpha_2 f_2) = \alpha_1 B(T)f_1 + \alpha_2 B(T)f_2$ in D a.s. Further, they satisfy the inequalities

$$\|B(T)f\|_{m,2,N}' \leq c_N'' \|f\|_{m+2}. \tag{2.11}$$

Consequently $B(T)f$ can be extended to a random linear map from $\mathbf{C}^{m+2}$ into $H_{m,2,N}$ for any N. Then $B(T)f(x)$ is a $\mathbf{C}_{loc}^{m-[d/2]-1}$-valued random variable by Sobolev's imbedding theorem.

Define a martingale by $B(t)f(x) = E[B(T)f(x)|\mathcal{F}_t]$ and set

$$A(t)f = \int_0^t L(s)f ds + B(t)f. \tag{2.12}$$

We shall prove that it is an infinitesimal generator of $T_{s,t}$. Associated with the partition $\Delta_m = \{s = t_0^{(m)} < \cdots < t_{n_m}^{(m)} = t\}$, set $T_{s,r}^{(m)} = T_{s,t_k^{(m)}}$ if $t_k^{(m)} \leq r < t_{k+1}^{(m)}$. Set

$$I_{m,n} = \sum_k T^{(m)}_{s,t_k^{n)}}(B^{(n)}(t_{k+1}^{(n)}) - B^{(n)}(t_k^{(n)}))f(x).$$

Then $I_{m,n}$ converges to $I_m = \int_s^t T_{s,r-}^{(m)} B(dr)f(x)$ weakly in L^2 as $n \to \infty$. Furthermore, we have $\lim_{m\to\infty} I_m = \int_s^t T_{s,r-}B(dr)f(x)$. We claim $\lim_{n\to\infty} I_{n,n} = \int_s^t T_{s,r-}B(dr)f(x)$. Note that $T_{s,t}$ is represented by a random positive kernel $T_{s,t}(x,dy)$, i.e.,

$$T_{s,t}f(x) = \int_{\mathbb{R}^d} T_{s,t}(x,dy)f(y).$$

Then we have

$$\begin{aligned} E[|I_{m,n} - I_{n,n}|^2] = &\sum_k E[(T^{(m)}_{s,t_k^{(n)}}(x,dy) - T^{(n)}_{s,t_k^{(n)}}(x,dy)) \\ &(T^{(m)}_{s,t_k^{(n)}}(x,dy') - T^{(n)}_{s,t_k^{(n)}}(x,dy')) \\ &(B^{(n)}(t_{k+1}^{(n)}f(y) - B^{(n)}(t_k^{(n)})f(y)) \\ &(B^{(n)}(t_{k+1}^{(n)})f(y') - B^{(n)}(t_k^{(n)})f(y'))]. \end{aligned}$$

It converges to 0 as $n \to \infty$ and $m \to \infty$. Consequently,

$$\begin{aligned} T_{s,t}f - f &= I_{n,n} + \sum_k T_{s,t_k^{(n)}}\left(U_{t_k^{(n)},t_{(k+1)}^{(n)}}f - f\right) \\ &\to \int_s^t T_{s,r-}B(dr)f + \int_s^t T_{s,r-}L(r)f\,dr, \end{aligned}$$

proving that $A(t)$ is an infinitesimal generator of the random positive semigroup $\{T_{s,t}\}$.

We shall next prove (2.6). Then the uniqueness of the random infinitesimal generator $A(t)$ follows. Note that

$$T_{t_k,t_{k+1}}f - f = \int_{t_k}^{t_{k+1}} T_{t_k,r-}B(dr)f + \int_{t_k}^{t_{k+1}} T_{t_k,r-}L(r)f\,dr. \tag{2.13}$$

It holds

$$\lim_{n\to\infty}\sum_{t_k\le t}\int_{t_k}^{t_{k+1}} T_{t_k,r-}L(r)f\,dr=\int_0^t L(r)f\,dr,$$

$$\lim_{n\to\infty}\sum_{t_k\le t}\int_{t_k}^{t_{k+1}} T_{t_k,r-}B(dr)f=B(t)f.$$

Then (2.6) follows.

The last statement of the theorem is immediate from (2.6). The proof is complete.

We shall give an another sufficient condition for the existence of the infinitesimal generator of the random positive semigroup. Let $0<\delta\le 1$. For a function f of $\mathbf{C}^m$, set

$$\|f\|_{m+\delta}:=\|f\|_m+\sum_{|\alpha|=m}\sup_{x\neq y}\frac{|D^\alpha f(x)-D^\alpha f(y)|}{|x-y|^\delta},$$

and denote by $\mathbb{C}^{m+\delta}$ the totality of $f\in\mathbf{C}^m$ such that $\|f\|_{m+\delta}<\infty$.

$(\mathbf{A.3})_\delta$. For each $s<t$, $T_{s,t}$ maps $\mathbf{C}^\delta$ into $\mathbf{C}^\delta_{loc}$ a.s. Further, for any positive integers p and N, there exists a positive constant $c_{p,N}$ such that

$$\sup_{|x|\le N} E[|T_{s,t}f(x)-f(x)|^{2p}|\mathcal{F}_s]\le c_{p,N}|t-s|\ \|f\|_{2+\delta}^{2p}\quad a.s., \tag{2.14}$$

$$\begin{aligned}&E[|T_{s,t}f(x)-f(x)-T_{s,t}f(y)+f(y)|^{2p}|\mathcal{F}_s]\\&\quad\le c_{p,N}|t-s\|\ |x-y|^{2p\delta}\|f\|_{2+\delta}^{2p}\quad a.s.\end{aligned} \tag{2.15}$$

holds for all $s<t, x,y\in B_N$ and $f\in\mathbf{C}^{2+\delta}$, where $B_N=\{x;|x|\le N\}$.

Remark. (1) The above assumption is checked more easily than $(A.2)_m$ in the case where the random positive semigroup has independent increments. Indeed the conditional expectations of the right hand sides of (2.14) and (2.15) can be replaced by the expectations, since $T_{s,t}$ and $\mathcal{F}_s$ are independent for any $s<t$.

(2) Inequality (2.15) is satisfied with $\delta=1$ if the following inequality holds.

$$\sup_{|x|\le N} E[\,|D^\alpha T_{s,t}f(x)-D^\alpha f(x)|^{2p}|\mathcal{F}_s]\le c_{p,N}|t-s|\ \|f\|_{|\alpha|+2}^{2p},\quad a.s. \tag{2.16}$$

holds for any α with $|\alpha|=1$, and $f\in\mathbf{C}^{|\alpha|+2}$.

Theorem 2.3. (c.f. [1]) *Let $\{T_{s,t}\}$ be a random positive semigroup satisfying $(A.1)$, $(A.3)_\delta$ for some $\delta>0$. Then it admits a unique random infinitesimal generator $A(t)$, which maps $\mathbf{C}^{2+\delta}$ into $\mathbf{C}^\gamma_{loc}$ for any $0<\gamma<\delta$. Further, the latter assertion of Theorem 2.1 is valid.*

For the proof of the theorem, we need a lemma.

Lemma 2.4. *Let $B^{(n)}(t)f$ be as in Lemma 2.2. For any positive integers p and N, there exists a positive constant $c'_{p,N}$ such that for any $f \in \mathbf{C}^{2+\delta}$,*

$$\sup_{|x|\le N} E[|B^{(n)}(t)f(x)|^{2p}] \le c'_{p,N} t \|f\|_{2+\delta}^{2p}, \tag{2.17}$$

$$E[|B^{(n)}(t)f(x) - B^{(n)}(t)f(y)|^{2p}] \le c'_{p,N} t |x-y|^{2p\delta} \|f\|_{2+\delta}^{2p}, \quad \forall x, y \in B_N \tag{2.18}$$

holds for all n.

Proof. We shall prove (2.18) only, since the proof of (2.17) is done similarly. We shall fix $f \in \mathbf{C}^{2+\delta}$ and $x, y \in B_N$. Set $M_t^{(n)} := B^{(n)}(t)f(x) - B^{(n)}(t)f(y)$. Then $M_t^{(n)}$, $t \ge 0$ is a real valued martingale. By Burkholder's inequality, there exists a positive constant c_1 (not depending on f, x, y, n) such that for any $t \in \Delta_n$,

$$E[|M_t^{(n)}|^{2p}] \le c_1 E\left[\left(\sum_{j=1}^{2^n t} |M_{t_{j+1}}^{(n)} - M_{t_j}^{(n)}|^2\right)^p\right], \tag{2.19}$$

where $t_{j+1} = j/2^n$. Set $N_k = \sum_{1\le j\le k} |M_{t_{j+1}}^{(n)} - M_{t_j}^{(n)}|^2$. It holds

$$N_j^p - N_{j-1}^p = \sum_{l=0}^{p-1} \binom{p}{l} N_{j-1}^l (N_j - N_{j-1})^{p-1}.$$

Since

$$E[(N_j - N_{j-1})^{p-l} | \mathcal{F}_{t_{j-1}}] \le c_{p,N} 2^{-n} |x-y|^{2(p-l)\delta} \|f\|_{2+\delta}^{2(p-l)} \quad a.s.$$

holds for any $l = 1, \ldots, p-1$ by $(A.3)_\delta$, we have

$$\begin{aligned} E[N_{[2^n t]}^p] &= \sum_{1\le j\le [2^n t]} E[N_j^p - N_{j-1}^p] \\ &\le c_{p,N} \sum_{1\le j\le [2^n t]} \sum_{l=0}^{p-1} \binom{p}{l} 2^{-n} |x-y|^{2(p-l)\delta} \|f\|_{2+\delta}^{2(p-l)} E[N_{j-1}^l]. \end{aligned} \tag{2.20}$$

If $p = 1$, the above inequality implies

$$E[N_{[2^n t]}] \le c'_{1,N} t |x-y|^{2\delta} \|f\|_{2+\delta}^2.$$

If $p = 2$, substituting the above to (2.20), we have

$$E[N_{[2^n t]}^2] \le c'_{2,N} t |x-y|^{4\delta} \|f\|_{2+\delta}^4.$$

Repeating this argument inductively, we arrive at

$$E[N^p_{[2^n t]}] \le c'_{p,N} t|x-y|^{2p\delta}\|f\|^{2p}_{2+\delta}.$$

We have thus proved the inequality (2.18). The proof is complete.

Proof of Theorem 2.3. Let T be an arbitrary positive constant. Let D be a countable dense linear subspace of $\mathbf{C}^{2+\delta}$. Then in view of the previous lemma, for any positive integer p we can choose a subsequence $\{n_i\}$ of $\{n\}$ such that $B^{(n_i)}(T)f(x),\ f \in D,\ x \in \mathbb{R}^d$ converge weakly in L^{2p}. Let $B(T)f(x),\ f \in D,\ x \in \mathbb{R}^d$ be their limits. They admit the linear property $B(T)(\alpha_1 f_+ \alpha_2 f_2)(x) = \alpha_1 B(T)f_1(x) + \alpha_2 B(T)f_2(x)$ in D for all x a.s. Further, it satisfies the inequality

$$E[|B(T)f(x) - B(T)f(y)|^{2p}] \le c'_{p,N} T|x-y|^{2\delta p}\|f\|^{2p}_{2+\delta} \quad \forall x, y \in B_N. \tag{2.21}$$

Therefore, by Kolmogorov's theorem, $B(T)f$ can be extended as a random linear map from $\mathbf{C}^{2+\delta}$ into $\mathbf{C}^{\gamma}_{loc}$ where $0 < \gamma < \delta$. See [3, p.31]. Define a martingale by $B(t)f(x) = E[B(T)f(x)|\mathcal{F}_t]$ and define $A(t)$ by (2.12). Then this $A(t)$ is the random infinitesimal generator of the given random positive semigroup as is shown in the proof of Theorem 2.1.

3. Representations of Random Infinitesimal Generators

We shall represent the random infinitesimal generator $A(t)$ of a random positive semigroup $\{T_{s,t}\}$. We denote by V^+ the totality of positive bounded linear operators T on $\mathbf{C}$ equipped with the strong topology, i.e. $T_n \to T$ in $\mathbf{V}+$ if and only if $T_n f \to T f$ in $\mathbf{C}$ for any $f \in \mathbf{C}$.

Theorem 3.1. *Assume* $(A.1)$ *and* $(A.2)_m,\ m \ge [d/2]+1$ *or* $(A.3)_\delta,\ 0 < \delta < 1$. *Then the random infinitesimal generator* $A(t)$ *is represented as a random integro-differential operator:*

$$\begin{aligned} A(t)f(x) = &\int_0^t L(s)f(x)ds + \sum_i F_i(x,t)\frac{\partial f}{\partial x_i}(x) + G(x,t)f(x) \\ &+ \int_{V^+} \{Tf(x) - f(x)\}\tilde{N}((0,t], dT). \end{aligned} \tag{3.1}$$

Here, each term of the above representation is interpreted as follows:

(i) $L(t)$ *is a second order integro-differential operator represented by*

$$\begin{aligned} L(t)f(x) = &\frac{1}{2}\sum_{i,j} a_{ij}(x,t)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_i b_i(x,t)\frac{\partial f}{\partial x_i}(x) + c(x,t)f(x) \\ &+ \int_{\mathbb{R}^d}(f(y) - f(x) - \sum_i \frac{y_i - x_i}{1+|y-x|^2}\frac{\partial f}{\partial x_i}(x))n_t(x,dy). \end{aligned} \tag{3.2}$$

Here $a_{ij}(x,t),\ b_i(x,t),\ c(x,t)$ *are* $\{\mathcal{F}_t\}$*-adapted and are continuous in* (x,t) *a.s. The matrix* $(a_{ij}(x,t))$ *is symmetric and nonnegative definite a.s.* $n_t(x,dy)$ *is an* $\{\mathcal{F}_t\}$*-adapted Lévy measure such that* $n_t(x,\{x\})=0$ *and* $\int_{\mathbb{R}^d}\phi_x(y)n_t(x,dy)<\infty$ *for any* t,x *a.s., where* ϕ_x *is a function of* $\boldsymbol{C}^2$ *such that* $\phi_x(x)=0,\ \phi_x(y)>0$ *if* $y\neq x$, *lim* $\inf_{y\to\infty}\phi_x(y)>0$ *and* $\phi_x(y)=O\left(|x-y|^2\right)$ *near* x.

(ii) $(F_1(x,t),\ldots,F_d(x,t))$ *is a continuous martingale with values in* $\mathbf{C}^{\gamma}_{loc}(\mathbb{R}^d,\mathbb{R}^d)$ *where* $\gamma=m-[d/2]-1$ *under* $(A.2)_m$ *and* $0<\gamma<\delta$ *under* $(A.3)_\delta$. *It is of mean* 0 *and the joint quadratic variation is written by*

$$\langle F_i(x,t),F_j(y,t)\rangle=\int_0^t f_{ij}(x,y,s)ds, \tag{3.3}$$

which are continuous in (x,y,t) *and* $\mathbf{C}^{\gamma}$*-functions of* (x,y). *Furthermore, the matrices* $(a_{ij}(x,t))-(f_{ij}(x,x,t))$ *are nonnegative definite for any* x *a.e.* t.

(iii) $G(x,t)$ *is a continuous martingale with values in* $\mathbf{C}^{\gamma}_{loc}(\mathbb{R}^d,\mathbb{R}^1)$ *with mean* 0.

(iv) $N(dtdT)$ *is a counting measure on* $\boldsymbol{V}^+$ *with an intensity measure of the form* $dt\mu_t(dT)$. *The meaures* $n_t(x,\cdot)-\int_{V^+}\mu_t(dT)T(x,dy)$ *are positive (nonnegative) for any* x *a.e.* t, *where* $T(x,dy)$ *is the kernel such that* $Tf(x)=\int T(x,dy)f(y)$ *holds for* $f\in\mathbf{C}$.

Furthermore, under $(A.2)_m$ *the intensity measure satisfies*

$$E\left[\int_0^t\int_{V^+}\sup_{|x|\le N}|D^\alpha Tf(x)-D^\alpha f(x)|^2\mu_s(dT)ds\right]\le c_N\|f\|^2_{m+2}, \tag{3.4}$$

for any α *with* $|\alpha|\le m-[d/2]-1$, *and under* $(A.3)_\delta$ *it satisfies*

$$\begin{aligned}&E\left[\int_0^t\int_{V^+}|Tf(x)-f(x)-Tf(y)+f(y)|^{2p}\mu_s(dT)ds\right]\\&\qquad\le c_{p,N}|x-y|^{2p\delta}\|f\|^{2p}_{2+\delta},\end{aligned} \tag{3.5}$$

for all $x,y\in B_N$.

In particular, if the random positive semigroup has independent increments, all coefficients and Lévy measures of the operator $L(t)$ *of* (3.2) *are deterministic. Furthermore,* $(F_1(x,t),\ldots,F_d(x,t),G(x,t))$ *is a Brownian motion with the spatial parameter* x, *and* $N(dtdT)$ *is a Poisson random measure with the deterministic intensity measures* μ_t.

Proof. We shall prove the theorem under conditions $(A.1)$ and $(A.2)_m$ only. The other case can be verified similarly. The representation of the operator $L(t)$ can be proved similarly as in Yosida [4] (XIII, Section 7) (cf. Kifer-Kunita [1]). Set $B(t)f=A(t)f-\int_0^t L(s)fds$ for $f\in\mathbf{C}^{m+2}$. Then it is decomposed as $B(t)f=B^c(t)f+B^d(t)f$, where $B^c(t)f$ is a $\mathbf{C}^{\gamma}$-valued continuous martingale and $B^d(t)f$ is a $\mathbf{C}^\gamma$-valued discontinuous martingale, where $\gamma = m - [d/2] - 1$. Define the counting measure $N(dtdT)$ on $[0,\infty) \times \mathbf{V}^+$ by

$$N((0,t] \times A) = \sharp\{s \in (0,t];\ T_{s,s+} \in E\}, \tag{3.6}$$

where E is a Borel set in $\mathbf{V}^+ - \{0\}$. Let $\mu(dtdT)$ be its compensator. We show later that the compensator is written as $\mu(dtdT) = dt\mu_t(dT)$. Set $\tilde{N}(dtdT) = N(dtdT) - \mu(dtdT)$. Then $B^d(t)f(x)$ is represented by

$$B^d(t)f(x) = \int_{V^+} \{Tf(x) - f(x)\}\tilde{N}((0,t], dT). \tag{3.7}$$

The bracket process of $D^\alpha B^d(t)f(x)$ is given by

$$\langle D^\alpha B^d(t)f(x)\rangle = \int_0^t \int_{V^+} (D^\alpha Tf(x) - D^\alpha f(x))^2 \mu(dtdT).$$

It satisfies

$$\|\langle D^\alpha B^d(t)f\rangle\|'_{m,2,N} = \|D^\alpha B^d(t)f\|'_{m,2,N} \le c''_N \|f\|_{m+2}$$

by (2.11). Then Sobolev's inequality implies (3.4).

Now observe that

$$\begin{aligned} f(x+y) = &f(x) + \sum_i (y_i - x_i)\frac{\partial f}{\partial x_i}(x) \\ &+ \frac{1}{2}\sum_{i,j}(y_i - x_i)(y_j - x_j)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \varphi_x(y). \end{aligned}$$

Let ψ_x be a $\mathbf{C}^{m+2}$-function with the compact support such that $\psi_x(y) = 1$ on a certain neighborhood of x. Set $g_x^i(y) = \psi_x(y)(y_i - x_i)$ and $A_{ij}^{(n)}(x,t) = A^{(n)}(t)(g_x^i g_x^j)(x)$, where $A^{(n)}(t)$ is defined by (2.5). Then the matrices $(A_{ij}^{(n)}(x,t))$ are nonnegative definite and are increasing in t with respect to the order of nonnegative definiteness for any x, n. Therfore $(A_{ij}(x,t)) \equiv (A(t)(g_x^i g_x^j)(x))$ is also nonnegative definite and increases with t for any x. This implies that its continuous martingale part of mean 0 is identically 0. Therefore we have $B^c(t)(\phi_x f)(x) = 0$ for any $f \in \mathbf{C}^{m+2}$ and $\phi_x(y)$ such that $\phi_x(y) = O(|x-y|^2)$ near x. Now set

$$F_i(x,t) = B^c(t)(g_x^i)(x), \quad G(x,t) = B^c(t)1(x).$$

These are $\mathbf{C}^\gamma_{loc}$-valued continuous martingales. These do not depend on the choice of the function ψ_x. In fact, let $\tilde{\psi}_x$ be another function with the same property as that of ψ_x. Then $B^c((\psi_x - \psi_x)f)(x) = 0$, since $\psi_x(y) - \psi_x(y) = O(|x-y|^2)$. Therefore $B^c(t)f$ is represented by

$$B^c(t)f(x) = \sum_i F_i(x,t)\frac{\partial f}{\partial x_i}(x) + G(x,t)f(x). \tag{3.8}$$

We have thus shown a representation of the infinitesimal generator $A(t)$.

We shall prove the latter assertion of (ii). Let $\Delta_n = \{0 = t_0 < t_1^{(n)} < \cdots t_k^{(n)} < \cdots\}$ be a sequence of partitions as before. We set $t_k = t_R^{(n)} = (k-1)/2^n$. Then we have

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{s\le t_k\le t} U_{t_k,t_{k+1}}(g_x^i g_x^j)(x) \\
&\quad = \int_s^t L(r)(g_x^i g_x^j)(x)dr \\
&\quad = \int_0^t a_{ij}(x,r)dr + \int_s^t \left(\int_{\mathbb{R}^d} n_r(x,dy)g_x^i(y)g_x^j(y)\right)dr,
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{s\le t_k\le t} E[T_{t_k,t_{k+1}}g_x^i(x)T_{t_k,t_{k,1}}g_x^j(x)T_{t_k,t_{k+1}}1(x)^{-1}|\mathcal{F}_{t_k}] \\
&\quad = \lim_{n\to\infty} \sum_{s\le t_k\le t} E[T_{t_k,t_{k+1}}g_x^i(x)T_{t_k,t_{k+1}}g_x^j(x)|\mathcal{F}_{t_k}] \\
&\quad = \langle (A(t)-A(s))g_x^i, (A(t)-A(s))g_x^j\rangle \\
&\quad = \langle F_i(x,t), F_j(x,t)\rangle - \langle F_i(x,s), F_j(x,s)\rangle \\
&\qquad + \int_0^t \int_{V^+} Tg_x^i(x)Tg_x^j(x)\mu(drdT).
\end{aligned} \tag{3.10}$$

Now, let $\xi_1, \ldots, \xi_n$ be any real numbers. Then, by applying Schwarz's inequality to the kernel $T_{t_k,t_{k+1}}(x,dy)$, we have

$$\begin{aligned}
&E\left[\left|\sum_i T_{t_k,t_{k+1}}g_x^i(x)\xi_i\right|^2 T_{t_k,t_{k+1}}1(x)^{-1}\middle|\mathcal{F}_{t_k}\right] \\
&\quad \le E\left[T_{t_k,t_{k+1}}\left|\sum_i g_x^i\xi_i\right|^2\middle|\mathcal{F}_{t_k}\right] \le U_{t_k,t_{k+1}}\left(\left|\sum_i g_x^i\xi_i\right|^2\right)
\end{aligned}$$

for any k. Take the summation of the each term of the above for k such that $s \le t_k \le t$ and let n tend to infiinity. Then we have the inequality for two matrices:

$$\begin{aligned}
&(\langle F_i(x,t), F_j(x,t)\rangle - \langle F_i(x,s), F_j(x,s)\rangle) \\
&\quad + \left(\int_0^t \int_{\mathbf{V}^+} Tg_x^i(x)Tg_x^j(x)\mu(drdT)\right) \\
&\quad \le \left(\int_s^t a_{ij}(x,r)dr\right) + \left(\int_s^t \left(\int_{\mathbb{R}^d} n_r(x,dy)g_x^i(y)g_x^j(y)\right)dr\right),
\end{aligned} \tag{3.11}$$

where "$\leq$" represents the order of matrices with respect to the positive definiteness. Now choose a sequence of functions $\psi_x^{(n)}(y)$ such that its supports is included in $U_n(x) = \{y : |y - x| \leq 1/n\}$ and set $g_x^{i,(n)}(y) = (y^i - x_i)\psi_x^{(n)}(y)$. Substitute them in place of g_x^i. in equality (3.11). Then, as $n \to \infty$, we find that the second member of the left hand side of (3.11) involving the Lévy measure $n_r(x, dy)$ and the second member of the right hand side involving the compensating measure $\mu(drdT)$ are both converging to 0. Consequently, we obtain

$$(\langle F_i(x,t),\ F_j(x,t)\rangle - \langle F_i(x,s),\ F_j(x,s)\rangle) \leq \left(\int_s^t a_{ij}(x,r)dr\right)$$

a.s. for any $t,\ s$. This proves that $\langle F_i(x,t),\ F_j(x,t)\rangle$ is absolutely continuous with respect to dt and the density function $(f_{ij}(x,x,t))$ satisfies $(f_{ij}(x,x,t)) \leq (a_{ij}(x,t))$ a.e. t. Finally the absolute continuity of $\langle F_i(x,t),\ F_j(y,t)\rangle$ with respect to dt follows from that of $\langle F_i(x,t),\ F_j(x,t)\rangle$, because

$$\begin{aligned} &|\langle F_i(x,t),\, F_j(y,t)\rangle - \langle F_i(x,s),\, F_j(y,s)\rangle| \\ &\quad \leq (\langle F_i(x,t)\rangle - \langle F_i(x,s)\rangle)^{1/2}(\langle F_j(y,t)\rangle - \langle F_j(y,s)\rangle)^{1/2}. \end{aligned}$$

We shall next prove (iv). Let $\epsilon > 0$ and $\tau_n^{(\epsilon)}$, $n = 1, 2, \ldots$ be the sequence of jumping times of the counting process $N_t^{(\epsilon)} = N((s,t], \{\|T - I\| > \epsilon\})$. Given a random positive semigroup $\{T_{s,t}\}$ with the random infinitesimal generator $A(t)$, define

$$\hat{T}_{s,t}^{(\epsilon)} = T_{s,\tau_1^{(\epsilon)}-} \cdots T_{\tau_{n-2}^{(\epsilon)},\tau_{n-1}^{(\epsilon)}-} T_{\tau_{n-1}^{(\epsilon)},t}\,, \quad if\ \tau_{n-1}^{(\epsilon)} \leq t < \tau_n^{(\epsilon)}. \tag{3.12}$$

Then $\{\hat{T}_{s,t}^{(\epsilon)}\}$ defines a random positive semigroup. It satisfies $(A.1)$ and $(A.2)_m$. Let $\hat{A}^{(\epsilon)}(t)$ be its infinitesimal generator. Then it is represented by

$$\begin{aligned} \hat{A}^{(\epsilon)}(t)f(x) = &\int_0^t L(s)f(x)ds + \sum_i F_i(x,t)\frac{\partial f}{\partial x_i}(x) + G(x,t)f(x) \\ &+ \int_{\|T-I\|\leq\epsilon} \{Tf(x) - f(x)\}\tilde{N}((0,t], dT) \\ &- \int_0^t \int_{\|T-I\|>\epsilon} \{Tf(x) - f(x)\}\mu(dsdT). \end{aligned}$$

Further,

$$\int_0^t L(s)fds - \int_{\|T-I\|>\epsilon} \{Tf(x) - f(x)\}\mu(dsdT)$$

is the infinitesimal generator of the conditional average $\hat{U}_{s,t}^{(\epsilon)} f = E[\hat{T}_{s,t}^{(\epsilon)} f|\mathcal{F}_s]$. Its Lévy measure $\hat{n}^{(\epsilon)}(x, dy)$ satisfies

$$\hat{n}_t^{(\epsilon)}(x,dy)dt = n_t(x,dy)dt - \int_{\|T-I\|>\epsilon} T(x,dy)\mu(dtdT).$$

It is a positive measure for any ϵ, x. This proves that $\mu(dtdT)$ is absolutely continuous with respect to dt and the density $\mu_t(dT)$ satisfies $n_t(x,dy) \geq \int T(x,dy)\mu_t(dT)$ a.e. t for any x. The proof is complete.

Now we shall consider a random positive semigroup $\{T_{s,t}\}$ with additional properties stated belows. It is *Markovian* if $T_{s,t}1 \equiv 1$ holds a.s. for any $s < t$. It is a *diffusion type* if (i) $T_{s,t}f$ is continuous in s, t a.s. for any $f \in \mathbf{C}$ and (ii) it has a *local property*, i.e., for any $x_0 \in \mathbb{R}^d$ and $\varphi_{x_0} \in \mathbf{C}$ such that $\varphi_{x_0}(x) = o(|x-x_0|^2)$ near x_0, $T_{s,t}\varphi_{x_0}(x_0)/(t-s)$ converges to 0 in $L^1(P)$ as $t - s \to 0$.

The following corollary can be verified easily.

Corollary 3.2. (1) $\{T_{s,t}\}$ *is Markovian if and only if* $c(x,t) \equiv 0$ *and* $G(x,t) \equiv 0$.

(2) $\{T_{s,t}\}$ *is of diffusion type if and only if its infinitesimal generator is a differential operator, i.e.,* $n_t(x,\cdot) \equiv 0$ *and* $\mu_t \equiv 0$.

4. Asymptotic Properties of Coefficients of Random Infinitesimal Generators

We shall discuss the intrinsic meaning of the coefficients of the infinitesimal generator $A(t)$ in the case of diffusion type. We assume that $T_{s,t}$ has the second order moment, i.e., $\int T_{s,t}(x,dy)|y-x|^2 < \infty$ a.s. for any x. Set

$$G_{s,t}(x) = \int_{\mathbb{R}^d} T_{s,t}(x,dy)1 - 1, \tag{4.1}$$

$$M^i_{s,t}(x) = \int_{\mathbb{R}^d} T_{s,t}(x,dy)(y_i - x_i), \tag{4.2}$$

$$V^{ij}_{s,t}(x) = \int_{\mathbb{R}^d} T_{s,t}(x,dy)(y_i - x_i)(y_j - x_j), \tag{4.3}$$

$$W^{ij}_{s,t}(x) = \int_{\mathbb{R}^d} T_{s,t}(x,dy)(y_i - x_i - M^i_{s,t}(x))(y_j - x_j - M^j_{s,t}(x)). \tag{4.4}$$

Two random fields $\Phi^{(\epsilon)}_t(x)$ and $\Psi^{(\epsilon)}_t(x)$ are called *asymptotically equal* at (x,t) as $\epsilon \to 0$ if

$$\lim_{\epsilon \to 0} E[|\Phi^{(\epsilon)}_t(x) - \Psi^{(\epsilon)}_t(x)|^2] = 0.$$

We will denote the relation by $\Phi^{(\epsilon)}_t(x) \asymp \Psi^{(\epsilon)}_t(x)$.

Theorem 4.1. *Let* $\{T_{s,t}\}$ *be a random positive semigroup of diffusion type satisfying* $(A.1)$ *and* $(A.2)_m$, $m \geq [d/2]+1$ *with the random infinitesimal* *generator represented by* (3.1). *Suppose that the coefficients of the random infinitesimal generator are continuous in* t. *Then we have for any* (x,t):

$$\frac{1}{\epsilon}G_{t,t+\epsilon}(x) \asymp c(x,t)+\frac{1}{\epsilon}(G(x,t+\epsilon)-G(x,t)), \tag{4.5}$$

$$\frac{1}{\epsilon}M^i_{t,t+\epsilon}(x) \asymp b_i(x,t)+\frac{1}{\epsilon}(F_i(x,t+\epsilon)-F_i(x,t)), \tag{4.6}$$

$$\frac{1}{\epsilon}V^{ij}_{t,t+\epsilon}(x) \asymp a_{ij}(x,t), \tag{4.7}$$

$$\frac{1}{\epsilon}W^{ij}_{t,t+\epsilon}(x) \asymp a_{ij}(x,t)-f_{ij}(x,x,t). \tag{4.8}$$

Proof. Note that

$$\begin{aligned} G_{t,t+\epsilon}(x) &= \int_t^{t+\epsilon} T_{t,r}A(dr)1(x) \\ &= \int_t^{t+\epsilon}\int_{\mathbb{R}^d} T_{t,r}c(r)(x)dr + \int_t^{t+\epsilon}\int_{\mathbb{R}^d} T_{t,r}(x,dy)G(y,dr), \end{aligned}$$

where $c(r)(x)=c(x,r)$. Then we have

$$\begin{aligned} &E\left[\left|G_{t,t+\epsilon}(x)-\left(\int_t^{t+\epsilon} c(x,r)dr+G(x,t+\epsilon)-G(x,t)\right)\right|^2\right] \\ &\quad \le 2\left\{E\left[\left|\int_t^{t+\epsilon}(T_{t,r}-I)c(r)(x)dr\right|^2\right]\right. \\ &\qquad +2\left\{E\left[\left|\int_t^{t+\epsilon}\int\int_{\mathbb{R}^d\times\mathbb{R}^d}(T_{t,r}-I)(x,dy)(T_{t,r}-I)(x,dy')g(y,y',dr)\right|^2\right]\right. \\ &\quad = o(\epsilon^2), \end{aligned}$$

where $g(y,y',t)=\langle G(y,t),\, G(y',t)\rangle$. This proves (4.5). Similarly, we have

$$\begin{aligned} &E\left[\left|M_{t,t+\epsilon}(x)-\int_t^{t+\epsilon} b(x,r)dr-F(x,t+\epsilon)+F(x,t)\right|^2\right] \\ &\quad \le 2E\left[\left|\left(\int_t^{t+\epsilon}(T_{t,r}-I)b(r)(x)dr\right|^2\right] \\ &\qquad +2E\left[\int_t^{t+\epsilon}\left(\int\int_{\mathbb{R}^d\times\mathbb{R}^d}(T_{t,r}-I)(x,dy)(T_{t,r}-I)(x,dy')f_{ij}(y,y',r)\right)dr\right] \\ &\quad = o(\epsilon^2), \end{aligned}$$

proving (4.6), and

$$\begin{aligned} E\left[\left|V^{ij}_{t,t+\epsilon}(x) - \int_t^{t+\epsilon} a_{ij}(x,r)dr\right|^2\right] \\ = E\left[\left|\int_t^{t+\epsilon} T_{t,r}(a_{ij}(r))(x)dr - \int_t^{t+\epsilon} a_{ij}(x,r)dr\right|^2\right] \\ = o(\epsilon^2), \end{aligned}$$

proving (4.7).

Now set $\varphi^i_x(y) = y_i - x_i$. Note that

$$W^{ij}_{t,t+\epsilon}(x) = \int_t^{t+\epsilon} T_{t,r}A(dr)(\varphi^i_x\varphi^j_x)(x) - M^i_{t,t+\epsilon}(x)M^j_{t,t+\epsilon}(x).$$

We have

$$\begin{aligned} &\int_t^{t+\epsilon} T_{t,r}A(dr)(\varphi^i_x\varphi^j_x)(x) \\ &\quad= \int_t^{t+\epsilon} T_{t,r}(a_{ij}(r))(x)dr \\ &\qquad+ \int_t^{t+\epsilon} T_{t,r}\{(b_i dr + F_i(dr))\varphi^j_x + (b_j dr + F_j(dr))\varphi^i_x)\}(x) \\ &\quad= J_1 + J_2. \end{aligned} \tag{4.9}$$

It holds $E[|J_2|^2] = o(\epsilon^2)$ because $T_{t,r}|\varphi^j_x|^2(x) = O(\epsilon^2)$ if $|r-t| \le \epsilon$ by (4.7). We have further,

$$\begin{aligned} M^i_{t,t+\epsilon}(x)M^j_{t,t+\epsilon}(x) &= \int_t^{t+\epsilon} M^j_{t,r}(x)T_{t,r}\{b_i(r)dr + F_i(dr)\}(x) \\ &\quad+ \int_t^{t+\epsilon} M^i_{t,r}(x)T_{t,r}\{b_j(r)dr + F_j(dr)\}(x) \\ &\quad+ \int_t^{t+\epsilon} \left(\int\int_{\mathbb{R}^d\times\mathbb{R}^d} T_{t,r}(x,dy)T_{t,r}(x,dy')f_{ij}(y,y',r)\right) dr \\ &= K_1 + K_2 + K_3. \end{aligned} \tag{4.10}$$

It holds $E[|K_1|^2] = o(\epsilon^2)$ and $E[|K_2|^2] = o(\epsilon^2)$, because $M^i_{t,r}(x) = O(\epsilon)$ if $|r-t| < \epsilon$ by (4.6). Further,

$$E\left[\left|J_1 - K_3 - \int_t^{t+\epsilon}(a_{ij}(x,r) - f_{ij}(x,x,r))dr\right|^2\right] = o(\epsilon^2).$$

Therefore (4.8) is verified.

The above theorem can be extended to a more general random positive semigroup.

Theorem 4.2. *Let $\{T_{s,t}\}$ be a random positive semigroup satisfying $(A.1)$ and $(A.2)_m$, $m \geq [d/2]+1$ with the infinitesimal generator represented by* (3.1)*. Suppose that the coefficients of the random infinitesimal generator and their characteristics are continuous in t. Then we have for any (x,t):*

$$\frac{1}{\epsilon}G_{t,t+\epsilon}(x) \asymp c(x,t)+\frac{1}{\epsilon}(G(x,t+\epsilon)-G(x,t)) + \frac{1}{\epsilon}\int_{V^+}\{T1(x)-1\}\tilde{N}((t,t+\epsilon],dT) \tag{4.11}$$

$$\frac{1}{\epsilon}M^i_{t,t+\epsilon}(x) \asymp b_i(x,t)+\frac{1}{\epsilon}(F_i(x,t+\epsilon)-F_i(x,t)) + \frac{1}{\epsilon}\int_{V^+}T\varphi^i_x(x)\tilde{N}((t,t+\epsilon],dT), \tag{4.12}$$

$$\frac{1}{\epsilon}W^{ij}_{t,t+\epsilon}(x) \asymp a_{ij}(x,t)+\int_{\mathbb{R}^d}n_t(x,dy)(y_i-x_i)(y_j-x_j), \tag{4.13}$$

$$\frac{1}{\epsilon}W^{ij}_{t,t+\epsilon}(x) \asymp a_{ij}(x,t)+\int_{\mathbb{R}^d}n_t(x,dy)(y_i-x_i)(y_j-x_j), -\left(f_{ij}(x,x,t)+\int_{V^+}T(\varphi^i_x\varphi^j_x)(x)\mu_t(dT)\right), \tag{4.14}$$

where $\varphi^i_x(y)=y_i-x_i$. Furthermore, for any (x,t)

$$\frac{1}{\epsilon}T_{t,t+\epsilon}f(x) \asymp \int_{\mathbb{R}^d}n_t(x,dy)f(y)-\int_{V^+}Tf(x)\mu_t(dT) + \frac{1}{\epsilon}\int_{V^+}Tf(x)N((t,t+\epsilon],dT), \tag{4.15}$$

holds for any $f\in\mathbf{C}^2$ such that $f(x)=0$ and $f(x)=o(|x-y|^2)$ near x.

The proof is omitted. It is left to the reader.

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